Petrov-Galerkin Crank-Nicolson Scheme for Parabolic Optimal Control Problems on Nonsmooth Domains

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Abstract. In this paper we transfer the a priori error analysis for the discretization of parabolic optimal control problems on domains allowing for H^2 regularity (i.e. either with smooth boundary or polygonal and convex) to a large class of nonsmooth domains. We show that a combination of two ingredients for the optimal convergence rates with respect to the spatial and the temporal discretization is required. First we need a time discretization scheme which has the desired convergence rate in the smooth case. Secondly we need a method to treat the singularities due to non-smoothness of the domain for the corresponding elliptic state equation. In particular we demonstrate this philosophy with a Crank-Nicolson time discretization and finite elements on suitably graded meshes for the spatial discretization. A numerical example illustrates the predicted convergence rates.

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1. Introduction

In this paper we extend the a priori error analysis for discretizations of a parabolic optimal control problem to the case of nonsmooth domains. The model problem under consideration is formulated as

$$\text{Minimize } J(q,u) = \frac{1}{2}\int_0^T \int_\Omega |u-\hat{u}|^2\,dx\,dt + \frac{\alpha}{2}\int_0^T \int_\Omega |q|^2\,dx\,dt, \tag{1.1a}$$

subject to the state equation

$$\begin{aligned} \partial_t u - \Delta u &= f + q && \text{in } (0,T) \times \Omega, \\ u &= 0 && \text{in } (0,T) \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{1.1b}$$

where $u = u(t,x)$ denotes the state variable and $q = q(t,x)$ is the control variable. A precise formulation of this problem including a functional analytic setting is given in the next section.

In the literature on a priori error estimates for this kind of problems, see, e.g., [24, 25, 26, 3, 27, 17, 18], the domain Ω is always assumed either to have a smooth boundary $\partial\Omega$ or to be polygonal and convex. Our main contribution is an extension of the results from [26] to a more general class of domains including polygonal or polyhedral domains with (non-convex) reentrant corners.

For optimal control problems governed by elliptic equations on non-convex domains there are several contributions establishing optimal order error estimates on properly chosen graded meshes, see [7, 6, 10, 5].

Our strategy is as follows. We formulate an assumption (see Assumption 4.2) on a family of finite element meshes ensuring optimal error estimates for the elliptic Ritz projection. This assumption is satisfied for different non-smooth domains with appropriate mesh grading, see Section 5 for details. Under this assumption, which is of "pure elliptic nature", we check, that all proofs from [26] can be directly extended. This means, that for getting optimal order error estimates for the parabolic optimal control problem, it is enough to check the approximation properties for the discretization of the corresponding elliptic equation. This philosophy explained here on the example of the discretization based Petrov-Galerkin Crank-Nicolson scheme in time and linear finite elements in space for the model problem mentioned above can be extended in several directions. First of all, one can include control constraints in the same fashion as in [26], also the consideration of a more general parabolic problem with variable coefficients is possible. For an extension of dG(r) (discontinuous Galerkin) discretizations, e.g., from [24, 25] an additional assumption on the L^2-projection similar to Assumption 4.2 is required. This additional assumption will be fulfilled on the same families of meshes as described in Section 5. Under this assumption we strongly expect that also the error estimates for a semi-linear parabolic equation, see [27], as well as for problems with state constraints, see [23, 17] can be covered.

The outline of the paper is as follows. In the next section we discuss the optimality conditions and the regularity issues for the optimal control under consideration. After the description of the discretization scheme in Section 3 we formulate and prove our main result on a priori error analysis in Section 4 under Assumption 4.2. Section 5 is devoted to the verification of this assumption for different situations. Finally, in Section 6 we present a numerical example illustration our results.

2. Continuous Problem

In this section, we briefly discuss the precise formulation of the optimization problem under consideration. Furthermore, we recall theoretical results on existence, uniqueness, and regularity of optimal solutions as well as optimality conditions. For this discussion, we explicitly take the possible non-smoothness of the domain Ω into account.

To set up a weak formulation of the state equation (1.1b), we introduce the following notation: For a polygonal or polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, we denote V to be $H_0^1(\Omega)$. Together with $H := L^2(\Omega)$, the Hilbert space V and its dual $V^* = H^{-1}(\Omega)$ build a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. Here and in what follows, we employ the usual notion for Lebesgue and Sobolev spaces. Furthermore, let D_Δ be the domain of the Laplacian given by $D_\Delta := \{\, v \in V \mid \Delta v \in H \,\}$.

Remark 2.1. If Ω is polygonal and convex or possesses a smooth boundary, then $D_\Delta = H^2(\Omega) \cap H_0^1(\Omega)$ (see e.g. [14, Theorem 4 in Section 6.3] for the case of a smooth boundary and [19, Remark 2.4.6 and Corollary 2.6.8.] for polygonal and convex domains). Since we do not assume the convexity of Ω the space D_Δ is in general not a subset of $H^2(\Omega)$.

