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Finite element error estimates on the boundary with application to optimal control

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Abstract This paper is concerned with the discretization of linear elliptic partial differential equations with Neumann boundary condition in polygonal domains. The focus is on the derivation of error estimates in the L^2-norm on the boundary for linear finite elements. Whereas common techniques yield only suboptimal results, a new approach in this context is presented which allows for quasi optimal ones, i.e., for domains with interior angles smaller than $2\pi/3$ a convergence order close to two can be achieved using quasi-uniform meshes. In the presence of corner singularities, which reduce the convergence rates, graded meshes are used to maintain the quasi optimal error bounds.

This result is applied to linear-quadratic Neumann boundary control problems with pointwise inequality constraints on the control. The approximations of the control are piecewise constant. The state and the adjoint state are discretized by piecewise linear finite elements. In a postprocessing step approximations of the continuous optimal control are constructed which possess superconvergence properties. Based on the improved error estimates on the boundary and optimal regularity in weighted Sobolev spaces almost second order convergence is proven for the approximations of the continuous optimal control problem. Mesh grading techniques are again used for domains with interior angles greater than $2\pi/3$.

Keywords: linear-quadratic Neumann boundary control problem, control constraints, corner singularities, weighted Sobolev spaces, finite element method, error estimates, boundary estimates, quasi-uniform meshes, graded meshes, postprocessing, superconvergence

AMS subject classification 65N30; 49M25, 65N50, 65N15

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1 Introduction

Let Ω be a bounded, two dimensional, polygonal domain with Lipschitz boundary Γ and m corner points $x^{(j)}$, $j = 1\ldots,m$, counting counterclockwise. In particular, Γ_j denotes the side on the boundary Γ which connects the corners $x^{(j)}$ and $x^{(j+1)}$, note that $x^{(m+1)} := x^{(1)}$. The angle between Γ_{j-1} and Γ_j is denoted by ω_j with the obvious modification for ω_1. In the first part of this paper, we will discretize the linear elliptic partial differential equation

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma_j, \quad j = 1,\ldots,m, \end{aligned} \tag{1.1}$$

with piecewise linear and continuous ansatz functions and focus on finite element error estimates in the L^2-norm on the boundary. Common approaches for this use the trace theorem or the Nitsche method in $L^2(\Gamma)$. This yields an error bound of $ch^{3/2}$. This estimate is sharp in case of $H^2(\Omega)$-regularity of the solution. For example such limited regularity can be caused by the presence of corner singularities or by right hand sides f and g in $L^2(\Omega)$ and $H^{1/2}(\Gamma)$, respectively. For more regular solutions in $W^{2,p}(\Omega)$ with $p > 2$ numerical examples indicate that the convergence order is better, cf. [18, 1]. This can be explained by first using the embedding $L^p(\Gamma) \hookrightarrow L^2(\Gamma)$ and then applying the trace theorem or the Nitsche method in $L^p(\Gamma)$. Similar techniques are used in [18]. By this one obtains an error bound of $ch^{2-1/p}$. Accordingly for problem (1.1) a convergence order close to 2 can only be expected if $\omega_j < \pi/2$ for $j = 1,\ldots,m$, since the corner singularities admit in general only a solution in $W^{2,\infty}(\Omega)$ for such domains. In the present work we will show that the estimates can even be improved assuming Hölder continuous right hand sides. Based on regularity results in weighted Sobolev spaces, techniques of [27, 28, 2] and local finite element error estimates as described in [26, 30, 9] a quasi optimal error bound of $ch^2|\ln h|^{1+\delta}$ with some $\delta \in [0,1/2]$ will be obtained for domains with interior angles smaller than $\pi/(2-\delta)$. Hence, in domains with interior angles smaller than $2\pi/3$ the error is definitely bounded by $ch^2|\ln h|^{3/2}$. These estimates hold still for quasi-uniform meshes. For domains with larger interior angles than $\pi/(2-\delta)$ mesh grading techniques will be applied to get the same result. In that case the mesh grading parameter μ_j, that determines the strength of the grading around the corner $x^{(j)}$, has to be chosen smaller than $(\delta + \pi/\omega_j)/2$, see Section 3 for details.

Error estimates of that kind are for example required for the numerical analysis of Neumann boundary control problems. The second part of this paper dedicated to this topic. We will consider the optimal control problem

$$\begin{aligned} J(\bar u) &= \min_{u\in U_{ad}} J(u), \\ J(u) &:= F(Su,u), \\ F(y,u) &:= \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \end{aligned} \tag{1.2}$$

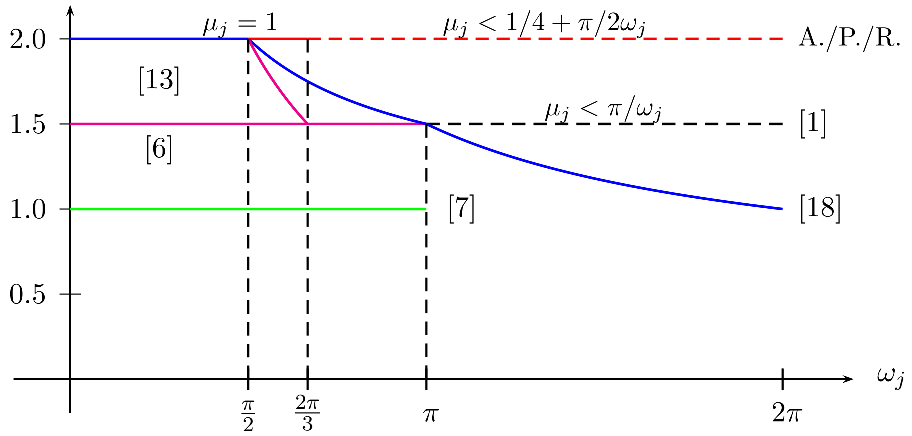

Figure 1: Convergence rates of the control in $L^2(\Gamma)$ depending on the interior angles ω_j. Solid lines: quasi-uniform meshes. Dashed lines: graded meshes.

where the associated state $y = Su$ to the control u is the weak solution of

$$\begin{aligned} -\Delta y + y &= 0 \quad \text{in } \Omega, \\ \partial_n y &= u \quad \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned} \tag{1.3}$$

the desired state y_d belongs to the Hölder space $C^{0,\sigma}(\bar{\Omega})$ with some $\sigma \in (0, 1)$ and the control variable is constrained by

$$a \leq u(x) \leq b \quad \text{for a.a. } x \in \Gamma.$$

We will focus on the full discretization of problem (1.2) combined with a postprocessing procedure, i.e., the state and the adjoint state are discretized by linear finite elements and the control is approximated by piecewise constant functions on the boundary. Afterwards, in a postprocessing step, piecewise linear and globally continuous controls are constructed, which possess superconvergence properties. This postprocessing approach is well known for linear optimal control problems, see [22, 3] for distributed controls and [18, 1] for controls located at the Neumann boundary. It is extended to semilinear control problems in [24]. Also different approaches for problem (1.2) have been discussed in the literature. The numerical analysis for the full discretization itself has been accomplished in [7]. In [6] the control is not discretized by piecewise constant functions but by piecewise linear and globally continuous functions. These two works attack semilinear elliptic problems. The variational discretization concept, originally suggested in [12] for distributed control problems, has been transferred to Neumann boundary control problems in [6] for semilinear and in [13, 18, 1] for linear elliptic equations. This approach only discretizes the state and adjoint state by linear finite elements but not the control. In Figure 1 the obtained convergence rates of the control in $L^2(\Gamma)$ for the different approaches are illustrated depending on the interior angles of the domain. In [7] a convergence order of 1 is obtained in convex domains for the full discretization of semilinear elliptic problems, where the control is discretized piecewise constantly. Approximating the control with piecewise linear and continuous functions implies in convex domains superlinear convergence and a convergence rate of 3/2 under an additional assumption on the control, see [6]. Furthermore, in [6] an error bound in $L^2(\Gamma)$ of $ch^{3/2-\epsilon}$ with some arbitrary $\epsilon > 0$ is established in convex domains for the concept of variational discretizations. This concept admit for linear problems in convex domains a convergence rate in $L^2(\Gamma)$ of 3/2 and an error bound of $ch^{2-2/p}|\log h|$ in $L^\infty(\Gamma)(\hookrightarrow L^2(\Gamma))$ with $2 < p < \min_j(2\omega_j/(2\omega_j - \pi))$, see [13]. An improved estimate for the concept of variational discretizations and the postprocessing approach can be found in [18]. There, the authors proved for convex domains an approximation rate of $2-1/p$ with the parameter p as before and an rate of $\min_j(1/2 + \pi/\omega_j)$ for nonconvex domains. They further establish better estimates using higher order finite elements for the discretization of the state and adjoint state (not contained in Figure 1). All the results, stated so far, have in common, that the convergence rates are lower than 3/2 in nonconvex domains. In [1] we have already proved that graded meshes with grading parameters $\mu_j < \pi/\omega_j$ can be used to maintain a convergence order of 3/2 in nonconvex domains. Using the optimal error estimates on the boundary of the first part and optimal regularity of the solution in weighted Sobolev spaces we will show in the present work, that the error is bounded by $ch^2|\ln h|^{3/2}$ using quasi-uniform meshes even if $\omega_j < 2\pi/3$. For larger interior angles we will prove that graded meshes with $\mu_j < 1/4 + \pi/2\omega_j$ imply the same convergence rate. For further literature concerning optimal control problems and its discretization we refer to [29, 14, 1, 18] and the references therein.

