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A PRIORI MESH GRADING FOR AN ELLIPTIC PROBLEM WITH DIRAC RIGHT-HAND SIDE*

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Abstract. The Green function of the Poisson equation in two dimensions is not contained in the Sobolev space $H^1(\Omega)$ such that finite element error estimates for the discretization of a problem with the Dirac measure on the right hand-side are nonstandard and quasi-uniform meshes are inappropriate. By using graded meshes L^2-error estimates of almost optimal order are shown. As a byproduct, we show for the Poisson equation with a right-hand side in L^2 that appropriate mesh refinement near some interior point diminishes the error at this point by nearly one order.

Key words. Dirac measure, Green function, fundamental solution, finite element method, graded mesh, error estimate

AMS subject classifications. 65N30, 65N15

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1. Introduction.

In this paper we consider the finite element solution of the elliptic boundary value problem

$$-\Delta u = \delta_{x_0} \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a convex, polygonal domain, and δ_{x_0} denotes the Dirac measure concentrated at $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, \partial\Omega) > 0$. Without loss of generality let $\operatorname{diam}(\Omega) \geq 1$. We restrict our considerations to convex domains. Singularities near concave corners are of local nature and can be treated by additional mesh refinement; see, e.g., [1].

Problems of this type occur in the simulation of field problems including point forces in linear elasticity or point charges in electrical field calculations. Furthermore, the application of the dual weighted residual method for estimating pointwise errors in a finite element discretization of a partial differential equation with smooth data leads to a problem of type (1.1); see [4]. Dirac measure terms can also be contained in the right-hand side of the adjoint problem in optimal control of elliptic partial differential equations with state constraints [7]. As a further example where point sources occur, we mention parameter identification problems with pointwise measurements [19].

For applying the finite element method, problem (1.1) is nonstandard since there is no $H^1(\Omega)$-solution. Hence the terms *solution* and *finite element solution* have to be defined carefully. The error analysis cannot start with the $H^1(\Omega)$-error such that also the Aubin–Nitsche method for obtaining an $L^2(\Omega)$-error estimate cannot be applied without modification.

Let us review the literature on a priori error estimates for problem (1.1) and start with discretizations on quasi-uniform meshes. Babuška [3] proved general er-

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ror estimates and specified for Dirac right-hand side and a two-dimensional smooth domain the almost optimal convergence order $h^{1-\varepsilon}$ for arbitrary $\varepsilon > 0$. Scott [21] considered differential equations of order $2m$ in smooth d-dimensional domains and finite elements of polynomial degree $k-1$. He proved for the error in the $H^s(\Omega)$-norm, $2m-k \le s \le 2m-\frac{d}{2}$, $k \ge m$, the convergence order $2m-\frac{d}{2}-s$, i.e., order $2-\frac{d}{2}$ in the $L^2(\Omega)$-norm for our problem. Casas [6] proved the same result for polygonal or polyhedral domains and general regular Borel measures on the right-hand side by using a different technique for the proof. To the best of our knowledge, Eriksson [10] was the first who used locally refined meshes near $x_0 \in \Omega$. By using techniques known from Schatz and Wahlbin [20] he proved convergence orders k and $k+1$ in the $W^{1,1}(\Omega)$- and $L^1(\Omega)$-norms, respectively, for approximations with piecewise polynomials of degree k. Eriksson [9] also analyzed a method where the numerical solution u_h is searched in the form $u_0 + v_h$ with u_0 being the fundamental solution.

With our contribution we continue the investigation of graded meshes with respect to the point $x_0 \in \Omega$ and prove two results. The first one is an intermediate step towards our final result but also of independent interest: We investigate the influence of the mesh refinement near x_0 on the finite element approximation of the solution of the Poisson equation with a right-hand side in $L^2(\Omega)$ and prove a convergence rate of $h^2\,|\ln h|^{3/2}$ for the error at the point x_0. Note that in this case the solution is in $H^2(\Omega)$ but in general not in $W^{2,\infty}(\Omega)$ such that one can expect only first order convergence in $L^\infty(\Omega)$ on quasi-uniform meshes. This means that the mesh grading improves the approximation quality at the point x_0 significantly. With this theorem we can derive the second result, an L^2-error estimate of order $h^2\,|\ln h|^{3/2}$ for the finite element approximation of problem (1.1).

The outline of this paper is as follows. In section 2 we introduce the graded meshes and state the results. We prove the main result on the basis of Theorem 2.1 whose proof is postponed to the third section. There, we formulate a couple of auxiliary results and conclude the result of Theorem 2.1 by using techniques developed by Frehse and Rannacher [11]. In section 4 we illustrate our theoretical findings by a numerical test.

Remark 1.1. One of our target applications is an optimal control problem with pointwise inequality constraints on the state variable (state constraints); see, e.g., [7]. If the state constraint is active at a singular point x_0, then the adjoint equation has typically the structure of problem (1.1). However, in this case the location of x_0 is a priori unknown, which makes the application of a priori graded meshes impossible. Nevertheless our main result (see Corollary 2.2) implies that the convergence order of almost $\mathcal{O}(N^{-1})$, where N is the number of elements, can, in principle, be achieved. This fact provides a target quality for application of adaptive procedures; see [5, 13, 14] for adaptive mesh refinement algorithms in the context of state constrained optimal control problems. In [5] a numerical example with one active point is discussed and approximation order $\mathcal{O}(N^{-1})$ is observed.

