Fakultät für Mathematik, Lehrstuhl M17
Technische Universität München
Thomas Wick

May 7, 2015

Exercise 3 - Numerical methods for fluid-structure interaction
(Summer term 2015)

Exercise 3.1: Let $\sigma \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $B \in \mathbb{R}^{n\times n}$ a matrix. Show

$$(\sigma, B) = (\sigma, \frac{1}{2}(B + B^T).$$

Here, $(\sigma, B) = \int_\Omega \sigma : B\,dx$.

Exercise 3.2: Show

$$|\hat{E}_{lin}(\hat{u})|_{L^2} \leq c\|\hat{u}\|_{H^1} \quad \forall \hat{u} \in H^1,$$

where $\hat{E}_{lin}(\hat{u}) := \frac{1}{2}(\hat{\nabla}\hat{u} + \hat{\nabla}\hat{u}^T)$, $|\cdot|$ is the L^2 semi-norm, and $c > 0$.

Exercise 3.3: Let Ω be a domain with sufficient regularity and the boundary decomposed into $\partial\Omega = \Gamma_D \cup \Gamma_N$. Let the linear boundary value problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_D, \\ a\nabla u \cdot n &= g \quad \text{on } \Gamma_N, \end{aligned}$$

be given. Here, the differential operator is given by

$$Lu = -\nabla \cdot (a\nabla u) + b\nabla u + cu,$$

where L is uniformly elliptic, i.e., there exists a $\theta > 0$, such that

$$\sum_{i,j}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2,$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

1. State the function spaces for u, a, b, c, f necessary for the weak formulation.
2. Derive the weak formulation assuming $\Gamma_N > 0$ and $\Gamma_D > 0$.
3. Is the resulting bilinear form $A(\cdot,\cdot)$ symmetric or non-symmetric? (Please give arguments/proofs for your answer);
4. Assuming $\Gamma_N = 0$, study existence and uniqueness of solutions. (Hint: verify the assumptions of the Lax-Milgram lemma; given on the back of the exercise sheet):
5. What is the challenge in the last term cu? (Hint: The difficulty is related to the V-ellipticity).

Please turn page.

Hint (will also be discussed in the next lecture on Mon, May 11, 2015):

Lax-Milgram theorem

Let W be a real Hilbert space with norm $\|\cdot\|$ and let $\langle\cdot\rangle$ denote the pairing of W with its dual space W^*. Furthermore, let $A(u,\varphi)$ be a bilinear continuous form on $W\times W$, i.e.,

$$|A(u,\varphi)| \le \alpha\|u\|\,\|\varphi\| \quad \text{for } u,\varphi\in W, \quad \alpha>0,$$

and which is coercive (V-elliptic), i.e.,

$$A(u,u) \ge \beta\|u\|^2, \quad \text{for } u\in W, \quad \beta>0.$$

Additionally, let $f\in W^*$ be a bounded linear functional on W, i.e,

$$|\langle f,\varphi\rangle| \le \gamma\|f\|\,\|\varphi\|, \quad \varphi\in W, \quad \gamma>0.$$

Then, there exists a unique element $u\in W$ such that

$$A(u,\varphi) = \langle f,\varphi\rangle \quad \forall\varphi\in W.$$

Discussion of exercises: Jun 1, 2015