The paper is organized as follows: In the next section we introduce weighted spaces and some of its properties, which are used for the analysis in the subsequent sections. Furthermore, we prove weighted $W^{2,\infty}$-regularity of the solution of problem (1.1) based on regularity results in weighted Hölder spaces. Section 3 is concerned with the discretization of problem (1.1) by linear finite elements using quasi-uniform and graded meshes. This section also includes the optimal finite element error estimates in the domain and on the boundary. In Section 4 we discuss the necessary and sufficient optimality condition of problem (1.2). As a key point for the subsequent numerical analysis we also prove optimal regularity in weighted Sobolev spaces for the solution of the optimal control problem. The discrete counterpart of the continuous optimal control problem (1.2) and its necessary and sufficient optimality condition is stated in Section 5. In Section 6 we prove interpolation error estimates on the boundary. These are special compared to the interpolation error estimates of Section 3, since the techniques used in Section 3 would only yield suboptimal results. Finally, optimal error estimates for the postprocessing approach are contained in Section 7. We emphasize, that the optimal control and the desired state are separated from the constants in all estimates.

In the sequel c denotes a generic constant, which is always independent of the discretization parameter.

2 Regularity in weighted Sobolev spaces

The regularity of the solution of the elliptic boundary value problem (1.1) is in general limited due to the corners of the domain even if the data f and g are smooth. If one uses classical Sobolev-Slobodetskij spaces $W^{s,p}(\Omega)$ to describe the regularity, then this limitation is given by a dependence of the parameters s and p on the interior angles of the domain, compare e.g. [11, 8]. Another possibility to state the regularity uses weighted Sobolev spaces, which incorporates better the singular behavior coming from the corners. To this end we introduce the circular sectors Ω_{R_j} and $\Omega_{R_j/64}$ which are centered at the corners $x^{(j)}$ and possess the opening angles ω_j and the radii R_j and $R_j/64$, respectively. The radii R_j can be chosen arbitrarily with the only restriction that the circular sectors Ω_{R_j} do not overlap. We denote by r_j and φ_j the polar coordinates located at the point $x^{(j)}$. The sides of the circular sectors Ω_{R_j}, which coincide with the boundary Γ locally, are defined by Γ_j^+ $(\varphi_j = \omega_j)$ and Γ_j^- $(\varphi_j = 0)$. We set $\Gamma_j^\pm = \Gamma_j^+ \cup \Gamma_j^-$. In the following the closure of some set G will be denoted either by $\bar{G}$ or by $\mathrm{cl}(G)$. We set

$$\Omega^0 = \Omega \setminus \bigcup_{j=1}^{m} \Omega_{R_j/64} \quad \text{and} \quad \Gamma^0 = \Gamma \cap \bar{\Omega}^0.$$

We define for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\vec{\beta} = (\beta_1, \ldots, \beta_m)^T \in \mathbb{R}^m$ the weighted Sobolev spaces $W^{k,p}_{\vec{\beta}}(\Omega)$ as the set of all functions on Ω with finite norm

$$\|v\|_{W^{k,p}_{\vec{\beta}}(\Omega)} \sim \|v\|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} \|v\|_{W^{k,p}_{\beta_j}(\Omega_{R_j})},$$

where the Sobolev spaces $W^{k,p}(\Omega)$ $(= H^k(\Omega)$ for $p = 2)$ are defined as usual and the weighted part in the norm is defined by

$$\|v\|_{W^{k,p}_{\beta_j}(\Omega_{R_j})} = \left(\sum_{|\alpha| \le k} \|r_j^{\beta_j} D^\alpha v\|^p_{L^p(\Omega_{R_j})} \right)^{1/p} \qquad \text{if } 1 \le p < \infty,$$

$$\|v\|_{W^{k,\infty}_{\beta_j}(\Omega_{R_j})} = \sum_{|\alpha| \le k} \|r_j^{\beta_j} D^\alpha v\|_{L^\infty(\Omega_{R_j})}$$

using standard multi-index notation. We denote by $\mathcal{C}$ the set of all corner points. The corresponding trace spaces $W^{k-1/p,p}_{\vec{\beta}}(\Gamma)$ $(k \ge 1)$ are equipped with the norm

$$\|v\|_{W^{k-1/p,p}_{\vec{\beta}}(\Gamma)} = \inf \left\{ \|u\|_{W^{k,p}_{\vec{\beta}}(\Omega)} : u \in W^{k,p}_{\vec{\beta}}(\Omega), u_{|\Gamma \setminus \mathcal{C}} = v \right\}.$$

Furthermore we define the space $W^{k,p}_{\vec{\beta}}(\Gamma)$ for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\vec{\beta} = (\beta_1, \ldots, \beta_m)^T \in \mathbb{R}^m$ as the space of all functions such that

$$\|v\|_{W^{k,p}_{\vec{\beta}}(\Gamma)} \sim \|v\|_{W^{k,p}(\Gamma^0)} + \sum_{j=1}^{m} \|v\|_{W^{k,p}_{\beta_j}(\Gamma_j^\pm)},$$

is finite, where

$$\|v\|_{W^{k,p}_{\beta_j}(\Gamma_j^\pm)} = \left(\sum_{|\alpha|\le k} \left(\|r_j^{\beta_j} \partial_t^\alpha v\|^p_{L^p(\Gamma_j^+)} + \|r_j^{\beta_j} \partial_t^\alpha v\|^p_{L^p(\Gamma_j^-)} \right) \right)^{1/p} \qquad \text{if } 1 \le p < \infty,$$

$$\|v\|_{W^{k,\infty}_{\beta_j}(\Gamma_j^\pm)} = \sum_{|\alpha|\le k} \left(\|r_j^{\beta_j} \partial_t^\alpha v\|_{L^\infty(\Gamma_j^+)} + \|r_j^{\beta_j} \partial_t^\alpha v\|_{L^\infty(\Gamma_j^-)} \right).$$

Note, that $\partial_t v$ denotes the tangential derivative of v. For the numerical analysis it is useful to introduce the semi-norms

$$|\cdot|_{W^{k,p}_{\vec\beta}(\Omega)} \text{ and } |\cdot|_{W^{k,p}_{\vec\beta}(\Gamma)},$$

which are defined in analogy to the classical Sobolev semi-norms.

The next two lemmas contain selected properties of the introduced weighted Sobolev spaces.