Remark 1.2. A first a posteriori error estimator which is proved to be reliable and efficient for elliptic problems with Dirac right-hand side was introduced by Araya, Behrens, and Rodríguez [2]. Numerical experiments with unstructured adaptive meshes yield the optimal convergence $\mathcal{O}(N^{-1})$ in the $L^2(\Omega)$-norm without the logarithmic factor. The experiments cover the cases where x_0 is or is not a mesh point.

In the more recent paper [8], Du and Xie constructed two a posteriori error estimators and proveed that one is reliable and efficient with respect to the broken

$W^{1,p}(\Omega)$-norm with $p \in (p^\Omega, 2)$, and the other is reliable in the $L^p(\Omega)$-norm with $p \in (p_\Omega, 2)$. Numerical experiments with these estimators as a guide for the adaptive mesh construction show a convergence of about $\mathcal{O}(N^{-1})$ in the $L^{1.5}(\Omega)$-norm and of about $\mathcal{O}(N^{-0.5})$ in the $W^{1,1.5}(\Omega)$-norm.

We end this section with the explanation of some notation. As usual, we denote by $W^{s,p}(\Omega)$ the Sobolev spaces and write $H^s(\Omega)$ for $W^{s,2}(\Omega)$. The scalar product in $L^2(\Omega)$ is denoted by $(.,.)$. The generic positive constant c is independent of h, and the notation $a \sim b$ means the existence of two constants c and C such that $ca \le b \le Ca$.

2. Main results. As mentioned in the introduction, problem (1.1) does not have an $H^1(\Omega)$-solution. Therefore we follow [2] and consider the solution u in the space

$$W_0^{1,q}(\Omega) := \{v \in W^{1,q}(\Omega) : v = 0 \text{ on } \partial\Omega \text{ in the sense of } L^q(\partial\Omega)\},$$

$q \in [1, 2)$, defined via

$$(\nabla u, \nabla v) = v(x_0) \quad \forall v \in W_0^{1,q'}(\Omega),$$

where $q' > 2$ satisfies $1/q + 1/q' = 1$.

For the approximate solution we introduce a family of graded triangulations $\mathcal{T}_h$ of Ω, where h is the discretization parameter. We assume that

$$h \le h_0 < 1$$

holds in order to ensure that $\ln h$ does not change sign. With r_T denoting the distance of an element (triangle) $T \in \mathcal{T}_h$ to x_0, we set the element sizes according to

$$h_T \sim \begin{cases} h^2 & \text{if } r_T = 0, \\ h r_T^{1/2} & \text{if } r_T > 0. \end{cases} \tag{2.1}$$

Notice that the number of elements of such a triangulation is of order h^{-2}; see, for example, [1]. Such meshes can be constructed via a coordinate transformation, see [17] or section 4, or by dyadic refinement, see, e.g., [12], or a combination of both, see [16].

The finite element space is then defined by

$$V_h := \{v_h \in C(\bar{\Omega}) : v_h|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \partial\Omega\},$$

where $\mathcal{P}_1(T)$ denotes the space of polynomials on T of degree at most one. The finite element solution $u_h \in V_h$ of (1.1) satisfies

$$(\nabla u_h, \nabla v_h) = v_h(x_0) \quad \forall v_h \in V_h. \tag{2.2}$$

The second problem considered is the Poisson problem with a right-hand side $f \in L^2(\Omega)$,

$$-\Delta z = f \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega, \tag{2.3}$$

which has no connection to the point $x_0 \in \Omega$ from the initial problem. In the following theorem we state that the proposed mesh grading (2.1) with respect to the point $x_0 \in \Omega$ yields an improvement in the convergence rate in the error at this point. Notice that one would expect only a convergence rate of h on quasi-uniform meshes.

THEOREM 2.1. *Let $f \in L^2(\Omega)$, $z \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of problem* (2.3), *and let z_h be a finite element approximation of z in the finite element space*

V_h defined above on a mesh that is graded according to condition (2.1). *Then the a priori estimate*

$$|(z - z_h)(x_0)| \le ch^2 |\ln h|^{3/2} \|z\|_{H^2(\Omega)}$$

holds for all $h \le h_0$.

The proof of the theorem is postponed to section 3.

From this theorem one can conclude the main result of this paper, an L^2-error estimate for problem (1.1).

COROLLARY 2.2. *Let u be the solution of* (1.1), *and $u_h \in V_h$ be its finite element approximation defined via* (2.2) *on a family of meshes that are graded according to condition* (2.1). *Then the a priori estimate*

$$\|u - u_h\|_{L^2(\Omega)} \le ch^2 |\ln h|^{3/2}$$

holds for all $h \le h_0$.

