Technische Universität München
Center for Mathematical Sciences, M17
Prof. Dr. Boris Vexler
Andreas Springer

Summer 2014

„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations": Sheet 3

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 3.1: Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $q_a, q_b \in L^2(\Omega)$ with $q_a \leq q_b$ almost everywhere in Ω. Show that the set

$$Q_{\text{ad}} = \{q \in L^2(\Omega) \mid q_a \leq q \leq q_b \text{ almost everywhere in } \Omega\}$$

is nonempty, bounded, convex and closed.

Exercise 3.2: Let $\Omega = (0, 1)$. Consider the optimization problem

$$\min_{(q,u)\in Q\times V} J(q, u) = \frac{1}{2}\|u - 1\|^2_{L^2(\Omega)}$$

$$\begin{array}{rrcll} \text{s.t.} & -u'' & = & q & \text{in} \quad \Omega \\ & u'(1) & = & 0 & \\ & u(0) & = & 0 & \end{array}$$

with $Q = L^2(\Omega)$.

Choose a suitable state space V together with the corresponding space W_{ad} of admissible pairs of controls and states and show on the one hand that

$$\overline{J} := \inf_{(q,u)\in W_{\text{ad}}} J(q, u) = 0$$

but that on the other hand the problem admits no global solution.

Exercise 3.3 (Distributed control with additional source term): Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Prove existence and uniqueness of a solution of the optimization problem

$$\min_{(q,u)\in Q\times V} J(q, u) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}$$

s. t.

$$\begin{cases} -\Delta u = f + q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{BVP1}$$

$$a \leq q(x) \leq b \qquad \text{for almost all } x \in \Omega$$

with $Q = L^2(\Omega)$, $V = H^1_0(\Omega)$, $a, b \in \mathbb{R}$, $a \leq b$, $f \in L^2(\Omega)$, $\alpha \geq 0$, $u_d \in L^2(\Omega)$.

Exercise 3.4 (Robin boundary control): Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Prove existence and uniqueness of a solution of the optimization problem

$$\min_{(q,u)\in Q\times V} J(q,u) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\partial\Omega)}$$

$$\text{s. t.} \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_n u + \beta u = q & \text{auf } \partial\Omega \end{cases} \tag{BVP2}$$

$$a \le q(s) \le b \quad \text{for almost all } s \in \partial\Omega$$

with $Q = L^2(\partial\Omega)$, $V = H^1(\Omega)$, $a, b \in \mathbb{R}$, $a \le b$, $\beta \in L^\infty(\partial\Omega)$, $\beta > 0$, $\alpha \ge 0$, $u_d \in L^2(\Omega)$.

Exercise 3.5 (Bilinear control): Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Show the existence of an optimal solution for the nonlinear optimal control problem

$$\min_{(q,u)\in Q\times V} J(q,u) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}$$

$$\text{s. t.} \quad \begin{cases} -\Delta u + qu = f & \text{in } \Omega, \\ u = 0 & \text{auf } \partial\Omega \end{cases} \tag{BVP3}$$

$$a \le q(x) \le b \quad \text{for almost all } x \in \Omega$$

with $Q = L^2(\Omega)$, $V = H^1_0(\Omega)$, $f \in L^2(\Omega)$, $\alpha > 0$, $a \in \mathbb{R}$ with $a \ge 0$, $b \in \overline{\mathbb{R}}$, $a \le b$. $u_d \in L^2(\Omega)$.