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„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations“: Sheet 1

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 1.1 (Existence of a solution): We consider the optimization problem

$$\text{Minimize } J(u) = \frac{1}{2}\int_0^1 x^2 u'(x)^2\, dx - \int_0^1 u(x)\, dx, \quad u \in H_0^1(I), \quad I = (0,1). \tag{P}$$

Show that this problem has no solution.

Hint: Start by showing

$$J(u) \geq -\frac{1}{2} \quad \text{for any } u \in H_0^1(I).$$

Proceed by constructing a sequence $(u_n)_{n\in\mathbb{N}}, u_n \in H_0^1(I)$ for $n \in \mathbb{N}$ such that

$$J(u_n) \to -\frac{1}{2}, \quad n \to \infty$$

and show that there is no function $u \in H_0^1(I)$ with

$$J(u) = -\frac{1}{2}.$$

Exercise 1.2 (A relative of Poincaré's inequality): Prove the following statement: There is a positive constant c_Ω such that for any function $u \in H^1(\Omega)$ with

$$\int_\Omega u(x)dx = 0,$$

the estimate

$$\|u\|_{L^2(\Omega)} \leq c_\Omega \|\nabla u\|_{L^2(\Omega)}$$

holds true.

Hint: Prove by contraposition and use that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, i. e., any sequence that is bounded in $H^1(\Omega)$ possesses a sub-sequence that converges in $L^2(\Omega)$.

Exercise 1.3*: Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\beta = (\beta_1, \beta_2) \in H^1(\Omega) \times H^1(\Omega)$ and $u \in H_0^1(\Omega)$.

(a) Show that the integral

$$\int_\Omega \beta(x) \nabla u(x)\, u(x)\, dx$$

attains a finite value.

Hint: Use embedding theorems and Hölder's inequality.

(b) Show that

$$\int_\Omega \beta(x) \nabla u(x)\, u(x)\, dx = -\frac{1}{2} \int_\Omega \operatorname{div} \beta(x)\, u(x)^2\, dx$$

holds true with $\operatorname{div} \beta = \partial_1 \beta_1 + \partial_2 \beta_2$.

Hint: Use integration by parts (Gauß's theorem).

(c) Consider the equation

$$\begin{cases} -\Delta u + \beta \cdot \nabla u + c\, u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$ and $c \in \mathbb{R}$, $c \geq 0$. Derive a *sufficient* condition for β and c that guarantees the existence of a weak solution for this equation through the Lax-Milgram theorem.