Proof. Denoting the error by $e := u - u_h$, we define the function $v \in H_0^1(\Omega)$ as the solution of

$$(\nabla v, \nabla \phi) = (e, \phi) \quad \forall \phi \in H_0^1(\Omega),$$

i.e., the weak solution of the boundary value problem

$$-\Delta v = e \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega.$$

Note that $v \in H^2(\Omega) \hookrightarrow W^{1,p}(\Omega)$ holds for any $p < \infty$. Its finite element approximation $v_h \in V_h$ is defined by

$$(\nabla v_h, \nabla \phi_h) = (e, \phi_h) \quad \forall \phi_h \in V_h.$$

With these auxiliary quantities we can estimate $\|e\|_{L^2(\Omega)}$ by utilizing Theorem 2.1:

$$\begin{aligned}
\|u - u_h\|_{L^2(\Omega)}^2 &= \|e\|_{L^2(\Omega)}^2 = (e, u) - (e, u_h) \\
&= (\nabla v, \nabla u) - (\nabla v, \nabla u_h) \\
&= v(x_0) - v_h(x_0) = (v - v_h)(x_0) \\
&\le ch^2 |\ln h|^{3/2} \left\|\nabla^2 v\right\|_{L^2(\Omega)} \\
&\le ch^2 |\ln h|^{3/2} \|e\|_{L^2(\Omega)} .
\end{aligned}$$

Dividing this inequality by $\|u - u_h\|_{L^2(\Omega)}$ gives the desired result. ∎

3. Proof of Theorem 2.1. We state a couple of auxiliary results in the forthcoming lemmas. At the end of the section we use these results to prove Theorem 2.1.

We split the domain Ω into the sets

$$\Omega_0 = \bigcup_{r_T = 0} T \quad \text{and} \quad \Omega_1 = \Omega \setminus \Omega_0$$

and choose an element $T^* \in \Omega_0$. Its diameter is $h_* \sim h^2$. A weight function $\sigma : \Omega \to \mathbb{R}$ is defined by

$$\sigma(x) := \left(|x - x_0|^2 + h_*^2\right)^{1/2} . \tag{3.1}$$

The following properties of σ can be proved by calculation.

LEMMA 3.1. *For the function σ defined in* (3.1) *the inequalities*

$$\begin{aligned} |\sigma| + |\nabla\sigma| &\le c \\ \left|\nabla^2\sigma\right| &\le c\sigma^{-1}, \\ \sigma^{-1}(x) &\le \begin{cases} h_*^{-1} & \text{if } x \in \{T\in\mathcal{T}_h : r_T = 0\}, \\ cr_T^{-1} & \text{if } x \in \{T\in\mathcal{T}_h : r_T > 0\} \end{cases} \end{aligned} \tag{3.2}$$

are valid.

For functions with elementwise H^2-regularity we will use the notation $\nabla_h v \in L^2(\Omega)$ and $\nabla_h^2 v \in L^2(\Omega)$ given through

$$\nabla_h v|_T = \nabla v|_T \quad \text{and} \quad \nabla_h^2 v|_T = \nabla^2 v|_T.$$

The nodal interpolant of a function $v \in H_0^1(\Omega)\cap C(\bar\Omega)$ is denoted by $\mathcal{I}_h v \in V_h$. We begin our considerations with an estimate of a weighted interpolation error.

LEMMA 3.2. *For any function v from the set*

$$\{v \in H_0^1(\Omega)\cap C(\bar\Omega) : v \in H^2(T)\ \forall T \in \mathcal{T}_h\}$$

the estimate

$$\left\|\sigma^{-1/2}\nabla(v-\mathcal{I}_h v)\right\|_{L^2(\Omega)} \le ch\left\|\nabla_h^2 v\right\|_{L^2(\Omega)}$$

holds on meshes of type (2.1). *For functions $v \in H_0^1(\Omega)\cap H^2(\Omega)$ this results in*

$$\left\|\sigma^{-1/2}\nabla(v-\mathcal{I}_h v)\right\|_{L^2(\Omega)} \le ch\left\|\nabla^2 v\right\|_{L^2(\Omega)}.$$

Proof. One can calculate by using (3.2) that

$$\begin{aligned} \left\|\sigma^{-1/2}\nabla(v-\mathcal{I}_h v)\right\|^2_{L^2(\Omega)} &= \sum_{T\subset\Omega_0}\int_T \sigma^{-1}\left|\nabla(v-\mathcal{I}_h v)\right|^2 + \sum_{T\subset\Omega_1}\int_T \sigma^{-1}\left|\nabla(v-\mathcal{I}_h v)\right|^2 \\ &\le \sum_{T\subset\Omega_0} ch_*^{-1}h_*^2\left\|\nabla^2 v\right\|^2_{L^2(T)} + \sum_{T\subset\Omega_1} cr_T^{-1}h_T^2\left\|\nabla^2 v\right\|^2_{L^2(T)} \\ &\le \sum_{T\subset\Omega} ch^2\left\|\nabla^2 v\right\|^2_{L^2(T)}. \end{aligned}$$

This proves the assertion. ∎

LEMMA 3.3. *For any function $v \in H_0^1(\Omega)\cap H^2(\Omega)$ the inequality*

$$\|\nabla(v-\mathcal{I}_h v)\|_{L^2(\Omega)} \le c\left\|\sigma\nabla^2 v\right\|_{L^2(\Omega)}$$

holds, provided the mesh is graded according to (2.1).