For a time interval $I = (0, T)$ we introduce the state space

$$X := W(0, T) = \left\{\, v \;\middle|\; v \in L^2(I, V) \text{ and } \partial_t v \in L^2(I, V^*) \,\right\}$$

and the control space $Q := L^2(I, H)$. We use the following notations for the inner products and norms on $L^2(\Omega)$ and $L^2(I, H)$:

$$\begin{aligned} (v, w) &:= (v, w)_{L^2(\Omega)}, & (v, w)_I &:= (v, w)_{L^2(I,H)}, \\ \|v\| &:= \|v\|_{L^2(\Omega)}, & \|v\|_I &:= \|v\|_{L^2(I,H)}. \end{aligned}$$

In this setting, a standard weak formulation of the state equation (1.1b) for given control $q \in Q$, $f \in L^2(I, H)$, and $u_0 \in H$ reads: Find a state $u \in X$ satisfying

$$\begin{aligned} (\partial_t u, \varphi)_I + (\nabla u, \nabla \varphi)_I &= (f + q, \varphi)_I \qquad \forall \varphi \in X, \\ u(0) &= u_0. \end{aligned} \tag{2.1}$$

Assumption 2.2. For our analysis, we will assume the following regularity properties of the data: $f, \hat{u} \in H^1(I, H)$ with $f(0), \hat{u}(T) \in V$ and $u_0 \in V$ with $\Delta u_0 \in V$.

Using this assumption, the following result on existence and regularity can be proved:

Proposition 2.3. *Under Assumption* 2.2 *and for fixed control* $q \in Q$*, there exists a unique solution* $u \in X$ *of problem* (2.1)*. Moreover, the solution exhibits the regularity*

$$u \in L^2(I, D_\Delta) \cap H^1(I, H) \cap C(\bar{I}, V)$$

with the estimate

$$\|\Delta u\|_I + \|\partial_t u\|_I + \|\nabla u(T)\| \le C\left\{\|f + q\|_I + \|\nabla u_0\|\right\}.$$

If additionally the fixed control q is in $H^1(I,H) \subset Q$, the state u exhibits the improved regularity

$$u \in H^1(I,D_\Delta) \cap H^2(I,H)$$

and the stability estimate

$$\|\partial_t \Delta u\|_I + \|\partial_t^2 u\|_I \le C\big\{\|f+q\|_{H^1(I,H)} + \|\nabla(f+q)(0)\| + \|\nabla \Delta u_0\|\big\}$$

holds.

Proof. Existence and the regularity $u \in L^2(I,D_\Delta) \cap H^1(I,H)$ is proven in [19, Theorem 5.1.1]. Then, the assertion $u \in C(\bar{I},V)$ and the corresponding estimates for can be obtained by choosing $-\Delta u \in L^2(I,H)$ and $\partial_t u \in L^2(I,H)$ as test function in (2.1).

The improved regularity $u \in H^1(I,D_\Delta) \cap H^2(I,H)$ can be proved as in [14] provided that the right-hand side $f+q$ exhibits the regularity $f+q \in H^1(I,H)$ with $(f+q)(0) \in V$ and the initial condition u_0 fulfills $\Delta u_0 \in V$. This is ensured by Assumption 2.2, the assumed regularity $q \in H^1(I,H)$, and the embedding $H^1(I,V) \hookrightarrow C(\bar{I},V)$. In contrast to [14], where $H^2(\Omega)$ regularity is used, one can not expect here the state variable to lie in $H^1(I;H^2(\Omega))$. However the proof of the stated results goes through. □

The weak formulation of the optimal control problem (1.1) is given as

$$\text{Minimize } J(q,u) := \frac{1}{2}\|u-\hat{u}\|_I^2 + \frac{\alpha}{2}\|q\|_I^2 \text{ s.t. (2.1) and } (q,u) \in Q \times X, \tag{2.2}$$

where $\alpha > 0$ is the regularization parameter.

Proposition 2.4. *For $\alpha > 0$ the optimal control problem* (2.2) *admits a unique solution $(\bar{q},\bar{u}) \in Q \times X$.*

Proof. For the standard proof we refer, e.g., to [22]. □

Utilizing the adjoint state equation for $z = z(q) \in X$ given by

$$\begin{aligned} -(\varphi,\partial_t z)_I + (\nabla\varphi,\nabla z)_I &= (\varphi,u(q)-\hat{u})_I \qquad \forall \varphi \in X, \\ z(T) &= 0, \end{aligned} \tag{2.3}$$

the optimality condition is given by

$$\bar{q} = -\alpha^{-1} z(\bar{q}). \tag{2.4}$$

Employing this optimality condition we obtain, the following regularity result:

Proposition 2.5. *Let $(\bar{q},\bar{u}) \in Q \times X$ be the solution of the optimization problem* (2.2) *and $\bar{z} = z(\bar{q}) \in X$ be the corresponding adjoint state. Then, there holds:*

$$\bar{q},\bar{u},\bar{z} \in H^1(I,D_\Delta) \cap H^2(I,H).$$

Furthermore, the following stability estimates are fulfilled:

$$\begin{aligned} \|\partial_t \Delta \bar{u}\|_I + \|\partial_t^2 \bar{u}\|_I &\le C\big\{\|f+\bar{q}\|_{H^1(I,H)} + \|\nabla(f+\bar{q})(0)\| + \|\nabla\Delta u_0\|\big\}, \\ \|\partial_t \Delta \bar{q}\|_I + \|\partial_t^2 \bar{q}\|_I &\le C(\alpha)\big\{\|\hat{u}\|_{H^1(I,H)} + \|\nabla\hat{u}(T)\| + \|f+\bar{q}\|_I + \|\nabla u_0\|\big\}. \end{aligned}$$

Proof. For $\bar{q} \in Q$, Proposition 2.3 implies that $\bar{u} \in L^2(I, D_\Delta) \cap H^1(I, H) \cap C(\bar{I}, V)$. This implies that the right-hand side of the adjoint equation (2.3) fulfills $\bar{u} - \hat{u} \in H^1(I, H)$ and $\bar{u}(T) - \hat{u}(T) \in V$. Consequently, we obtain by Proposition 2.3 that $\bar{z} \in H^1(I, D_\Delta) \cap H^2(I, H)$. This implies the stated regularity of $\bar{q}$.

The stability estimates for $\bar{u}$ follows directly from Proposition 2.3. For $\bar{z}$, Proposition 2.3 applied to the adjoint equation (2.3) implies

$$\|\partial_t \Delta \bar{z}\|_I + \|\partial_t^2 \bar{z}\|_I \leq C\big\{\|\bar{u}\|_{H^1(I,H)} + \|\hat{u}\|_{H^1(I,H)} + \|\nabla \bar{u}(T)\| + \|\nabla \hat{u}(T)\|\big\}$$

and the estimate for $\bar{u}$ from Proposition 2.3 together with the optimality condition (2.4) yields the assertion. □

3. Discretization

In this section, we describe the space-time finite element discretization of the optimal control problem (2.2).

3.1. Semidiscretization in time

At first, we present the semidiscretization in time of the state equation by continuous Galerkin methods. We consider a partitioning of the time interval $\bar{I} = [0, T]$ as

$$\bar{I} = \{\, 0 \,\} \cup I_1 \cup I_2 \cup \cdots \cup I_M$$

with subintervals $I_m = (t_{m-1}, t_m]$ of size k_m and time points

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T.$$

We define the discretization parameter k as a piecewise constant function by setting $k\big|_{I_m} = k_m$ for $m = 1, 2, \ldots, M$. Moreover, we denote by k the maximal size of the time steps, i.e., $k = \max_{m=1,2,\ldots,M} k_m$. We impose the following conditions on the time mesh:

(i) There is a constant $\kappa > 0$ (independent of k) such that for all $m = 1, 2, \ldots, M-1$

$$\kappa^{-1} \leq \frac{k_m}{k_{m+1}} \leq \kappa$$

holds.

(ii) There is a constant $\gamma > 0$ (independent of k) such that

$$k \leq \gamma \min_{m=1,2,\ldots,M} k_m.$$

The semidiscrete trial space is given as

$$X_k = \left\{\, v_k \in C(\bar{I}, V) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}_1(I_m, V),\ m = 1, 2, \ldots, M \,\right\},$$

while the test space consisting of discontinuous piecewise polynomials of order 0 is defined as

$$\widetilde{X}_k = \left\{\, v_k \in L^2(I, V) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}_0(I_m, V),\ m = 1, 2, \ldots, M,\ v_k(0) \in V \,\right\}.$$

Here, $\mathcal{P}_r(I_m, V)$ denotes the space of polynomials up to order r defined on I_m with values in V. We use the notations

$$(v, w)_{I_m} := (v, w)_{L^2(I_m,H)} \quad \text{and} \quad \|v\|_{I_m} := \|v\|_{L^2(I_m,H)}.$$

To define the continuous Galerkin approximation (so-called cG(1) approximation) using the spaces X_k and $\widetilde{X}_k$ we use for $v_k \in X_k$ the abbreviation $v_{k,m} := v_k(t_m)$ and for $w_k \in \widetilde{X}_k$ we set $w_{k,m} = \lim_{t\uparrow t_m} w_k(t)$. The bilinear form $B(\cdot,\cdot)$ for $u_k \in X_k$ and $\varphi \in \widetilde{X}_k$ is then defined by

$$B(u_k, \varphi) := (\partial_t u_k, \varphi)_I + (\nabla u_k, \nabla\varphi)_I + (u_{k,0}, \varphi_0).$$

The cG(1) semidiscretization of the state equation (2.1) for a given control $q \in Q$ reads: Find a state $u_k = u_k(q) \in X_k$ such that

$$B(u_k, \varphi) = (f + q, \varphi)_I + (u_0, \varphi_0) \quad \forall \varphi \in \widetilde{X}_k. \tag{3.1}$$

The existence and uniqueness of solutions to (3.1) can be directly shown by "elliptic" arguments. For the general cG(r) case we refer to [31].