Lemma 2.1. *Let $\mathcal{G}$ be a polygonal domain Ω with m boundary corner points or its boundary Γ. Furthermore, let n be the dimension of $\mathcal{G}$ and l, k nonnegative integers. Then the following three assertions hold:*

1. *Let $\beta'_j > -n/p$ and $\beta_j - \beta'_j \le k$ for $j = 1, \dots, m$ and $1 \le p < \infty$. Then the continuous embedding $W^{l+k,p}_{\vec\beta}(\mathcal{G}) \hookrightarrow W^{l,p}_{\vec\beta'}(\mathcal{G})$ holds.*

2. *Let $n/q - n/p > \beta_j - \beta'_j$ for $j = 1, \dots, m$ and $1 \le q < p \le \infty$. Then the continuous embedding $W^{l,p}_{\vec\beta}(\mathcal{G}) \hookrightarrow W^{l,q}_{\vec\beta'}(\mathcal{G})$ is valid.*

3. *Let $\beta'_j > -n/p$ and $\beta_j - \beta'_j < 1$ for $j = 1, \dots, m$ and $1 \le p < \infty$. Then the compact embedding $W^{l+1,p}_{\vec\beta}(\mathcal{G}) \overset{c}{\hookrightarrow} W^{l,p}_{\vec\beta'}(\mathcal{G})$ holds.*

Proof. 1. Let $\gamma_j := \beta'_j + k$ and $\vec\gamma = (\gamma_1, \dots, \gamma_m)$. By Hardy's inequality applied k-times and embeddings in usual Sobolev spaces one obtains for $\beta'_j > -n/p$ that

$$W^{l+k,p}_{\vec\gamma}(\mathcal{G}) \hookrightarrow W^{l,p}_{\vec\beta'}(\mathcal{G}),$$

cf. Lemma 7.1.5 in [16] for the two dimensional case with $p = 2$. Now, the first assertion follows immediately since

$$W^{l+k,p}_{\vec\beta}(\mathcal{G}) \hookrightarrow W^{l+k,p}_{\vec\gamma}(\mathcal{G})$$

for $\beta_j \le \gamma_j$ which is equivalent to $\beta_j - \beta'_j \le k$.
2. This is a consequence of the Hölder inequality.
3. For three space dimensions this is proven in Lemma 8.1.2 in [19]. In one and two space dimensions it can be proven analogously using the continuous embedding of 1. □

Lemma 2.2. *Let $\mathcal{G}$ be a polygonal domain Ω with m corner points or its boundary Γ. Further, let n be the dimension of $\mathcal{G}$, $q \in [1,\infty)$, $-n/q < \beta_j < n - n/q + 1$ for $j = 1,\dots,m$, $k \geq 0$ and $v \in W_{\beta}^{k+1,q}(\mathcal{G})$. Then the norm equivalence*

$$\|v\|_{W_{\vec{\beta}}^{k+1,q}(\mathcal{G})} \sim |v|_{W_{\vec{\beta}}^{k+1,q}(\mathcal{G})} + \sum_{|\delta|\leq k} \left| \int_{\mathcal{G}} D^{\delta} v \, dx \right| \tag{2.1}$$

is valid.

Proof. This assertion has already been proven in Lemma 2.2 of [4], where the authors assume that $1 - 2/q < \beta_j \leq 1$. Let $\vec{1} = (1,\dots,1) \in \mathbb{R}^m$. According to Lemma 2.1 one has

$$W_{\vec{\beta}}^{k+1,q}(\mathcal{G}) \hookrightarrow W_{\vec{1}}^{k+1,1}(\mathcal{G}) \hookrightarrow W^{k,1}(\mathcal{G}) \text{ and } W_{\vec{\beta}}^{k+1,q}(\mathcal{G}) \overset{c}{\hookrightarrow} W_{\vec{\beta}}^{k,q}(\mathcal{G}) \tag{2.2}$$

for $-n/q < \beta_j < n - n/q + 1$. These two embeddings are essential to prove the norm equivalence (2.1). In fact, tracing through the proof of Lemma 2.2 in [4] reveals that the condition $1 - 2/q < \beta_j \leq 1$ can simply be replaced by $-n/q < \beta_j < n - n/q + 1$ by means of (2.2). □

As usual, the space $C^k(\bar{\Omega})$ denotes the set of all functions on Ω with bounded and uniformly continuous derivatives up to order k. The norm in $C^k(\bar{\Omega})$ is defined by

$$\|v\|_{C^k(\bar{\Omega})} = \sum_{|\alpha|\leq k} \sup_{x\in\Omega} |D_x^{\alpha} v(x)|.$$

Functions belonging to the Hölder space $C^{k,\sigma}(\bar{\Omega})$ additionally possess bounded derivatives of order k which are Hölder continuous with exponent $\sigma \in (0,1)$. The norm in the Hölder space $C^{k,\sigma}(\bar{\Omega})$ is given by

$$\|v\|_{C^{k,\sigma}(\bar{\Omega})} = \|v\|_{C^k(\bar{\Omega})} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega} \frac{|D_x^{\alpha} v(x) - D_y^{\alpha} v(y)|}{|x-y|^{\sigma}}.$$

The regularity of the solution of problem (1.1) will be described in the introduced weighted Sobolev spaces. All the numerical analysis in the sequel uses only the aforementioned spaces either. However, for technical reasons, when analyzing and proving the regularity, it is useful to have a second type of weighted Sobolev space and appropriate weighted Hölder spaces available. If one is only interested in the regularity results itself, then one can directly jump to the statements of Lemma 2.4 and Lemma 2.6.

The spaces $V_{\vec{\beta}}^{k,p}(\Omega)$ and $V_{\vec{\beta}}^{k,p}(\Gamma)$ are defined analogously to the first kind of weighted Sobolev spaces, except that the weighting functions $r_j^{\beta_j}$ in the definition of the norms are substituted by $r_j^{\beta_j - k + |\alpha|}$. Note that the the classical Sobolev spaces $W^{k,p}(\Omega)$ are included in the weighted Sobolev spaces $W_{\vec{\beta}}^{k,p}(\Omega)$ by setting $\beta_j = 0$ for $j = 1,\dots,m$, whereas they do not belong automatically to the scale of the weighted spaces $V_{\vec{\beta}}^{k,p}(\Omega)$. But there is a relation between the space $W_{\vec{\beta}}^{k,p}(\Omega)$ and the space $V_{\vec{\beta}}^{k,p}(\Omega)$. For that reason, we recall Lemma 2.1 of [1], which represents this relation for $k = p = 2$. For general $k \in \mathbb{N}_0$ and $1 < p < \infty$ we refer to Theorem 2.1 of [20].

Lemma 2.3. *Let η_j, $j = 1, \ldots, m$, be infinitely differentiable cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/64}$ and $\operatorname{supp} \eta_j \subset \Omega_{R_j}$. For $\beta \in (0,1)^m$ one has*

$$W_\beta^{2,2}(\Omega) = V_\beta^{2,2}(\Omega) \oplus \eta_1 \mathcal{P}_0(\Omega) \oplus \cdots \oplus \eta_m \mathcal{P}_0(\Omega),$$

where $\mathcal{P}_0(\Omega)$ is the set of constant polynomials on Ω. In particular, for any $v \in W_\beta^{2,2}(\Omega)$ one can write $v = v_s + \sum_{j=1}^m \eta_j v(x^{(j)})$ with $v_s \in V_\beta^{2,2}(\Omega)$. Moreover, the norm equivalence

$$\|v\|_{W_\beta^{2,2}(\Omega)} \sim \|v_s\|_{V_\beta^{2,2}(\Omega)} + \sum_{j=1}^m |v(x^{(j)})|$$

is valid.

Finally, we introduce the weighted Hölder spaces $N_{\vec{\beta}}^{k,\sigma}(\Omega)$, where $k \in \mathbb{N}_0$, $\sigma \in (0,1)$ and $\vec{\beta} = (\beta_1, \ldots, \beta_m)^T \in \mathbb{R}^m$. These spaces consist of all k times continuously differentiable functions in $\bar{\Omega} \backslash \mathcal{C}$ such that

$$\|v\|_{N_{\vec{\beta}}^{k,\sigma}(\Omega)} \sim \|v\|_{C^{k,\sigma}(\bar{\Omega}^0)} + \sum_{j=1}^m \|v\|_{N_{\beta_j}^{k,\sigma}(\Omega_{R_j})} \tag{2.3}$$

is finite, where

$$\begin{aligned} \|v\|_{N_{\beta_j}^{k,\sigma}(\Omega_{R_j})} = & \sum_{|\alpha| \le k} \|r_j^{\beta_j - \sigma - k + |\alpha|} D^\alpha v\|_{C^0(\bar{\Omega}_{R_j})} \\ & + \sum_{|\alpha| = k} \sup_{x,y \in \Omega_{R_j}} \frac{\left| r_j(x)^{\beta_j} D_x^\alpha v(x) - r_j(y)^{\beta_j} D_y^\alpha v(y) \right|}{|x-y|^\sigma}. \end{aligned}$$

The trace spaces $N_{\vec{\beta}}^{k,\sigma}(\Gamma)$ are defined in the same manner.