Proof. With the help of the function σ we can estimate the element size on the two subdomains. On Ω_0 there follows directly

$$h_*^2 \le \sigma^2(x) \qquad \forall x \in \Omega_0. \tag{3.3}$$

On Ω_1 one has $\sigma(x) \ge r_T$ and $\sigma(x) \ge h_*$. Since there holds $h_T \sim hr_T^{1/2}$ the relation $h_T^2 \sim h^2 r_T \sim h_* r_T$ is used to conclude

$$h_T^2 \le c\sigma^2(x) \qquad \forall x \in \Omega_1. \tag{3.4}$$

Now we can estimate

$$\|\nabla(v-\mathcal{I}_h v)\|^2_{L^2(\Omega)} \le c\sum_T \int_T h_T^2 \left|\nabla^2 v\right|^2 = c\sum_{T\subset\Omega_0}\int_T h_*^2 \left|\nabla^2 v\right|^2 + c\sum_{T\subset\Omega_1}\int_T h_T^2 \left|\nabla^2 v\right|^2 .$$

With the estimates (3.3), (3.4) one can continue with

$$\|\nabla(v-\mathcal{I}_h v)\|^2_{L^2(\Omega)} \le c\sum_T \int_T \sigma^2 \left|\nabla^2 v\right|^2 = c\left\|\sigma\nabla^2 v\right\|^2_{L^2(\Omega)},$$

and the assertion is proved. ∎

LEMMA 3.4. *Let the function $y \in H_0^1(\Omega)\cap H^2(\Omega)$ be the solution of the boundary value problem*

$$-\Delta y = w \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega \tag{3.5}$$

with a given right-hand side $w \in L^2(\Omega)$. Then for $h \le h_0$ the estimate

$$\left\|\sigma\nabla^2 y\right\|_{L^2(\Omega)} \le c\left|\ln h\right| \|\sigma w\|_{L^2(\Omega)}$$

holds, where σ is the weight function defined in (3.1).

Proof. Set $\xi := x - x_0$ and denote by ξ_1, ξ_2 its components. By the chain rule it holds

$$\left\|\xi_i\nabla^2 y\right\|_{L^2(\Omega)} \le \left\|\nabla^2(\xi_i y)\right\|_{L^2(\Omega)} + c\|\nabla y\|_{L^2(\Omega)}, \qquad i = 1, 2.$$

With the definition of σ and the a priori estimate $\left\|\nabla^2 y\right\|_{L^2(\Omega)} \le c\|\Delta y\|_{L^2(\Omega)}$ this yields

$$\begin{aligned}\left\|\sigma\nabla^2 y\right\|^2_{L^2(\Omega)} &= \sum_{i=1}^2 \left\|\xi_i\nabla^2 y\right\|^2_{L^2(\Omega)} + h_*^2\left\|\nabla^2 y\right\|^2_{L^2(\Omega)}\\ &\le \sum_{i=1}^2\left(\left\|\nabla^2(\xi_i y)\right\|^2_{L^2(\Omega)} + c\|\nabla y\|^2_{L^2(\Omega)}\right) + ch_*^2\|\Delta y\|^2_{L^2(\Omega)}.\end{aligned}$$

With the use of $h_* \le \sigma$ we continue and get

$$\begin{aligned}\left\|\sigma\nabla^2 y\right\|^2_{L^2(\Omega)} &\le c\sum_{i=1}^2\|\Delta(\xi_i y)\|^2_{L^2(\Omega)} + c\|\nabla y\|^2_{L^2(\Omega)} + c\|\sigma\Delta y\|^2_{L^2(\Omega)}\\ &\le c\sum_{i=1}^2\|\xi_i\Delta y\|^2_{L^2(\Omega)} + c\|\nabla y\|^2_{L^2(\Omega)} + c\|\sigma w\|^2_{L^2(\Omega)}\\ &\le c\|\sigma\Delta y\|^2_{L^2(\Omega)} + c\|\nabla y\|^2_{L^2(\Omega)} + c\|\sigma w\|^2_{L^2(\Omega)}\\ &\le c\|\sigma w\|^2_{L^2(\Omega)} + c\|\nabla y\|^2_{L^2(\Omega)},\end{aligned} \tag{3.6}$$

where we have used inequality (3.2) and the definition (3.5) of y. It remains to show that $\|\nabla y\|_{L^2(\Omega)} \le |\ln h|\,\|\sigma w\|_{L^2(\Omega)}$. Following an argument taken from [18] we consider

$$\|\nabla y\|^2_{L^2(\Omega)} = |(\Delta y, y)| \le \|\sigma\Delta y\|_{L^2(\Omega)}\left\|\sigma^{-1}y\right\|_{L^2(\Omega)} = \|\sigma w\|_{L^2(\Omega)}\left\|\sigma^{-1}y\right\|_{L^2(\Omega)}. \tag{3.7}$$

The last factor will be estimated by using its representation in polar coordinates (r,θ) with respect to x_0. In the following we use the observation

$$\sigma(r) = \left(r^2 + h_*^2\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}(\ln\sigma(r) - \ln\sigma(0)) = \frac{r}{\sigma^2} \tag{3.8}$$

and the inequality

$$\left|\frac{\ln\sigma(r) - \ln\sigma(0)}{r}\right| \le \frac{c}{\sigma}\,|\ln h| \qquad \text{for } h \le h_0, \tag{3.9}$$

which is proved later. Furthermore for simplicity of notation we replace the integration domain Ω by a disc of radius $R = \operatorname{diam}(\Omega) \ge 1$ with the center in x_0, such that this disc contains Ω. We continue the function y with $y = 0$ outside the domain Ω such that this extension of the domain does not change the value of any quantities involved. With the observation (3.8), partial integration with respect to the radius r, and estimate (3.9), one can conclude