The semi-discrete optimization problem for the cG(1) time discretization has the following form:

$$\text{Minimize } J(q_k, u_k) \text{ subject to (3.1) and } (q_k, u_k) \in Q \times X_k. \tag{3.2}$$

The uniquely determined optimal solution of (3.2) is denoted by $(\bar{q}_k, \bar{u}_k) \in Q \times X_k$.

Remark 3.1. Note, that the optimal control $\bar{q}_k$ is searched for in the continuous space Q, and the subscript k indicates only the usage of the semidiscretized state equation.

Similarly to the continuous case, the optimality condition can be formulated as

$$\bar{q}_k = -\alpha^{-1} z_k(\bar{q}_k),$$

where $z_k = z_k(q) \in \widetilde{X}_k$ denotes the solution of the semidiscrete adjoint equation

$$B(\varphi, z_k) = (\varphi, u_k(q) - \hat{u})_I \quad \forall \varphi \in X_k.$$

This yields that $\bar{q}_k$ is piecewise constant in time, i.e., that $\bar{q}_k \in \widetilde{X}_k$.

Additionally to the partition of $\bar{I}$ introduced at the beginning of this section, we consider a "dual" partition of the time interval $\bar{I}$ defined by

$$\bar{I} = \{\, 0 \,\} \cup I_1^* \cup I_2^* \cup \cdots \cup I_{M+1}^*$$

with $I_m^* := (t_{m-1}^*, t_m^*]$ for $m = 1, 2, \ldots, M+1$ and

$$t_0^* := t_0, \quad t_m^* := \frac{t_{m-1} + t_m}{2} \text{ for } m = 1, 2, \ldots, M, \quad \text{and} \quad t_{M+1}^* := t_M.$$

On this partition, we define the space Q_k by

$$Q_k := \left\{ w_k \in C(\bar{I}, V) \;\middle|\; w_k\big|_{I_m^*} \in \mathcal{P}_1(I_m^*, V),\ m = 1, 2, \ldots, M+1 \right\}$$

and the interpolation $\pi_k \colon C(\bar{I}, V) \cup \widetilde{X}_k \to Q_k$ as follows:

1. For $J = I_1^* \cup I_2^*$

$$\pi_k v(t)\big|_J := v(t_1^*) + \frac{t - t_1^*}{t_2^* - t_1^*}(v(t_2^*) - v(t_1^*))$$

2. For $J = I_m^*$ with $m = 3, 4, \ldots, M-1$:

$$\pi_k v(t)\big|_J := v(t_{m-1}^*) + \frac{t - t_{m-1}^*}{t_m^* - t_{m-1}^*}(v(t_m^*) - v(t_{m-1}^*))$$

3. For $J = I_M^* \cup I_{M+1}^*$:

$$\pi_k v(t)\big|_J := v(t_{M-1}^*) + \frac{t - t_{M-1}^*}{t_M^* - t_{M-1}^*}(v(t_M^*) - v(t_{M-1}^*)).$$

3.2. Discretization in space

To define the finite element discretization in space, we consider a family of two or three dimensional finite element meshes $\{\mathcal{T}_h\}_{h>0}$, see, e.g., [13]. A mesh consists of triangular, quadrilateral, tetrahedral, or hexahedral cells K, which constitute a non-overlapping cover of the computational domain Ω. The corresponding mesh is denoted by $\mathcal{T}_h = \bigcup\{K\}$, where we define the discretization parameter h as a cellwise constant function by setting $h\big|_K = h_K$ with the diameter h_K of the cell K. We use the symbol h also for the maximal cell size, i.e., $h = \max h_K$.

Remark 3.2. We do not assume the family of meshes $\{\mathcal{T}_h\}_{h>0}$ to be neither shape-regular nor quasi-uniform. For dealing with corner or edge singularities, we will use graded meshes, see Section 5 for details.

On the mesh $\mathcal{T}_h$ we construct a conforming finite element space $V_h \subset V$ in a standard way:

$$V_h = \left\{ v \in V \;\middle|\; v\big|_K \in \mathcal{Q}_1(K) \text{ for } K \in \mathcal{T}_h \right\}.$$

Here, $\mathcal{Q}_1(K)$ consists of shape functions obtained via (bi-/tri-)linear transformations of (bi-/tri-)linear polynomials defined on the reference cell. To obtain the fully discretized versions of the time discretized state equation (3.1), we utilize the space-time finite element spaces

$$X_{k,h} = \left\{ v_{kh} \in C(\bar{I}, V_h) \;\middle|\; v_{kh}\big|_{I_m} \in \mathcal{P}_1(I_m, V_h) \right\} \subset X_k$$

and

$$\widetilde{X}_{k,h} = \left\{ v_{kh} \in L^2(I, V_h) \;\middle|\; v_{kh}\big|_{I_m} \in \mathcal{P}_0(I_m, V_h) \text{ and } v_{kh}(0) \in V_h \right\} \subset \widetilde{X}_k.$$

