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„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations“: Sheet 6

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 6.1 (Optimality condition for the linear quadratic case): We consider the optimal control problem

$$\min_{(q,u)\in Q\times V} J(q,u) = \frac{\alpha}{2}\|q\|^2_{L^2(\Omega_q)} + \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)}$$
$$\text{s.t.} \begin{cases} -\Delta u + \partial_x u = Bq + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\Omega \subseteq \mathbb{R}^d$ a bounded Lipschitz-domain, $\Omega_q \subseteq \Omega$, $\alpha > 0$, $Q = L^2(\Omega_q)$, $V = H^1_0(\Omega)$, $B\colon Q \to L^2(\Omega)$ a bounded linear operator and $u_d, f \in L^2(\Omega)$

(a) Write down the reduced cost functional j using the solution operator $S\colon L^2(\Omega) \to L^2(\Omega)$ of the state equation that maps the right hand side $Bq + f$ to the corresponding state u.

(b) Compute the directional derivative of j in direction δq with the chain rule.

(c) From the result of (b), derive a representation of the gradient $\nabla j(q)$ with respect to the L^2 inner product (*Hint: use adjoint operators*).

(d) Give a weak formulation of the state equation and use it to assemble the Lagrangian.

(e) Derive a first order optimality system from the Lagrangian.

(f) Show that the solution operator $\hat{S}$ of the adjoint equation is in fact the adjoint S^* of the solution operator S, i. e., for any $f, g \in L^2(\Omega)$, the identity

$$(Sf, g)_{L^2(\Omega)} = (f, \hat{S}g)_{L^2(\Omega)}$$

holds true.

(g) Use the solution operators S and S^* to eliminate u and z from the first order optimality system. Compare the resulting optimality condition for q to the gradient computed in (c).

Exercise 6.2 (Derivative representations for a semilinear problem): Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Consider the following optimization problem with boundary control:

$$\min_{(q,u)\in Q\times V} J(q,u) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\partial\Omega)}$$

$$\text{s. t.} \quad \begin{cases} -\Delta u + u + u^3 = f & \text{in } \Omega, \\ \partial_n u = q & \text{on } \partial\Omega \end{cases} \tag{BVP}$$

$$a(s) \le q(s) \le b(s) \text{ for almost all } s \in \partial\Omega\,,$$

where $Q = L^2(\partial\Omega)$, $V = H^1(\Omega)$, $f \in L^2(\Omega)$. $a, b \in L^2(\partial\Omega)$, $a(s) < b(s)$ almost everywhere on $\partial\Omega$, $\alpha > 0$, and $u_d \in L^2(\Omega)$.

(a) Give a weak formulation of the state equation.

(b) Compute the partial derivatives of J and of the semilinear form a that results from the weak formulation of (BVP).

Next, we check the requirements for Theorem 5.8 from the lecture. Proceed as follows.

(c) Show that the map $F : H^1(\Omega) \to H^1(\Omega)^*,\ u \mapsto a(q,u)$ is continuously Fréchet differentiable.

(d) Show that the Operator $G : H^1(\Omega) \to H^1(\Omega)^*,\ \delta u \mapsto a'_u(q,u)(\delta u)$ possesses a continuous inverse.

Having verified the requirements of Theorem 5.8, we apply Theorem 5.9.

(e) Give a weak and a strong formulation of the tangent equation.

(f) Give the tangent-based representation of the derivative of the reduced cost functional j.

Alternatively we consider the adjoint based representation of the first derivative (see Theorem 5.10).

(g) Give a weak and a strong formulation of the adjoint equation.

(h) Give the adjoint-based representation formula for the first derivative of j.

(i) Give the optimality system (necessary first order optimality condition).

Exercise 6.3 (Parameter control): Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Give the optimality system for the optimal control problem

$$\min_{(q,u)\in Q\times V} J(q.u) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)}$$

$$\text{s. t.} \quad \begin{cases} -\Delta u + q_0 e^{-q_1 u} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $Q = \mathbb{R}^2$, $V = H^1_0(\Omega)$, $u_d \in L^2(\Omega)$.