$$\begin{aligned}
\left\|\sigma^{-1}y\right\|_{L^2(\Omega)}^2 &= \int_\Omega \sigma^{-2}y^2 = \int_0^{2\pi}\int_0^R r\sigma^{-2}y^2\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^R \frac{|\ln\sigma(r) - \ln\sigma(0)|}{r} r\,2y\partial_r y\,\mathrm{d}r\,\mathrm{d}\theta \\
&\le \int_0^{2\pi}\int_0^R \frac{c}{\sigma}\,|\ln h|\,r\,|y\partial_r y|\,\mathrm{d}r\,\mathrm{d}\theta \\
&\le c\,|\ln h| \int_0^{2\pi}\int_0^R \sigma^{-1}r\,|y|\,|\nabla y|\,\mathrm{d}r\,\mathrm{d}\theta \\
&\le c\,|\ln h|\,\left\|\sigma^{-1}y\right\|_{L^2(\Omega)} \|\nabla y\|_{L^2(\Omega)}\,.
\end{aligned}$$

Dividing by $\left\|\sigma^{-1}y\right\|_{L^2(\Omega)}$ yields

$$\left\|\sigma^{-1}y\right\|_{L^2(\Omega)} \le c\,|\ln h|\,\|\nabla y\|_{L^2(\Omega)}\,.$$

Inserting this into (3.7) and dividing by $\|\nabla y\|_{L^2(\Omega)}$ yields

$$\|\nabla y\|_{L^2(\Omega)} \le c\,|\ln h|\,\|\sigma w\|_{L^2(\Omega)}$$

and thus with (3.6) the claim of the lemma.

It remains to prove inequality (3.9). To this end, we distinguish the cases $r > h_*$ and $r \le h_*$ and begin with the case $r > h_*$. Since $\sigma(r)$ is strictly monotone and positive the function $|\ln\sigma(r)|$ takes its maximum at the left or right boundary of $[0,R]$. For $h \le h_0$ these values can be estimated by

$$|\ln\sigma(0)| = |\ln h_*| \le c\,|\ln h| \ \text{ and} \tag{3.10}$$

$$|\ln\sigma(R)| = \left|\ln\sqrt{R^2 + h_*^2}\right| \le c\,|\ln h|\,, \tag{3.11}$$

since $\ln\sqrt{R^2 + h_*^2} \le \ln\sqrt{R^2 + h_0^2} = c\,|\ln h_0| \le c\,|\ln h|$ for $c = \ln\sqrt{R^2 + h_0^2}/\,|\ln h_0|$. Thus it follows that

$$|\ln\sigma(r) - \ln\sigma(0)| \le 2\max_{0\le r\le R}|\ln\sigma(r)| \le c\,|\ln h|\,,$$

again for $h \leq h_0$. Since $1/r \leq c/\sigma$, the inequality (3.9) is proved.

For the case $r \leq h_*$ we can conclude by the mean value theorem that

$$\left|\frac{\ln\sigma(r) - \ln\sigma(0)}{r}\right| \leq \max_{0\leq s\leq h_*} |(\ln\sigma)'(s)| = \max_{0\leq s\leq h_*} \frac{s}{\sigma(s)^2}.$$

As the last function is monotonically increasing on $[0, h_*]$, it takes its maximum at the end of the interval. This means by using $h_* \leq \sigma(r) \leq \sqrt{2}h_*$ that

$$\left|\frac{\ln\sigma(r) - \ln\sigma(0)}{r}\right| \leq \frac{h_*}{2h_*^2} \leq \frac{\sqrt{2}}{2}\frac{1}{\sigma}$$

and inequality (3.9) is also proved in this case. ◻

For our further considerations we introduce a regularized Dirac function (see, e.g., [11]),

$$\delta^h := \begin{cases} |T^*|^{-1}\,\mathrm{sign}(z - z_h) & \text{in } T^*, \\ 0 & \text{elsewhere,} \end{cases}$$

where z is the solution of (2.3) and z_h is the corresponding finite element approximation from Theorem 2.1. Notice that $\delta^h \in L^2(\Omega)$. The corresponding regularized Green function $g^h \in H_0^1(\Omega) \cap H^2(\Omega)$ is defined by

$$-\Delta g^h = \delta^h \text{ in } \Omega, \qquad g^h = 0 \text{ on } \partial\Omega. \tag{3.12}$$

Besides, we define the function $g_h^h \in V_h$ as the Ritz projection of g^h onto V_h, i.e.,

$$(\nabla g_h^h, \nabla\phi_h) = (\nabla g^h, \nabla\phi_h) \quad \forall\phi_h \in V_h. \tag{3.13}$$

LEMMA 3.5. *For the regularized Green function g^h defined in* (3.12) *the estimate*

$$\left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)} \leq c\,|\ln h|^{1/2}$$

holds for $h \leq h_0$.