The so-called cG(1)cG(1) discretization of the state equation for given control $q \in Q$ has the form: Find a state $u_{kh} = u_{kh}(q) \in X_{k,h}$ such that

$$B(u_{kh}, \varphi) = (f + q, \varphi)_I + (u_0, \varphi_0) \quad \forall \varphi \in \widetilde{X}_{k,h}. \tag{3.3}$$

Then, the corresponding optimal control problem is given as

$$\text{Minimize } J(q_{kh}, u_{kh}) \text{ subject to (3.3) and } (q_{kh}, u_{kh}) \in Q \times X_{k,h}. \tag{3.4}$$

The uniquely determined optimal solution of (3.4) is denoted by $(\bar{q}_{kh}, \bar{u}_{kh}) \in Q \times X_{k,h}$. As before, the optimality condition can be formulated as

$$\bar{q}_{kh} = -\alpha^{-1} z_{kh}(\bar{q}_{kh}), \tag{3.5}$$

where $z_{kh} = z_{kh}(q) \in \widetilde{X}_{k,h}$ denotes the solution of the discrete adjoint equation

$$B(\varphi, z_{kh}) = (\varphi, u_{kh}(q) - \hat{u})_I \quad \forall \varphi \in X_{k,h}.$$

By inspection of the optimality condition (3.5), we obtain that $\bar{q}_{kh} \in \widetilde{X}_{k,h}$ and so the control does not need to be discretized explicitly, cf., e.g., [20].

Finally, on the "dual" partition, we define the discrete space $Q_{k,h}$ by

$$Q_{k,h} := \left\{ w_k \in C(\bar{I}, V_h) \;\middle|\; w_k\big|_{I_m^*} \in \mathcal{P}_1(I_m^*, V_h),\ m = 1, 2, \dots, M+1 \right\}$$

and note that $\pi_k(\widetilde{X}_{k,h}) \subset Q_{k,h}$.

4. Error Analysis

In this section, we prove the main result of this article, namely an $\mathcal{O}(k^2 + h^2)$ estimate for the error $\|\bar{q} - \tilde{q}_{kh}\|_I$ between the continuous solution $\bar{q} \in Q$ of (2.2) and the postprocessed discrete solution $\tilde{q}_{kh} \in Q_{k,h}$ defined by

$$\tilde{q}_{kh} = -\alpha^{-1} \pi_k z_{kh}(\bar{q}_{kh}), \tag{4.1}$$

where $\bar{q}_{kh} \in \widetilde{X}_{k,h}$ is the solution of (3.4). The asserted estimate follows directly by the triangle inequality from the Theorems 4.1 and 4.4 below.

4.1. Estimates for the error due to time discretization

Theorem 4.1. *Let Assumption* 4.2 *be fulfilled. For the solution* $\bar{q} \in Q$ *of* (2.2) *and* $\tilde{q}_k$ *defined by*

$$\tilde{q}_k = -\alpha^{-1} \pi_k z_k(\bar{q}_k)$$

with the solution $\bar{q}_k \in Q$ *of* (3.2)*, it holds*

$$\|\bar{q} - \tilde{q}_k\|_I \leq C(\alpha) k^2 \big\{ \|f + \bar{q}\|_{H^1(I,H)} + \|\hat{u}\|_{H^1(I,H)} + \|\nabla (f + \bar{q})(0)\| + \|\nabla \hat{u}(T)\| + \|\nabla \Delta u_0\| \big\}.$$

Proof. This result can be proved following the lines of the proof of Theorem 6.6 in [26]. There, the domain is assumed to be polygonal and convex. However, the proof of Theorem 6.6 there does not exploit H^2 regularity. It requires only the regularity stated in Proposition 2.3. □

4.2. Estimates for the error due to space discretization

For the error analysis derived here, we will make use of the spatial Ritz projection $R_h \colon V \to V_h$ defined by

$$(\nabla R_h v, \nabla \varphi) = (\nabla v, \nabla \varphi) \quad \forall \varphi \in V_h. \tag{4.2}$$

Assumption 4.2. The family of spatial meshes $\{\,\mathcal{T}_h\,\}_h$ is constructed such that for the Ritz projection defined by (4.2) the estimate

$$h\|\nabla(v - R_h v)\| + \|v - R_h v\| \le Ch^2\|\Delta v\|$$

holds for all $v \in D_\Delta$.

Remark 4.3. If the domain Ω is polygonal and convex, then this assumption holds on shape-regular, quasi-uniform meshes by standard finite element theory with

$$h\|\nabla(v - R_h v)\| + \|v - R_h v\| \le Ch^2\|v\|_{H^2(\Omega)} \le Ch^2\|\Delta v\|,$$

where the H^2 regularity is used in the last step. In the case of a non-smooth domain, however, this assumption is typically not fulfilled on quasi-uniform meshes. In Section 5 we discuss several situations, where this assumption holds, if the family of meshes is constructed using an appropriate mesh grading.

