Technische Universität München
Center for Mathematical Sciences, M17
Prof. Dr. Boris Vexler
Andreas Springer
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„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations“: Sheet 7

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 7.1: Show that the KKT conditions can be stated equivalently with a single Lagrangian multiplier $\mu \in L^2(\Omega)$ (instead of two multipliers as in Theorem 5.16):

Let $Q = L^2(\Omega)$, $a, b \in Q$ with $a < b$ almost everywhere in Ω and

$$Q_{ad} = \{q \in Q |\ a(x) \leq q(x) \leq b(x) \text{ for almost all } x \in \Omega\}.$$

Let $\bar{q} \in Q_{ad}$, $\bar{z} \in L^2(\Omega)$. Show that the variational inequality

$$(\alpha\bar{q} + \bar{z}, p - \bar{q}) \geq 0 \qquad \forall p \in Q_{ad}$$

is equivalent to the existence of a Lagrangian multiplier $\mu \in L^2(\Omega)$ satisfying

$$\mu = -\alpha\bar{q} - \bar{z},$$

$$\begin{aligned} \mu(x) &\leq 0 \quad \text{for almost all } x \in A_- = \{x \in \Omega |\ \bar{q}(x) = a(x)\}, \\ \mu(x) &\geq 0 \quad \text{for almost all } x \in A_+ = \{x \in \Omega |\ \bar{q}(x) = b(x)\}, \\ \mu(x) &= 0 \quad \text{for almost all } x \in I = \Omega \setminus (A_- \cup A_+). \end{aligned}$$

Exercise 7.2: Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We consider the following optimal control problem with finite dimensional control:

$$\min_{(q,u) \in Q \times V} J(q, u) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left(q_0^2 + q_1^2\right)$$

s. t.

$$\begin{cases} -\Delta u + e^{q_0} u^3 = q_1 f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where $Q = \mathbb{R}^2$, $V = H_0^1(\Omega)$, and $\alpha > 0$.

(a) Derive the adjoint-based gradient representation for the reduced cost functional of this problem.

(b) Give the representation formula for the second derivative and the corresponding auxiliary equations as discussed in the lecture.

(c) Formulate Newton's method with cg as linear solver.

(d) Assuming n_{cg} is the number of cg iterations required for a sufficiently good approximation of the Newton update, count the number of linear and non-linear PDE solves required per Newton step. How large is n_{cg} at most for our problem?

It was mentioned in the lecture that for low dimensional problems it can be more efficient to assemble the Hessian of the reduced cost functional explicitly.

(e) From the Lagrangian, derive a representation of the derivative $j''(q)(\delta q, \tau q)$ that makes use of the solutions δu and τu of the tangent equations for directions δq and τq.
Hint: Use the representation of the second derivative of j in terms of second partial derivatives of the Lagrangian that we considered in the lecture and see what result you get when choosing $v = \delta u$.

(f) Formulate the algorithm for Newton's method in this setting.

(g) How many PDEs have to be solved per Newton iteration for this variant of the algorithm?

(h) Generalize your considerations in (d) and (g) to higher dimension of the control space Q. Depending on the dimension N of the control space, for which maximum number of cg iterations n_{cg} is the matrix-free algorithm cheaper than assembling the full Hessian?

(i) How does the memory consumption of the two algorithms compare when assuming that the size of the discrete state space is much greater than N?