Proof. The assertion follows from setting $\rho = h_*$ in [11, Theorem B4]. In this paper, a $C^{1,1}$-domain Ω is considered, but this assumption is not necessary for the result of this lemma. For completeness we briefly discuss the proof. Let $\xi \in T_*$, and let $g_\xi \in W_0^{1,q}(\Omega)$, $q \in [1, 2)$, be the Green function fulfilling

$$-\Delta g_\xi = \delta_\xi \text{ in } \Omega, \qquad g_\xi = 0 \text{ on } \partial\Omega.$$

The estimate

$$|g_\xi(x)| \leq c\,(\ln|x - \xi| + 1) \tag{3.14}$$

is proved using the maximum principle. We obtain

$$\left|g^h(\xi)\right| = \left|(\nabla g_\xi, \nabla g^h)\right| = \left|(\delta^h, g_\xi)\right| \leq |T^*|^{-1}\int_{T^*} g_\xi\,dx.$$

Using (3.14) we get

$$\sup_{\xi\in T_*}\left|g^h(\xi)\right| \leq c(|\ln h| + 1) \leq c\,|\ln h|$$

for $h \le h_0$. It follows that

$$\left\|\nabla g^h\right\|^2_{L^2(\Omega)} = (\delta_h, g^h) \le \sup_{\xi\in T_*} \left|g^h(\xi)\right| \le c\left|\ln h\right|.$$

As in the proof of Lemma 3.4 we obtain

$$\left\|\sigma\nabla^2 g^h\right\|^2_{L^2(\Omega)} \le c\left\|\sigma\delta_h\right\|^2_{L^2(\Omega)} + c\left\|\nabla g^h\right\|^2_{L^2(\Omega)}.$$

The desired estimate follows directly. ∎

LEMMA 3.6. *For the regularized Green function g^h and its Ritz projection g^h_h defined in* (3.12) *and* (3.13), *respectively, the estimate*

$$\left\|\sigma^{-1}(g^h - g^h_h)\right\|_{L^2(\Omega)} \le c\left|\ln h\right|^{3/2}$$

holds for $h \le h_0$.

Proof. We introduce the abbreviation $e_g := g^h - g^h_h$ and consider the auxiliary equation

$$-\Delta y = \frac{\sigma^{-2}e_g}{\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)}} \text{ in } \Omega, \qquad y = 0 \text{ on } \partial\Omega.$$

Its weak form can be written as

$$(\nabla y, \nabla\varphi) = \frac{(\sigma^{-1}e_g, \sigma^{-1}\varphi)}{\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)}} \quad \forall\phi \in H^1_0(\Omega).$$

The choice $\varphi = e_g$ yields

$$\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)} = (\nabla e_g, \nabla y) = (\nabla e_g, \nabla(y - \mathcal{I}_h y)) \le \left\|\nabla e_g\right\|_{L^2(\Omega)} \left\|\nabla(y - \mathcal{I}_h y)\right\|_{L^2(\Omega)}. \tag{3.15}$$

For the first term on the right-hand side we use Lemma 3.3 with the choice $v = g^h$ and conclude with the result from Lemma 3.5 that

$$\left\|\nabla e_g\right\|_{L^2(\Omega)} \le c\left\|\nabla(g^h - \mathcal{I}_h g^h)\right\|_{L^2(\Omega)} \le c\left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)} \le c\left|\ln h\right|^{1/2}. \tag{3.16}$$

For the second term on the right-hand side of inequality (3.15) we write with Lemmas 3.3 and 3.4

$$\left\|\nabla(y - \mathcal{I}_h y)\right\|_{L^2(\Omega)} \le c\left\|\sigma\nabla^2 y\right\|_{L^2(\Omega)} \le c\left|\ln h\right| \left\|\sigma\frac{\sigma^{-2}e_g}{\left\|\sigma^{-1}e_g\right\|}\right\|_{L^2(\Omega)} = c\left|\ln h\right|. \tag{3.17}$$

Inequality (3.15) together with estimates (3.16) and (3.17) yields the assertion of this lemma. ∎

LEMMA 3.7. *For the regularized Green function g^h and its Ritz projection g^h_h defined in* (3.12) *and* (3.13), *respectively, the inequality*

$$\left\|\nabla^2_h(\sigma(g^h - g^h_h))\right\| \le c\left|\ln h\right|^{3/2}$$

is satisfied for $h \le h_0$.

Proof. We use again the abbreviation $e_g := g^h - g_h^h$, apply the product rule on every element $T \in \mathcal{T}_h$, and get

$$\nabla^2(\sigma e_g)|_T = (\nabla^2\sigma)e_g|_T + 2\nabla\sigma|_T \cdot \nabla e_g|_T + \sigma(\nabla^2 e_g)|_T.$$

This results with Lemma 3.1 in the estimate

$$\left\|\nabla_h^2(\sigma e_g)\right\|_{L^2(\Omega)}^2 \le c\left(\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)}^2 + \left\|\nabla e_g\right\|_{L^2(\Omega)}^2 + \left\|\sigma(\nabla_h^2 e_g)\right\|_{L^2(\Omega)}^2\right). \tag{3.18}$$

The first term on the right-hand side of this inequality is estimated in Lemma 3.6, giving a contribution of $c\,|\ln h|^3$. The second term is estimated in (3.16). Since the equality $\nabla^2(g_h^h|_T) = 0$ holds for linear elements on every element T it follows for the third term with application of Lemma 3.5 that

$$\left\|\sigma(\nabla_h^2 e_g)\right\|_{L^2(\Omega)}^2 = \left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)}^2 \le c\,|\ln h|\,. \tag{3.19}$$

This means Lemma 3.6 together with the inequalities (3.18), (3.16), and (3.19) yields the assertion. ∎

LEMMA 3.8. *For the regularized Green function g^h and its Ritz projection g_h^h defined in* (3.12) *and* (3.13), *respectively, the inequality*

$$\left\|\sigma^{1/2}\nabla(g^h - g_h^h)\right\|_{L^2(\Omega)} \le ch\,|\ln h|^{3/2}$$

holds for $h \le h_0$.