Now we have everything at hand to prove the regularity results. However, for the sake of clarity let us first declare the concept of a generalized (or weak) solution and its specialties with corner domains. Let f and g be elements of the dual spaces of $H^1(\Omega)$ and $H^{1/2}(\Gamma)$, respectively. Then, the generalized solution of (1.1) is the unique element $y \in H^1(\Omega)$ that satisfies

$$a(y,v) = (f,v)_{L^2(\Omega)} + (g,v)_{L^2(\Gamma)} \quad \forall v \in H^1(\Omega), \tag{2.4}$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(y,v) = \int_\Omega (\nabla y \cdot \nabla v + yv) \, \mathrm{d}x. \tag{2.5}$$

If the boundary of the domain Ω is smooth enough, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, then one can show that the generalized solution is even an element of $H^2(\Omega)$. In polygonal domains this statement fails. In general the weak solution of (1.1) does not belong to $H^2(\Omega)$ if $\omega_j > \pi$ for some j. Instead, one can show that the solution has the asymptotics

$$y = \sum_{j=1}^{m} \eta_j c_{0,j} + \sum_{j:\omega_j>\pi} \eta_j c_{1,j} r_j^{\lambda_j} \cos(\lambda_j \varphi_j) + y_{reg},$$

where η_j denote the cut-off functions introduced in Lemma 2.3, $c_{0,j}$ and $c_{1,j}$ are some constants, $\lambda_j = \pi/\omega_j$ and the function y_{reg} belongs to $H^2(\Omega)$. For more general $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ with $\max(0, 1-\lambda_j) < \beta_j < 1$ the asymptotic representation

$$y = \sum_{j=1}^{m} \eta_j c_{0,j} + y_{sing},$$

holds, where y_{sing} belongs to $V^{2,2}_{\vec{\beta}}(\Omega)$ (cf. [23, Chapter 2]). Based on such representations regularity results in weighted Sobolev spaces can be proven.

Lemma 2.4. *Let $\lambda_j = \pi/\omega_j$ and let β_j satisfy the condition*

$$1 > \beta_j > \max(0, 1-\lambda_j) \quad \text{or} \quad \beta_j = 0 \text{ and } 1 - \lambda_j < 0$$

for $j = 1, \ldots, m$. Furthermore, let $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$. Then problem (1.1) *has a unique generalized solution $y \in W^{2,2}_{\vec{\beta}}(\Omega)$ and the a priori estimate*

$$\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le c \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right)$$

is valid.

Proof. We get from the Lax-Milgram Theorem the unique solvability of problem (1.1) in $H^1(\Omega)$ if $\beta_j < 1$. The unique solvability in $W^{2,2}_{\vec{\beta}}(\Omega)$ and the validity of the a priori estimate for $\max(0, 1-\lambda_j) < \beta_j < 1$ is then a consequence of Lemma 6.3.3 of [19] by using a partition of unity method (The aforementioned asymptotic representation is used in the proof of that lemma). In case that $\beta_j = 0$ and $1 - \lambda_j < 0$ we can deduce the unique solvability from Corollary 4.4.2.8 of [11]. The a priori estimate holds in that case according to Theorem 4.3.1.4 of [11] and the Lax-Milgram Theorem. □

Next, we would like to get regularity results in $W^{2,\infty}_{\vec{\beta}}(\Omega)$ and $W^{2,\infty}_{\vec{\beta}}(\Gamma)$. To our knowledge there is no reference where this is done directly. Instead, we use regularity results in weighted Hölder spaces for that purpose. The following Lemma represents parts of Theorem 1.4.5 of [17], which has been adapted to our setting (compare also [20]). Note, that asymptotic representations of the solution are again used in its proof.

Lemma 2.5. *Let $u \in V^{2,2}_{\vec{\beta}}(\Omega)$ with $\beta_j = 1/2$ for $j = 1, \ldots, m$ be a solution of*

$$\begin{aligned} -\Delta u + u &= F \quad \text{in } \Omega \\ \partial_n u &= G \quad \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned}$$

where $F \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ and $G \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$. If $0 < 2 + \sigma - \delta_j < \lambda$ for $j = 1, \ldots, m$, then u belongs to $N^{2,\sigma}_{\vec{\delta}}(\Omega)$ and the a priori estimate

$$\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} \le c\left(\|F\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|G\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|u\|_{L^1(\Omega)}\right) \tag{2.6}$$

is valid.

Lemma 2.6. *Let $\lambda_j = \pi/\omega_j$ and let γ_j satisfy the condition*

$$2 > \gamma_j > \max(0, 2 - \lambda_j) \quad \text{or} \quad \gamma_j = 0 \text{ and } 2 - \lambda_j < 0$$

for $j = 1, \ldots, m$. Moreover, let $f \in C^{0,\sigma}(\bar{\Omega})$ with $\sigma \in (0,1)$ and $g \equiv 0$. Then the unique generalized solution y of problem (1.1) fulfills the a priori estimate

$$\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} \le c\left(\|y\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^{m}\sum_{|\alpha|\le 2}\|r_j^{\gamma_j} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})}\right) \le c\|f\|_{C^{0,\sigma}(\bar{\Omega})}.$$

Proof. From Lemma 2.4 we know that the solution y of (1.1) belongs to $W^{2,2}_{\vec{\beta}}(\Omega)$ if $1 > \beta_j > \max(0, 1 - \lambda_j)$. In the following we choose $\beta_j = \frac{1}{2}$, which is possible since $\lambda_j > 1/2$ for every $\omega_j \in [0, 2\pi)$. Next, we would like to apply Lemma 2.5, but $y \notin V^{2,2}_{\vec{\beta}}(\Omega)$. Instead, we first use Lemma 2.3. This yields the splitting

$$u := y - \sum_{j=1}^{m} \eta_j y(x^{(j)}) \in V^{2,2}_{\vec{\beta}}(\Omega), \tag{2.7}$$

where η_j denote the cut-off functions introduced in Lemma 2.3. Furthermore, u solves

$$\begin{aligned} -\Delta u + u &= f - \sum_{j=1}^{m} \eta_j y(x^{(j)}) + \sum_{j=1}^{m} y(x^{(j)})\Delta\eta_j =: F \quad && \text{in } \Omega, \\ \partial_n u &= -\sum_{j=1}^{m} y(x^{(j)})\partial_n \eta_j =: G && \text{on } \Gamma_k,\ k = 1, \ldots, m. \end{aligned}$$

Let $\delta_j \ge \sigma$ be real numbers for $j = 1, \ldots, m$. Then, one can show for any function $w \in C^{0,\sigma}(\bar{\Omega})$ that

$$\begin{aligned} \|w\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} &\le \|w\|_{C^{0,\sigma}(\bar{\Omega}_0)} + c\sum_{j=1}^{m}\left(\|r_j^{\delta_j-\sigma} w\|_{C^0(\bar{\Omega}_{R_j})} + \sup_{\substack{x,y\in\Omega_{R_j}\\ |x-y|\le\frac{1}{2}r_j(x)}} r_j(x)^{\delta_j}\frac{|w(x)-w(y)|}{|x-y|^\sigma}\right) \\ &\le c\|w\|_{C^{0,\sigma}(\bar{\Omega})} \end{aligned} \tag{2.8}$$

(cf. Section 1.1 in [25] for technical details or Section 5 of [20]). Thus, the functions f, η_j and $\Delta\eta_j$ belong to $N^{0,\sigma}_{\vec{\delta}}(\Omega)$ and $\partial_n\eta_j$ to $N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ ($\Delta\eta_j$ and $\partial_n\eta_j$ even vanish in the neighborhood of every corner). Based on this we can conclude for $\delta_j - \sigma \ge 0$

$$F \in N^{0,\sigma}_{\vec{\delta}}(\Omega) \text{ and } G \in N^{1,\sigma}_{\vec{\delta}}(\Gamma).$$