Theorem 4.4. *For* $\tilde{q}_k$ *defined by* $\tilde{q}_k = -\alpha^{-1}\pi_k z_k(\bar{q}_k)$ *from Theorem* 4.1 *with the solution* $\bar{q}_k \in Q$ *of* (3.2) *and* $\tilde{q}_{kh}$ *defined by* (4.1) *with the solution* $\bar{q}_{kh} \in Q$ *of* (3.4)*, it holds under Assumption* 4.2 *that*

$$\|\tilde{q}_k - \tilde{q}_{kh}\|_I \le C(\alpha)\bigl\{k^2 + h^2\bigr\}\bigl\{\|f + \bar{q}_k\|_I + \|\nabla u_0\| + \|u_0\|\bigr\} + k^2\|\partial_t \hat{u}\|_I + h^2\|\hat{u}\|_I.$$

Proof. Under Assumption 4.2 it is possible to extend the proof of Theorem 6.10 in [26] to the case of a nonsmooth domain. The main component of this proof is Lemma 5.7 in [26], which shows, that a certain discretization error can be bounded by the error with respect to the Ritz projection R_h. Using Assumption 4.2, this lemma can be directly extended to the case considered here and the above result follows. □

5. Verification of Assumption 4.2

Now we discuss cases for which the Assumption 4.2 is fulfilled. It is well known, that for convex polygonal or polyhedral domains the Assumption 4.2 holds for finite element approximations on shape-regular, quasiuniform meshes (see Remark 4.3). So we focus on examples of nonconvex domains for which the Assumption 4.2 is also fulfilled.

5.1. Nonconvex polygonal domains

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with one non-convex interior angel $\omega > \pi$, located at the origin. Further we introduce the distance of a finite element τ of the triangulation $\mathcal{T}_h$ to the origin as $r_\tau = \inf_{(x_1,x_2)\in\tau}\sqrt{x_1^2 + x_2^2}$. Assume that the family of shape-regular triangulation $\{\mathcal{T}_h\}_h$ fulfills the conditions

$$\left.\begin{aligned} c_1 h^{1/\nu} \le h_\tau \le c_2 h^{1/\nu}, &\quad \text{for } r_\tau = 0,\\ c_1 h r_\tau^{1-\nu} \le h_\tau \le c_2 h r_\tau^{1-\nu}, &\quad \text{for } r_\tau > 0, \end{aligned}\right\} \tag{5.1}$$

with $\nu < \frac{\pi}{\omega}$ and $h_\tau = \operatorname{diam}(\tau)$.

Remark 5.1. For meshes which fulfill the condition 5.1 the number of elements is of order h^{-2}. Therefore the number of elements is of the same order as in a quasiuniform triangulation (see e.g. [8, Remark 3.1] or [28, 29]).

On meshes fulfilling (5.1) it is known that Assumption 4.2 is fulfilled, see e.g. [11, Theorem 5.1], [28, Theorem 1] or [29, Theorem 2].

Remark 5.2. The results can be transfered to domains with a corner with interior angle $\omega > \pi$ and smooth boundary everywhere else e.g. a segment of a disk with a reentrant corner.

Remark 5.3. As the singularities show local behavior, therefore more general two dimensional domains with more than one non-convex corner can be treated in a similar fashion, as we can write the solution as the sum of a regular part and the singularity functions of each non convex corner (see e.g. [21, Section 1.4])

5.2. Prismatic domains

Let $\Omega = G \times Z \subset \mathbb{R}^3$ be a bounded prismatic domain, where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $Z = (0, z_0)$ is an interval. Again we assume that G has one corner with interior angel $\omega > \pi$ located at the origin.

As in [32, Section 2.3.2] we construct the triangulation of Ω in the following way: Assume that the triangulation of G is constructed such that the condition (5.1) is fulfilled. From this triangulation we get a triangulation of the domain Ω by extruding the triangles in x_3 direction quasiuniform with mesh size h. This gives a mesh of triangular prisms, to get an anisotropic thetraedal mesh each prism is divided into thetraheda. For elements of this mesh the following estimates hold

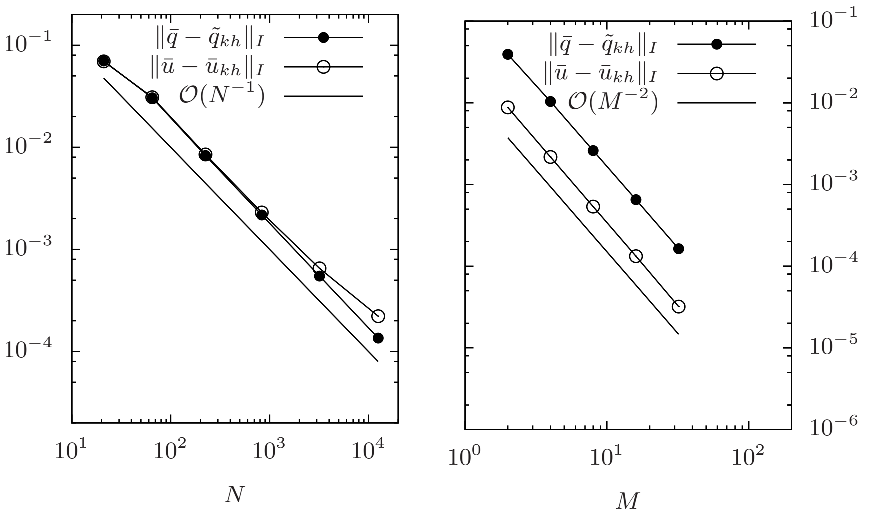
$$\left.\begin{aligned} c_1 h^{1/\nu} \leq h_{\tau,i} &\leq c_2 h^{1/\nu}, && \text{for } r_\tau = 0 \text{ and } i = 1,2,\\ c_1 h r_\tau^{1-\nu} \leq h_{\tau,i} &\leq c_2 h r_\tau^{1-\nu}, && \text{for } r_\tau > 0 \text{ and } i = 1,2,\\ c_1 h \leq h_{\tau,3} &\leq c_2 h, \end{aligned}\right\}$$

