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„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations“: Sheet 5

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 5.1 (Differentiability of superposition operators): Let $\Omega = (0,1)$ and

$$\Phi\colon L^p(\Omega) \to L^p(\Omega),\ u \mapsto \sin(u)\,.$$

(a) Show that Φ is Gâteaux differentiable for any p satisfying $1 \le p \le \infty$.

(b) Show that, despite the sine function being smooth, Φ is not Fréchet differentiable for $p \in [1, \infty)$.
Hint: Make use of the sequence

$$\delta u_n(x) = \begin{cases} 1, & \text{if } x < \frac{1}{n}, \\ 0, & \text{if } x \ge \frac{1}{n}. \end{cases}$$

(c) Is Φ Fréchet differentiable if $p = \infty$?

Exercise 5.2: Let $\Omega = (0,1)$, $Q = L^2(\Omega)$. We consider the minimization problem

$$\min_{q \in Q_{ad}} j(q) = -\int_0^1 \cos(q(x))\, dx$$

with $Q_{ad} = \{q \in Q |\ 0 \le q(x) \le 2\pi \text{ for almost all } x \in \Omega\}$.

Show that:

(a) j is twice Gâteaux differentiable, for $\overline{q} = 0$

$$j'(\overline{q})(p - \overline{q}) \ge 0 \quad \forall p \in Q_{ad}$$

and there is a $\gamma > 0$ such that

$$j''(\overline{q})(p,p) \ge \gamma \|p\|_Q^2 \quad \forall p \in Q\,. \tag{1}$$

(b) $\overline{q}$ is not a strict local minimizer of j.

(c) j is not twice Fréchet differentiable in $\overline{q}$ with respect to the L^2 norm.

(d) j is twice Fréchet differentiable on $L^\infty(\Omega)$.

(e) the coercivity condition (1) is not satisfied in $L^\infty(\Omega)$.

Remark: Here we have an example of the so called *two-norm discrepancy*, i.e., in $L^2(\Omega)$ $j''(\overline{q})$ is positive definite but j does not satisfy the necessary differentiability conditions. In $L^\infty(\Omega)$ the situation is the other way around. j is twice Fréchet differentiable but $j''(\overline{q})$ is not positive definite.

(f) nevertheless, $\overline{q}$ is a local optimum with respect to $L^\infty(\Omega)$.