Proof. We use the abbreviation $e_g := g^h - g_h^h$. With the product rule we observe

$$\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 = (\nabla e_g, \sigma\nabla e_g) = (\nabla e_g, \nabla(\sigma e_g)) - (\nabla e_g, e_g\nabla\sigma). \tag{3.20}$$

For the first term on the right-hand side we apply the Galerkin orthogonality and estimate

$$\begin{aligned}
(\nabla e_g, \nabla(\sigma e_g)) &= (\nabla e_g, \nabla(\sigma e_g - \mathcal{I}_h(\sigma e_g))) \\
&= (\sigma^{1/2}\nabla e_g, \sigma^{-1/2}\nabla(\sigma e_g - \mathcal{I}_h(\sigma e_g))) \\
&\le \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + \left\|\sigma^{-1/2}\nabla(\sigma e_g - \mathcal{I}_h(\sigma e_g))\right\|_{L^2(\Omega)}^2 \\
&\le \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + ch^2\left\|\nabla_h^2(\sigma e_g)\right\|_{L^2(\Omega)}^2 \\
&\le \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + ch^2\,|\ln h|^3\,, \qquad (3.21)
\end{aligned}$$

where we have used Lemmas 3.2 and 3.7 in the last two steps, respectively.

For estimating the second term on the right-hand side of (3.20) we consider another auxiliary equation,

$$-\Delta y = \frac{e_g}{\|e_g\|_{L^2(\Omega)}} \;\text{ in } \Omega, \qquad y = 0 \;\text{ on } \partial\Omega.$$

Utilizing the weak form of this equation with e_g as the test function, and later on

Lemma 3.2, we can write

$$
\begin{aligned}
\|e_g\|_{L^2(\Omega)} &= (\nabla e_g, \nabla y) = (\nabla e_g, \nabla(y - \mathcal{I}_h y)) \\
&\le \left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)} \left\|\sigma^{-1/2}\nabla(y-\mathcal{I}_h y)\right\|_{L^2(\Omega)} \\
&\le \left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)} ch \left\|\nabla^2 y\right\|_{L^2(\Omega)} \\
&\le ch \left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)} \qquad (3.22)
\end{aligned}
$$

since the L^2-norm of $e_g / \|e_g\|_{L^2(\Omega)}$ is one. With this result the second term on the right-hand side of (3.20) can be estimated with the help of Lemma 3.1 as

$$
\begin{aligned}
(\nabla e_g, e_g \nabla\sigma) &= (\sigma^{1/2}\nabla e_g, \sigma^{-1/2} e_g \nabla \sigma) \\
&\le \left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)} \left\|\sigma^{-1/2} e_g \nabla\sigma\right\|_{L^2(\Omega)} \\
&\le \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|^2_{L^2(\Omega)} + c\left\|\sigma^{-1/2} e_g\right\|^2_{L^2(\Omega)} \\
&\le \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|^2_{L^2(\Omega)} + c(e_g, \sigma^{-1} e_g) \\
&\le \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|^2_{L^2(\Omega)} + c\left\|e_g\right\|_{L^2(\Omega)} \left\|\sigma^{-1} e_g\right\|_{L^2(\Omega)}.
\end{aligned}
$$

With estimate (3.22) and Lemma 3.6 one can conclude

$$
\begin{aligned}
(\nabla e_g, e_g \nabla\sigma) &\le \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|^2_{L^2(\Omega)} + ch\left|\ln h\right|^{3/2}\left\|\sigma^{1/2}\nabla e_g\right\| \\
&\le \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|^2_{L^2(\Omega)} + ch^2\left|\ln h\right|^3 \qquad (3.23)
\end{aligned}
$$

by applying Young's inequality in the last step. With (3.20) the assertion follows from inequalities (3.21) and (3.23). ∎

Now we are able to prove Theorem 2.1.

Proof. Let T^* denote an element that contains x_0, and set $\tilde{e} := z - z_h$. By using the nodal interpolant $\mathcal{I}_h$ we estimate

$$
\begin{aligned}
|(z - z_h)(x_0)| &\le \max_{T^*} |\tilde{e}| \\
&\le \max_{T^*} |z - \mathcal{I}_h z| + \max_{T^*} |\mathcal{I}_h \tilde{e}| \\
&\le \max_{T^*} |z - \mathcal{I}_h z| + c\,|T^*|^{-1} \int_{T^*} |\mathcal{I}_h \tilde{e}| \\
&\le \max_{T^*} |z - \mathcal{I}_h z| + c\,|T^*|^{-1} \left(\int_{T^*} |z - \imath z| + \int_{T^*} |\tilde{e}| \right) \\
&\le c \max_{T^*} |z - \mathcal{I}_h z| + c\,|T^*|^{-1} \int_{T^*} |\tilde{e}| \\
&\le ch_* \left\|\nabla^2 z\right\|_{L^2(T^*)} + c\,|T^*|^{-1} \int_{T^*} |\tilde{e}|. \qquad (3.24)
\end{aligned}
$$

Since $h_* \sim h^2$ it remains to estimate

$$
|T^*|^{-1} \int_{T^*} |\tilde{e}|.
$$

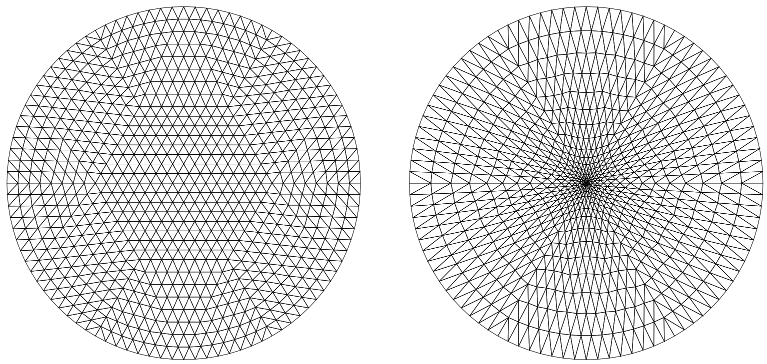

FIG. 4.1. *Quasi-uniform mesh and graded mesh according to* (2.1).