Now we can apply Lemma 2.5. We obtain that u belongs to $N^{2,\sigma}_{\vec{\delta}}(\Omega)$ if $0 < 2+\sigma-\delta_j < \lambda_j$ for $j = 1, \ldots, m$. Furthermore, the a priori estimate

$$\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} \le c \left(\|F\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|G\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|u\|_{L^1(\Omega)} \right) \tag{2.9}$$

is valid. By setting $\gamma_j = \delta_j - \sigma \ge 0$ we obtain

$$\begin{aligned}
\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} &\le c \left(\|y\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})} \right) \\
&\le c \Bigg(\|u\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha u\|_{C^0(\bar{\Omega}_{R_j})} \\
&\quad + \sum_{j=1}^{m} \left| y(x^{(j)}) \right| \left[\|\eta_j\|_{C^2(\bar{\Omega}^0)} + \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha \eta_j\|_{C^0(\bar{\Omega}_{R_j})} \right] \Bigg) \\
&\le c \left(\|u\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha u\|_{C^0(\bar{\Omega}_{R_j})} + \sum_{j=1}^{m} \left| y(x^{(j)}) \right| \right)
\end{aligned}$$

where we inserted (2.7) and used that $r_j^{\gamma_j}$ (for $\gamma_j \ge 0$) and $|D^\alpha \eta_j|$ is bounded by a constant. Since $\gamma_j = \delta_j - \sigma$ and $2 - |\alpha| \ge 0$ for $|\alpha| \le 2$ we can conclude

$$\begin{aligned}
&\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} \\
&\qquad \le c \left(\|u\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\le 2} \|r_j^{\delta_j-\sigma-2+|\alpha|} D^\alpha u\|_{C^0(\bar{\Omega}_{R_j})} + \sum_{j=1}^{m} \left| y(x^{(j)}) \right| \right) \\
&\qquad \le c \left(\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} + \sum_{j=1}^{m} \left| y(x^{(j)}) \right| \right).
\end{aligned} \tag{2.10}$$

Next, we apply the a priori estimate (2.9), which holds for $\gamma_j > 2 - \lambda_j$, and insert the definitions of F and G and (2.7). This yields

$$\begin{aligned}
\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} &\le c \Bigg(\|f + \sum_{j=1}^{m} y(x^{(j)}) \left(\Delta\eta_j - \eta_j \right) \|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \| \sum_{j=1}^{m} y(x^{(j)}) \partial_n \eta_j \|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \\
&\quad + \|y - \sum_{j=1}^{m} \eta_j y(x^{(j)})\|_{L^1(\Omega)} \Bigg) \\
&\le c \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|y\|_{L^1(\Omega)} + \sum_{j=1}^{m} \left| y\left(x^{(j)}\right) \right| \right).
\end{aligned} \tag{2.11}$$

Again we used that $|D^\alpha \eta_j|$ is bounded by a constant. The last two terms in (2.11) can be estimated by applying the Sobolev embedding theorem, Lemma 2.1 and the a priori estimate of Lemma 2.4. By this we obtain with some $\epsilon \in (1, 4/3)$

$$\begin{aligned}
\|y\|_{L^1(\Omega)} + \sum_{j=1}^{m} \left| y\left(x^{(j)}\right)\right| &\le c\|y\|_{C^0(\bar\Omega)} \le c\|y\|_{W^{2,1+\epsilon}(\bar\Omega)} \le c\|y\|_{W^{2,2}_{\vec\beta}(\bar\Omega)} \\
&\le c\|f\|_{W^{0,2}_{\vec\beta}(\bar\Omega)} \le c\|f\|_{L^2(\Omega)} \le c\|f\|_{C^{0,\sigma}(\bar\Omega)}. \qquad (2.12)
\end{aligned}$$

Finally, the inequalities (2.8), (2.10), (2.11) and (2.12) yield the a priori estimate of the assertion. □

3 Finite element error estimates on the boundary

We will now discretize the boundary value problem (1.1) by a finite element method. For this purpose we introduce a family of graded triangulations $\{\mathcal{T}_h\}$ of the domain Ω where h denotes the global mesh parameter (cf. Definition 4.4.13 in [5]). We assume $h \le h_0 < 1$. Note, that a segmentation $\mathcal{E}_h$ of the boundary is naturally introduced by the triangulation $\mathcal{T}_h$. We denote by $\mu_j \in (0,1]$, $j = 1, \dots, m$, the mesh grading parameters which are collected in the vector $\vec\mu$. The distance of the triangle T to the corner $x^{(j)}$ is defined by $r_{T,j} := \inf_{(x_1,x_2)\in T} |x - x^{(j)}|$. We assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$\begin{array}{lll}
c_1 h^{1/\mu_j} \le h_T \le c_2 h^{1/\mu_j} & \text{for } r_{T,j} = 0, \\
c_1 h r_{T,j}^{1-\mu_j} \le h_T \le c_2 h r_{T,j}^{1-\mu_j} & \text{for } 0 < r_{T,j} \le R_j, \\
c_1 h \le h_T \le c_2 h & \text{for } r_{T,j} > R_j
\end{array} \qquad (3.1)$$

for $j = 1, \dots, m$ and the radii R_j which we have defined in Section 2. Next, we introduce the space V_h as the space of all piecewise linear and globally continuous functions in $\bar\Omega$,

$$V_h(\Omega) := \{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h\},$$

where $\mathcal{P}_1(T)$ denotes the space of polynomials of degree less than or equal to one on T.

The discrete solution is the unique element $y_h \in V_h(\Omega) \subset H^1(\Omega)$ that satisfies

$$a(y_h, v_h) = (f, v_h)_{L^2(\Omega)} + (g, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h(\Omega). \qquad (3.2)$$

We set $\vec\lambda = (\lambda_1, \dots, \lambda_m)^T = (\pi/\omega_1, \dots, \pi/\omega_m)^T$ and $\vec a = (a, \dots, a)^T \in \mathbb{R}^m$ for any real number a, e.g. $\vec 1 = (1, \dots, 1)^T \in \mathbb{R}^m$. Furthermore, all inequalities in the sequel containing vectorial parameters must be understood component-by-component. We proved in [1] the following estimates for the discretization error in the domain.

Lemma 3.1. *Let y and y_h be the solutions of* (2.4) *and* (3.2)*, respectively. The discretization error can be estimated by*

$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{W^{1,2}(\Omega)} \le ch^2\|y\|_{W^{2,2}_{\vec\beta}(\Omega)} \le ch^2\left(\|f\|_{W^{0,2}_{\vec\beta}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\beta}(\Gamma)}\right),$$

provided that $\vec 1 - \vec\lambda < \vec\beta \le \vec 1 - \vec\mu$, $\vec\beta \ge \vec 0$, $f \in W^{0,2}_{\vec\beta}(\Omega)$ and $g \in W^{1/2,2}_{\vec\beta}(\Gamma)$.

We will need this lemma to prove the following estimate on the boundary.

Theorem 3.2. *Let y and y_h be the solution of (2.4) and (3.2), respectively, and $g \equiv 0$. The finite element error on the boundary admits for some arbitrary $\delta \in [0, 1/2]$ the estimate*

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{1+\delta} \|f\|_{C^{0,\sigma}(\bar{\Omega})}$$

provided that $\vec{\mu} < \vec{\delta}/2 + \vec{\lambda}/2$, $\vec{\mu} \in (\delta/2, 1]^m$ and $f \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0, 1)$.