where $h_{\tau,i}$ is the length of the projection of the element τ to the x_i-axis (for $= 1, 2, 3$), $r_\tau = \inf_{(x_1,x_2)\in\tau} \sqrt{x_1^2 + x_2^2}$ the distance of the element τ to the x_3-axis and $\nu < \frac{\pi}{\omega}$. On such meshes the Assumption 4.2 is fulfilled, see [9, Theorem 5.2].

Remark 5.4. In [2, Corollary 4.1] the validity of Assumption 4.2 is shown for prismatic domains with Neumann boundary conditions on $G \times \{0, z_0\}$ and Dirichlet conditions on the remaining part of $\partial\Omega$.

5.3. General polyhedral domains

For the solution of the Dirichlet problem for the Poisson equation on general polyhedral domains $\Omega \subset \mathbb{R}^3$ we refer to [4], where a refinement strategy is proposed that the Assumption 4.2 holds (see [4, Corollary 3.12]). As the grading strategy and construction of corresponding meshes is more complicated as in the previous cases, we omit the details here and refer for details to [4].



(A) Convergence with respect to spatial refinement with fixed time step size.

(B) Convergence with respect to the time step size with fixed spatial mesh.

FIGURE 1. Observed convergence of the numerical example. Here, N with $N^{-1} = \mathcal{O}(h^2)$ (cf. Remark 5.1) denotes the number of cells in the spatial mesh and M with $M^{-1} = k$ is the number of times steps

6. Numerical Results

For the numerical verification we consider the optimal control Problem (1.1) with $\alpha = 1$ on a L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]$ and the unit time interval $(0,T) = (0,1)$. The remaining data f and $\hat{u}$ are chosen, such that the exact solution in polar coordinates $(r,\varphi) \in \mathbb{R}_+ \times [0, 2\pi)$ is given by

$$\bar{u}(r,\varphi,t) = (\mathrm{e}^{\lambda t} - 1) \cdot u_s(r,\varphi),$$
$$\bar{z}(r,\varphi,t) = (\mathrm{e}^{\lambda(1-t)} - 1) \cdot u_s(r,\varphi),$$

with $\lambda = \frac{2}{3}$ and

$$u_s(r,\varphi) = r^\lambda \sin(\lambda\varphi) \cdot (r\cos\varphi - 1)(r\cos\varphi + 1)(r\sin\varphi + 1)(r\sin\varphi - 1).$$

For the grading parameter $\nu = 0.6$ in (5.1), Figure 1 depicts the behavior of the errors $\|\bar{q} - \tilde{q}_{kh}\|_I$ and $\|\bar{u} - \bar{u}_{kh}\|_I$ for a sequence of temporal and spacial meshes. They exhibit the proved convergence order $\mathcal{O}(k^2 + h^2)$.

Remark 6.1. For the Crank-Nicolson time stepping discretizations of [3] similar convergence results can be observed.

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The function u_s for the numerical example in Section 6 was taken from a presentation by T. Apel and J. Pfefferer. The numerical experiments in Section 6 are carried out using both a MATLAB-FEM implementation based on [1, 12, 15] and the software packages RoDoBo [30] and Gascoigne [16].

References

[1] J. Alberty, C. Carstensen, and S. A. Funken: *Remarks around 50 lines of Matlab: short finite element implementation.* Numerical Algorithms **20** (1999), 117–137.

[2] T. Apel: *Anisotropic finite elements: Local estimates and applications.* Advances in Numerical Mathematics. Teubner, Stuttgart, 1999.

[3] T. Apel and T. G. Flaig: *Crank-Nicolson schemes for optimal control problems with evolution equations.* SIAM Journal on Numerical Analysis **50** (2012), 1484–1512.

[4] T. Apel, A. L. Lombardi, and M. Winkler: *Anisotropic mesh refinement in polyhedral domains: error estimates with data in $L^2(\Omega)$*, 2013. URL `http://arxiv.org/abs/1303.2960`.

[5] T. Apel, J. Pfefferer, and A. Rösch: *Finite element error estimates for Neumann boundary control problems on graded meshes.* Comput. Optim. Appl. **52** (2012), 3–28.

