Exercises for Advanced Topics in High Performance Scientific Computing
WS 15/16, Nr.: 0000003213, 02.10.011, Rechnerraum
Exercise Sheet 5 (until November 18, 2015)

Exercise 5* Use the finite element method with linear elements to solve the Laplace equation

$$\Delta u(x, y) = 0, \quad (x, y) \in \Omega \tag{1}$$

on the domain Ω with the boundary conditions

$$u(x, y, t) = \begin{cases} +1, & (x, y) \in \Gamma^+ \\ -1, & (x, y) \in \Gamma^- \\ 0, & (x, y) \in \Gamma^0 \end{cases} . \tag{2}$$

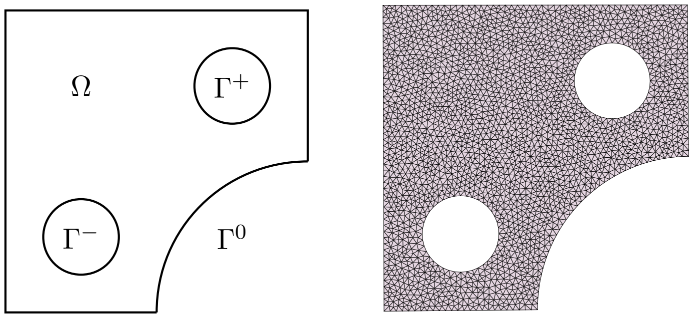

Figure 1: Domain Ω with boundaries Γ^+, Γ^-, and Γ^0.

Refer to the paper "Remarks around 50 lines of Matlab: short finite element implementation" for a condensed overview of the finite element method. a) Follow chapter 4 "Data representation of the triangulation Ω" for an explanation of the mesh data format. Download the data files `Domain.zip` from the Moodle platform. The binary file `Coordinates.bin` stores the 2D coordinates of the nodes

$$(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in \mathbb{R}^{2n} \tag{3}$$

as double precision floating point numbers. The node index is not stored explicitly, but is implied by to the position of the coordinate pair in the vector. The binary file `Elements.bin` contains for each triangle the node numbers a, b, c of the vertices as 32 bit integers.

$$(a_0, b_0, c_0, a_1, b_1, c_1, \ldots, a_{m-1}, b_{m-1}, c_{m-1}) \in \mathbb{N}^{3m} \tag{4}$$

b) Chapter 5 "Assembling the stiffness matrix" introduces formulas to compute an element matrix M_{ij} with $i, j \in \{a, b, c\}$ associated to the triangle (a, b, c).

$$G := \begin{pmatrix} y_b - y_c & x_c - x_b \\ y_c - y_a & x_a - x_c \\ y_a - y_b & x_b - x_a \end{pmatrix}, \quad A := \det \begin{pmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{pmatrix}, \quad M_{ij} := \frac{1}{4A} G G^T, \ i, j \in \{a, b, c\} \tag{5}$$

The system matrix A is computed as the sum of all element matrices M. Use the compressed row storage (CRS) format to store the matrix A. The right hand side of the Laplace equation is zero but it has to be updated with the Dirichlet values in the next step. c) Chapter 7

"Incorporating Dirichlet condition" explains the update of the right hand side of $Au = f$. The Dirichlet nodes are stored in the binary file `Dirichlet.bin` as 32 bit integers.

$$(d_0, d_1, \ldots, d_{l-1}) \in \mathbb{N}^l \tag{6}$$

The Dirichlet values are stored in `Values.bin` as double precision numbers.

$$(v_0, v_1, \ldots, v_{l-1}) \in \mathbb{R}^l \tag{7}$$

d) Solve the resulting linear system $Au = f$ with the Jacobi method with an initial guess $u^{(0)} = u_0$ and for $k > 0$ calculate

$$u_i^{(k+1)} := \frac{1}{a_{ii}} \left(f_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} u_j^{(k)} \right), \; i = 1, \ldots, n \,. \tag{8}$$

Terminate the Jacobi iteration if the norm $\|f - Au\|_2 < 10^{-6}$ and visualize the solution u with Matlab.

* Place all source files of the exercises in a folder named `Exercise5` in your home directory on the `mephisto.uni-graz.at` cluster.