Remark 3.3. *To get optimal discretization error estimates in the domain one only needs a graded mesh with grading parameters $\vec{\mu} < \vec{\lambda}$ if the largest interior angle in the domain is larger than π. However, the stronger condition $\vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$ is required to guarantee a finite element error estimate on the boundary of order $O(h^2 |\ln h|^{3/2})$. Numerical examples also indicate that this condition is sharp, cf. [1]. Mesh grading with the stronger refinement condition is indeed necessary in domains which have interior angles greater than $2\pi/3$.*

Remark 3.4. *In Theorem 3.2 a homogeneous boundary datum g is assumed. However, the assertion can easily be generalized to the inhomogeneous case. Then one only needs regularity results as in Lemma 2.6 which incorporate an inhomogeneous boundary datum g. The following proof remains unchanged.*

Remark 3.5. *Optimal finite element error estimates in the L^2-norm on a strip at the boundary with width h are closely related to the error estimate of Theorem 3.2. In [21] the authors prove an optimal estimate on a strip for the Dirichlet problem in convex polygonal and polyhedral domains using quasi-uniform meshes. Whereas the general approach in [21] as well as in the present work relies on local finite element error estimates as described in [30, 9], the regularity theory used for the numerical analysis differs fundamentally. In [21] weighted and anisotropic spaces are used, which employ the distance to the boundary. In contrast, our analysis is based on weighted spaces with respect to the corners, which allow the usage of graded meshes.*

The remainder of this section is devoted to the proof of Theorem 3.2. In Section 2 we have already introduced circular sectors Ω_{R_j} and $\Omega_{R_j/64}$ which are centered at the corners $x^{(j)}$ and possess the radii R_j and $R_j/64$, compare also the mesh condition (3.1). Next, we introduce the circular sectors $\Omega_{R_j/2}$, $\Omega_{R_j/4}$, $\Omega_{R_j/8}$, $\Omega_{R_j/16}$ and $\Omega_{R_j/32}$ with the radii $R_j/2$, $R_j/4$, $R_j/8$, $R_j/16$ and $R_j/32$, respectively. The sides of the circular sectors $\Omega_{R_j/16}$ which coincide with the boundary Γ are denoted by $\Gamma^+_{R_j/16}$ ($\varphi_j = \omega_j$) and $\Gamma^-_{R_j/16}$ ($\varphi_j = 0$). The union of both is denoted by $\Gamma^\pm_{R_j/16}$. Furthermore we define $\tilde{\Omega}^0 = \Omega \backslash \bigcup_{j=1}^m \Omega_{R_j/16}$, $\tilde{\Gamma}^0 = \Gamma \cap \mathrm{cl}(\tilde{\Omega}^0)$ and $\check{\Omega}^0 = \Omega \backslash \bigcup_{j=1}^m \Omega_{R_j/32}$. Figure 2 illustrates exemplarily such a partition of the domain. The dashed and the dotted lines indicate (not to scale) the domains $\Omega_{R_j/8}$ and Ω_{R_j}, respectively.

Now we will proceed for every corner in the same way. Let the corner $x^{(j_0)}$ be the corner under consideration. We assume for the sake of simplicity but without loss of generality that the corner $x^{(j_0)}$ is located at the origin and $R_{j_0} = 1$. Furthermore we

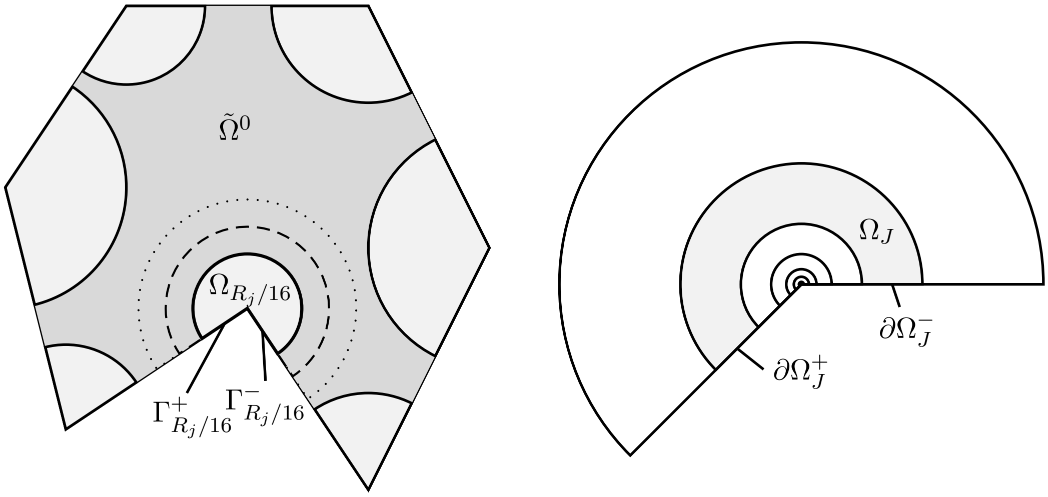

Figure 2: Partition of Ω with subdomains $\Omega_{R_j/16}$ (left) and partition of Ω_R with subdomains Ω_J (right)

suppress the subscript j_0 in the following, i.e. $\Omega_R = \Omega_{R_{j_0}}$, $\Omega_{R/2} = \Omega_{R_{j_0}/2}$, etc. We divide the domain Ω_R into subsets Ω_J,

$$\Omega_R = \bigcup_{J=0}^{I} \Omega_J$$

where $\Omega_J = \{x : d_{J+1} \leq |x| \leq d_J\}$ for $J = 0, \ldots, I-1$ and $\Omega_I = \{x : |x| \leq d_I\}$. The radii d_J are set to 2^{-J} and the index I is chosen such that

$$2^{-(I+k+1)} \leq c_2 h^{1/\mu} \leq 2^{-(I+k)}$$

for some fixed $k \in \mathbb{N}_0$ and c_2 from (3.1). Thus, $I \sim \log \frac{1}{h}$ for some $h \leq h_0 < 1$. Obviously, there exists some constant $c_I \in \mathbb{R}$ with

$$c_2 2^k \leq c_I \leq c_2 2^{k+1} \tag{3.3}$$

such that

$$d_I = 2^{-I} = c_I h^{1/\mu}. \tag{3.4}$$

For the moment we only assume that the parameter k is chosen such that $c_I \geq 1$. It will be exactly specified in the proof of Lemma 3.10. The boundary parts of Ω_J which coincide with the boundary of Ω_R are denoted by $\partial\Omega_J^+$ for $\varphi = \omega$ and by $\partial\Omega_J^-$ for $\varphi = 0$. We set $\partial\Omega_J^\pm = \partial\Omega_J^+ \cup \partial\Omega_J^-$. Figure 2 shows such an division. Note, that

$$\Omega_{R/2} = \bigcup_{J=1}^{I} \Omega_J,\ \Omega_{R/4} = \bigcup_{J=2}^{I} \Omega_J,\ \Omega_{R/8} = \bigcup_{J=3}^{I} \Omega_J,\ \text{etc.}$$

Next we introduce the extended subsets Ω'_J for $J \geq 1$ and Ω''_J for $J \geq 2$ by

$$\Omega'_J = \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1}$$

and

$$\Omega''_J = \Omega'_{J-1} \cup \Omega'_J \cup \Omega'_{J+1}$$

with the obvious modifications for $J = I-1, I$. The boundary parts $\partial\Omega^{\pm}_J{}'$ are analogously defined with respect to Ω'_J.

Before going into detail let us elucidate the structure of our proof. As we will see on page 25, $L^2(\Gamma^{\pm}_{R/16})$-discretization error estimates are crucial ingredients of the proof of Theorem 3.2. These are established in Lemma 3.12. The proof requires $L^\infty(\Omega'_J)$-interpolation error estimates, see Lemma 3.7 and Remark 3.8, the weighted finite element error estimate of Lemma 3.10, and some kind of an inverse inequality provided in Lemma 3.11. The proof of Lemma 3.10 relies on a kick back argument, which is established by the special partition of the domain Ω_R, the $H^1(\Omega_J)$-interpolation error estimates of Lemma 3.7, and local $H^1(\Omega_J)$-finite element error estimates provided by Lemma 3.9. Lemma 3.7 and Remark 3.8 are also used in the proof of Lemma 3.9. All these arguments rely on the property that the mesh is quasi-uniform in the strips Ω'_J which we are going to prove first.

Lemma 3.6. *The element size h_T of the elements $T \subset \Omega'_J$ satisfies*

$$2^{-2(1-\mu)} c_1 h d_J^{1-\mu} \le h_T \le 2^{1-\mu} c_2 h d_J^{1-\mu}, \qquad \text{if } 1 \le J \le I-2, \quad (3.5)$$

$$c_1 h^{1/\mu} \le h_T \le 2^{2(1-\mu)} c_2 h d_I^{1-\mu} = 2^{2(1-\mu)} c_2 c_I^{1-\mu} h^{1/\mu}, \quad \text{if } J = I, I-1 \quad (3.6)$$

with constants c_1 and c_2 from (3.1) *and c_I from* (3.4).