[6] T. Apel, A. Rösch, and D. Sirch: *L^∞-error estimates on graded meshes with application to optimal control.* SIAM J. Control Optim. **48** (2009), 1771–1796.

[7] T. Apel, A. Rösch, and G. Winkler: *Optimal control in non-convex domains: a priori discretization error estimates.* Calcolo **44** (2007), 137–158.

[8] T. Apel, A.-M. Sändig, and J. R. Whiteman: *Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains.* Math. Methods Appl. Sci. **19** (1996), 63–85.

[9] T. Apel and D. Sirch: *L^2-error estimates for Dirichlet and Neumann problems on anisotropic finite element meshes.* Applications of Mathematics **56** (2011), 177–206.

[10] T. Apel and G. Winkler: *Optimal control under reduced regularity.* Appl. Numer. Math. **59** (2009), 2050–2064.

[11] I. Babuška, R. Kellogg, and J. Pitkäranta: *Direct and inverse error estimates for finite elements with mesh refinements.* Numerische Mathematik **33** (1979), 447–471.

[12] L. Chen and C.-S. Zhang: *AFEM@MATLAB: A MATLAB package of adaptive finite element methods.* Technical report, University of Maryland, 2006. URL `www.math.umd.edu/~zhangcs/paper/AFEM@matlab.pdf.gz`.

[13] P. G. Ciarlet: *The finite element method for elliptic problems.* North-Holland Publishing Company, Amsterdam, 1979.

[14] L. C. Evans: *Partial Differential Equations*, volume 19 of *Grad. Stud. Math.* AMS, Providence, 2002.

[15] S. Funken, D. Praetorius, and P. Wissgott: *Efficient implementation of adaptive P1-FEM in Matlab*. Computational Methods in Applied Mathematics **11** (2011), 460–490.

[16] *The finite element toolkit* GASCOIGNE. `http://www.gascoigne.de`.

[17] W. Gong and M. Hinze: *Error estimates for parabolic optimal control problems with control and state constraints*. Comput. Optim. Appl. **56** (2013), 131–151.

[18] W. Gong, M. Hinze, and Z. J. Zhou: *Space-time finite element approximation of parabolic optimal control problems*. J. Numer. Math. **20** (2012), 111–145.

[19] P. Grisvard: *Singularities in boundary value problems*, volume 22 of *Research Notes in Applied Mathematics*. Springer, New York, 1992.

[20] M. Hinze: *A variational discretization concept in control constrained optimization: The linear-quadratic case*. Comput. Optim. Appl. **30** (2005), 45–61.

[21] A. Kufner and A.-M. Sändig: *Some Applications of Weighted Sobolev Spaces*. Teubner, Leipzig, 1987.

[22] J.-L. Lions: *Optimal Control of Systems Governed by Partial Differential Equations*, volume 170 of *Grundlehren Math. Wiss.* Springer, Berlin, 1971.

[23] D. Meidner, R. Rannacher, and B. Vexler: *A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time*. SIAM J. Control Optim. **49** (2011), 1961–1997.

[24] D. Meidner and B. Vexler: *A Priori Error Estimates for Space-Time Finite Element Discretization of Parabolic Optimal Control Problems. Part I: Problems without Control Constraints*. SIAM Journal on Control and Optimization **47** (2008), 1150 – 1177.

[25] D. Meidner and B. Vexler: *A priori error estimates for space-time finite element approximation of parabolic optimal control problems. Part II: Problems with control constraints*. SIAM J. Control Optim. **47** (2008), 1301–1329.

[26] D. Meidner and B. Vexler: *A priori error analysis of the Petrov-Galerkin Crank-Nicolson scheme for parabolic optimal control problems*. SIAM J. Control Optim. **49** (2011), 2183–2211.

[27] I. Neitzel and B. Vexler: *A priori error estimates for space-time finite element discretization of semilinear parabolic optimal control problems*. Numer. Math. **120** (2012), 345–386.

[28] L. A. Oganesyan and L. A. Rukhovets: *Variational-difference schemes for linear second-order elliptic equations in a two-dimensional region with piecewise smooth boundary*. Zh. Vychisl. Mat. Mat. Fiz. **8** (1968), 97–114. In Russian. English translation in USSR Comput. Math. and Math. Phys., 8 (1968) 129–152.

[29] G. Raugel: *Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone*. C. R. Acad. Sci. Paris, Sér. A **286** (1978), 791–794.

[30] RoDoBo. *A C++ library for optimization with stationary and nonstationary PDEs with interface to* GASCOIGNE [16]. `http://www.rodobo.org`.

[31] F. Schieweck: *A-stable discontinuous Galerkin-Petrov time discretization of higher order*. Journal of Numerical Mathematics **18** (2010), 2557.

[32] D. Sirch: *Finite Element Error Analysis for PDE-constrained Optimal Control Problems: The Control Constrained Case Under Reduced Regularity*. Ph.D. thesis, Technische Universität München, 2010.

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