To this end, we consider the auxiliary problem (3.12). From the weak form of this boundary value problem it is easy to see that

$$(\nabla g^h, \nabla \tilde{e}) = (\delta^h, \tilde{e}) = |T^*|^{-1} \int_{T^*} |\tilde{e}| \tag{3.25}$$

is the term left to consider. With the Ritz projection g_h^h defined in (3.13) we can write

$$\begin{aligned} (\nabla g^h, \nabla \tilde{e}) &= (\nabla(z - z_h), \nabla g^h) \\ &= (\nabla(z - z_h), \nabla(g^h - g_h^h)) \\ &= (\nabla(z - \mathcal{I}_h z), \nabla(g^h - g_h^h)) \\ &\leq \left\| \sigma^{-1/2} \nabla(z - \mathcal{I}_h z) \right\|_{L^2(\Omega)} \left\| \sigma^{1/2} \nabla(g^h - g_h^h) \right\|_{L^2(\Omega)}, \end{aligned} \tag{3.26}$$

using Galerkin orthogonality. The application of Lemmas 3.2 and 3.8 together with (3.25) yields the assertion. ◻

4. Numerical examples. In this section we illustrate our theoretical findings by numerical examples. For the computation of the finite element approximations we used the finite element library MooNMD [15]. In the numerical examples we choose $x_0 = (0, 0)$ and

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 < 1\}.$$

We consider quasi-uniform meshes and meshes that are graded according to condition (2.1). In Figure 4.1 one can see both versions of mesh for $h = 1/16$. The graded mesh is generated by transforming the uniform mesh using the mapping

$$T(x) = x \|x\|.$$

As the first example we consider the boundary value problem (1.1) and its finite element solution according to (2.2). The exact solution u is given as the fundamental solution of the Laplace equation,

$$u(x) = -\frac{1}{2\pi} \ln \sqrt{x_1^2 + x_2^2}.$$

Table 4.1 shows the estimated order of convergence (eoc) for quasi-uniform and graded meshes, respectively. In the case of quasi-uniform meshes one can see a convergence

Table 4.1
L^2-error and estimated order of convergence for quasi-uniform and graded meshes, first example.

N_{nodes}	Quasi-uniform mesh $\|u-u_h\|_{L^2(\Omega)}$	eoc	Graded mesh $\|u-u_h\|_{L^2(\Omega)}$	eoc
817	3.04e−03		8.79e−04	1.96
3169	1.52e−03	1.02	2.34e−04	1.95
12481	7.60e−04	1.01	6.17e−05	1.95
49537	3.80e−04	1.01	1.61e−05	1.94
197377	1.90e−05	1.00	4.21e−06	1.95

Table 4.2
Error in x_0 and estimated order of convergence for quasi-uniform and graded meshes, second example.

N_{nodes}	Quasi-uniform mesh $\|(z-z_h)(x_0)\|$	eoc	Graded mesh $\|(z-z_h)(x_0)\|$	eoc
817	1.15e−03		2.07e−03	2.09
3169	7.01e−04	0.73	5.11e−04	2.06
12481	3.68e−04	0.94	1.25e−04	2.05
49537	1.82e−04	1.02	3.08e−05	2.04
197377	8.80e−05	1.05	7.57e−06	2.03

rate of 1 in h as proved in [21]. For meshes that are graded according to condition (2.1) one observes a convergence rate slightly smaller than 2. This confirms our theoretical results. Notice that the curved boundary is approximated by straight lines. The additional error introduced by this has no influence on our results since $\text{meas}(\Omega\backslash\Omega_h) \leq ch^2$ due to the smoothness of the boundary, $\|u\|_{L^\infty(\Omega\backslash\Omega_h)} \leq ch^2|u|_{W^{1,\infty}(\Omega\backslash\Omega_h)}$ due to the boundary condition and the smoothness of u, such that with $u_h \equiv 0$ in $\Omega \setminus \Omega_h$ the estimate $\|u-u_h\|_{L^2(\Omega\backslash\Omega_h)} \leq ch^3|u|_{W^{1,\infty}(\Omega\backslash\Omega_h)}$ holds.

A second example illustrates Theorem 2.1. We choose $f \in L^2(\Omega)$ such that the exact solution is

$$z = 1 + r\left(\ln\left(\frac{r}{\mathrm{e}}\right)\right)^{-1} \in W^{2,p}(\Omega), \quad p \leq 2,$$

where we use $r := \|x\|$. Note that homogeneous Dirichlet boundary conditions are satisfied. The corresponding results are shown in Table 4.2. As expected the convergence order is one in h for quasi-uniform meshes and about two in h for the graded meshes.

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