Proof. For any element $T \subset \Omega'_J$ and $J \le I-2$ one has $d_{J+2} < r_T < d_{J-1}$. Thus, assertion (3.5) follows immediately with $d_{J+2} = 2^{-2} d_J$, $d_{J-1} = 2d_J$ and the mesh condition (3.1). Assertion (3.6) holds analogously since for any element $T \subset \Omega_J$, $J = I, I-1$, one has $0 \le r_T \le d_{I+2} = 2^2 d_I = 2^2 c_I h^{1/\mu}$. □

As indicated above we will use a kick back argument in the proof of Lemma 3.10. This depends on the size of the constant c_I. For that purpose we distinguish between the generic constant c and the constant c_I in the following two lemmas.

Lemma 3.7. *Let $\mu \in (0,1]$, $v_1 \in W^{2,p}_\alpha(\Omega'_J)$, $v_2 \in W^{2,\infty}_\alpha(\Omega'_J)$ and $l = 0, 1$.*

1. *For $1 \le J \le I-2$ the estimates*

$$\|v_1 - I_h v_1\|_{W^{l,2}(\Omega_J)} \le c h^{2-l} d_J^{(2-l)(1-\mu)+1-2/p-\alpha} |v_1|_{W^{2,p}_\alpha(\Omega'_J)}, \quad (3.7)$$

$$\|v_2 - I_h v_2\|_{L^\infty(\Omega_J)} \le c h^2 d_J^{2-2\mu-\alpha} |v_2|_{W^{2,\infty}_\alpha(\Omega'_J)} \quad (3.8)$$

   are valid for $2 \le p \le \infty$ and $\alpha \in \mathbb{R}$.

2. *Let $1 < q \le \infty$, $\theta_l := \max\{0, (3-l-2/p)(1-\mu)-\alpha)\}$ and $\theta_\infty := \max\{0, 2-2\mu-\alpha\}$. Then for $J = I, I-1$ the inequalities*

$$\|v_1 - I_h v_1\|_{W^{l,2}(\Omega_J)} \le c c_I^{\theta_l+1-2/p} h^{(3-l-\alpha-2/p)/\mu} |v_1|_{W^{2,p}_\alpha(\Omega'_J)}, \quad (3.9)$$

$$\|v_2 - I_h v_2\|_{L^\infty(\Omega_J)} \le c c_I^{\theta_\infty} h^{(2-\alpha)/\mu} |v_2|_{W^{2,\infty}_\alpha(\Omega'_J)} \quad (3.10)$$

   hold if $\max(2, q) \le p \le \infty$ and $-2/q < \alpha < 2 - 2/q$.

Proof. We begin with estimating $\|v_1 - I_h v_1\|_{W^{l,2}(\Omega_J)}$, $J = 0, \dots, I$, and $l = 0, 1$. Let $T \in \mathcal{T}_h$ be a triangle with $T \cap \Omega_J \neq \emptyset$. We first introduce the polynomial $p_1 \in \mathcal{P}_1(T)$ and use the triangle inequality. This yields

$$\begin{aligned}\|v_1 - I_h v_1\|_{W^{l,2}(T)} &= \|v_1 - p_1 - I_h(v_1 - p_1)\|_{W^{l,2}(T)} \\ &\leq \|v_1 - p_1\|_{W^{l,2}(T)} + \|I_h(v_1 - p_1)\|_{W^{l,2}(T)}. \end{aligned} \tag{3.11}$$

We can conclude for the first term in (3.11) after the transformation to the reference element $\hat{T}$ and using the embedding $W^{2,q'}(\hat{T}) \hookrightarrow W^{l,2}(\hat{T})$ which holds for $q' \geq 1$,

$$\|v_1 - p_1\|_{W^{l,2}(T)} \leq c|T|^{1/2} h_T^{-l} \|\hat{v}_1 - \hat{p}_1\|_{W^{l,2}(\hat{T})} \leq c|T|^{1/2} h_T^{-l} \|\hat{v}_1 - \hat{p}_1\|_{W^{2,q'}(\hat{T})}. \tag{3.12}$$

To estimate the second term in (3.11) we use an inverse inequality for functions in finite dimensional spaces (only for $l = 1$) and the embedding $L^\infty(\hat{T}) \hookrightarrow L^2(\hat{T})$ to get

$$\begin{aligned}\|I_h(v_1 - p_1)\|_{W^{l,2}(T)} &\leq c|T|^{1/2} h_T^{-l} \|\hat{I}_h(\hat{v}_1 - \hat{p}_1)\|_{W^{l,2}(\hat{T})} \leq c|T|^{1/2} h_T^{-l} \|\hat{I}_h(\hat{v}_1 - \hat{p}_1)\|_{L^2(\hat{T})} \\ &\leq c|T|^{1/2} h_T^{-l} \|\hat{I}_h(\hat{v}_1 - \hat{p}_1)\|_{L^\infty(\hat{T})} \leq c|T|^{1/2} h_T^{-l} \|\hat{v}_1 - \hat{p}_1\|_{L^\infty(\hat{T})} \\ &\leq c|T|^{1/2} h_T^{-l} \|\hat{v}_1 - \hat{p}_1\|_{W^{2,q'}(\hat{T})}. \end{aligned} \tag{3.13}$$

The last steps hold due to the boundedness of the interpolation operator I_h from $L^\infty(\hat{T})$ to $L^\infty(\hat{T})$ and the embedding $W^{2,q'}(\hat{T}) \hookrightarrow L^\infty(\hat{T})$, which is valid for $q' > 1$. The inequalities (3.12) and (3.13) yield together with (3.11) and the Deny-Lions Lemma [10]

$$\|v_1 - I_h v_1\|_{W^{l,2}(T)} \leq c|T|^{1/2} h_T^{-l} \|\hat{v}_1 - \hat{p}_1\|_{W^{2,q'}(\hat{T})} \leq c|T|^{1/2} h_T^{-l} |\hat{v}_1|_{W^{2,q'}(\hat{T})}, \tag{3.14}$$

which holds for $q' > 1$. Now we distinguish triangles T with $r_T = 0$ and $r_T > 0$. For triangles with $r_T = 0$ we choose $q' = 4q/(2q + 2 + \alpha q) > 1$ with some $q > 1$ and $-2/q < \alpha < 2 - 2/q$. Using the embedding $W^{2,q}_\alpha(\hat{T}) \hookrightarrow W^{2,q'}(\hat{T})$ according to Lemma 2.1 we obtain

$$\|v_1 - I_h v_1\|_{W^{l,2}(T)} \leq c|T|^{1/2} h_T^{-l} |\hat{v}_1|_{W^{2,q}_\alpha(\hat{T})}.$$

The transformation back to the world element yields

$$\|v_1 - I_h v_1\|_{W^{l,2}(T)} \leq c h_T^{3-l-\alpha-2/q} |v_1|_{W^{2,q}_\alpha(T)} \leq c h^{(3-l-\alpha-2/q)/\mu} |v_1|_{W^{2,q}_\alpha(T)}, \tag{3.15}$$

since $\hat{r}^\alpha \sim h_T^{-\alpha} r^\alpha$, $|T| \sim h_T^2$ and $h_T \sim h^{1/\mu}$ (cf. mesh condition (3.1)). In case that $r_T > 0$ we easily get from (3.14) for any $q' = q > 1$ and (3.1)

$$\|v_1 - I_h v_1\|_{W^{l,2}(T)} \leq c h_T^{3-l-2/q} |v_1|_{W^{2,q}(T)} \leq c h^{3-l-2/q} r_T^{(3-l-2/q)(1-\mu)-\alpha} |v_1|_{W^{2,q}_\alpha(T)}. \tag{3.16}$$

So far we have derived estimates for $\|v_1 - I_h v_1\|_{W^{l,2}(T)}$. Next, we consider the domains Ω_J. We distinguish between $1 \leq J \leq I - 2$ and $J = I - 1, I$. In the former case we get from (3.16)

$$\begin{aligned}\|v_1 - I_h v_1\|_{W^{l,2}(\Omega_J)} &\le \left(\sum_{T\subset\Omega'_J} \|v_1 - I_h v_1\|^2_{W^{l,2}(T)} \right)^{1/2} \\ &\le c \left(\sum_{T\subset\Omega'_J} \left(h^{2-l} r_T^{(2-l)(1-\mu)-\alpha} |v_1|_{W^{2,2}_\alpha(T)} \right)^2 \right)^{1/2} \\ &\le c h^{2-l} d_J^{(2-l)(1-\mu)-\alpha} |v_1|_{W^{2,2}_\alpha(\Omega'_J)} \\ &\le c h^{2-l} d_J^{(2-l)(1-\mu)+1-2/p-\alpha} |v_1|_{W^{2,p}_\alpha(\Omega'_J)},\end{aligned}$$

where we used $r_T \sim d_J$ (cf. Lemma 3.6), the Hölder inequality with some $p \ge 2$ and $|\Omega'_J| \sim d_J^2$ in the last steps. In case that $J = I-1, I$ we can conclude using (3.15) and (3.16) with some $q > 1$, $p \ge \max(2,q)$ and $-2/q < \alpha < 2 - 2/q$

$$\begin{aligned}&\|v_1 - I_h v_1\|_{W^{l,2}(\Omega_J)} \le \left(\sum_{\substack{T\subset\Omega'_J \\ r_T=0}} \|v_1 - I_h v_1\|^2_{W^{l,2}(T)} + \sum_{\substack{T\subset\Omega'_J \\ r_T>0}} \|v_1 - I_h v_1\|^2_{W^{l,2}(T)} \right)^{1/2} \\ &\le c \left(\sum_{\substack{T\subset\Omega'_J \\ r_T=0}} \left(h^{(3-l-\alpha-2/p)/\mu} |v_1|_{W^{2,p}_\alpha(T)} \right)^2 + \sum_{\substack{T\subset\Omega'_J \\ r_T>0}} \left(h^{3-l-2/p} r_T^{(3-l-2/p)(1-\mu)-\alpha} |v_1|_{W^{2,p}_\alpha(T)} \right)^2 \right)^{1/2} \\ &\le c \left(\sum_{\substack{T\subset\Omega'_J \\ r_T=0}} \left(h^{(3-l-\alpha-2/p)/\mu} |v_1|_{W^{2,p}_\alpha(T)} \right)^2 + \sum_{\substack{T\subset\Omega'_J \\ r_T>0}} \left(c_I^{\theta_l} h^{(3-l-\alpha-2/p)/\mu} |v_1|_{W^{2,p}_\alpha(T)} \right)^2 \right)^{1/2} \\ &\le c c_I^{\theta_l} h^{(3-l-\alpha-2/p)/\mu} \left(\sum_{T\subset\Omega'_J} |v_1|^2_{W^{2,p}_\alpha(T)} \right)^{1/2} \\ &\le c c_I^{\theta_l} h^{(3-l-\alpha-2/p)/\mu} |v_1|_{W^{2,p}_\alpha(\Omega'_J)} \left(\sum_{T\subset\Omega'_J} 1 \right)^{1/2-1/p}, \qquad (3.17)\end{aligned}$$

where we used $h^{1/\mu} \le r_T \le c d_I = c c_I h^{1/\mu}$, if $r_T > 0$, and the discrete Hölder inequality. Since $|\Omega'_J| \sim d_I^2$, $d_I = c_I h^{1/\mu}$ and $\min_{T\subset\Omega'_J} h_T \sim h^{1/\mu}$ for $J = I, I-1$ we get that

$$\left(\sum_{T\subset\Omega'_J} 1 \right)^{1/2-1/p} \le \left(\frac{|\Omega'_J|}{\min_{T\subset\Omega'_J} h_T^2} \right)^{1/2-1/p} \le c \left(c_I^2 \right)^{1/2-1/p} = c c_I^{1-2/p}.$$

Thus, we obtain for $q > 1$, $p \ge \max(2,q)$ and $-2/q < \alpha < 2 - 2/q$

$$\|v_1 - I_h v_1\|_{W^{l,2}(\Omega_J)} \le c c_I^{\theta_l+1-2/p} h^{(3-l-\alpha-2/p)/\mu} |v_1|_{W^{2,p}_\alpha(\Omega'_J)}.$$

It remains to prove the L^∞-error estimates. Let T be a triangle with $T \cap \Omega_J \neq \emptyset$. As in (3.11) we first insert an arbitrary polynomial $p_1 \in \mathcal{P}_1(T)$. This yields for some $q' > 1$

$$\begin{aligned}\|v_2 - I_h v_2\|_{L^\infty(T)} &\le \|v_2 - p_1\|_{L^\infty(T)} + \|I_h(v_2 - p_1)\|_{L^\infty(T)}\\ &\le c\|\hat{v}_2 - \hat{p}_1\|_{L^\infty(\hat{T})} \le c|\hat{v}_2|_{W^{2,q'}_\alpha(\hat{T})},\end{aligned}$$

cf. (3.13) and (3.14). If $r_T = 0$ we get as in (3.15) for $q > 1$ and $-2/q < \alpha < 2 - 2/q$

$$\|v_2 - I_h v_2\|_{L^\infty(T)} \le c h_T^{2-\alpha-2/q} |v_2|_{W^{2,q}_\alpha(T)} \le c h^{(2-\alpha)/\mu} |v_2|_{W^{2,\infty}_\alpha(T)} \tag{3.18}$$

In case that $r_T > 0$ we can conclude as in (3.16)

$$\|v_2 - I_h v_2\|_{L^\infty(T)} \le c h_T^2 |v_2|_{W^{2,\infty}(T)} \le c h^2 r_T^{2-2\mu-\alpha} |v_2|_{W^{2,\infty}_\alpha(T)}. \tag{3.19}$$

Now we suppose that $v_2 - I_h v_2$ admits its maximum in Ω_J at some point $x_0 \in \bar{T}_* \subset \Omega_J'$. If $1 \le J \le I-2$ we obtain using (3.19) and $r_T \sim d_J$

$$\begin{aligned}\|v_2 - I_h v_2\|_{L^\infty(\Omega_J)} = \|v_2 - I_h v_2\|_{L^\infty(T_*)} &\le c h^2 d_J^{2-2\mu-\alpha} |v_2|_{W^{2,\infty}_\alpha(T_*)}\\ &\le c h^2 d_J^{2-2\mu-\alpha} |v_2|_{W^{2,\infty}_\alpha(\Omega_J')}.\end{aligned}$$

In case that $J = I-1, I$ we get for $r_{T_*} = 0$ according to (3.18)

$$\|v_2 - I_h v_2\|_{L^\infty(\Omega_J)} \le c h_{T_*}^{2-\alpha-2/q} |v_2|_{W^{2,q}_\alpha(T_*)} \le c h_{T_*}^{2-\alpha} |v_2|_{W^{2,\infty}_\alpha(T_*)} \le c h_{T_*}^{2-\alpha} |v_2|_{W^{2,\infty}_\alpha(\Omega_J')},$$

which holds for some $q > 1$ and $-2/q < \alpha < 2 - 2/q$. Since $h_{T_*} \sim h^{1/\mu}$ we can continue with

$$\|v_2 - I_h v_2\|_{L^\infty(\Omega_J)} \le c h^{(2-\alpha)/\mu} |v_2|_{W^{2,\infty}_\alpha(\Omega_J')}.$$

In case that $r_{T_*} > 0$ we can conclude analogously to (3.17) using (3.19)

$$\begin{aligned}\|v_2 - I_h v_2\|_{L^\infty(\Omega_J)} &= \|v_2 - I_h v_2\|_{L^\infty(T_*)} \le c h_{T_*}^2 |v_2|_{W^{2,\infty}(T_*)}\\ &\le c h^2 r_{T_*}^{2-2\mu-\alpha} |v_2|_{W^{2,\infty}_\alpha(T_*)} \le c c_I^{\theta_\infty} h^2 h^{(2-2\mu-\alpha)/\mu} |v_2|_{W^{2,\infty}_\alpha(\Omega_J')}\\ &= c c_I^{\theta_\infty} h^{(2-\alpha)/\mu} |v_2|_{W^{2,\infty}_\alpha(\Omega_J')}.\end{aligned}$$

□

Remark 3.8. *The inequalities* (3.7)–(3.10) *hold as well if we replace Ω_J with Ω_J' and Ω_J' with Ω_J'', respectively. In that case we have to distinguish $2 \le J \le I-3$ and $J = I-2, I-1, I$.*

Lemma 3.9. *Let $\mu \in (0,1]$ and $y \in W^{2,\infty}_\alpha(\Omega_R)$.*

1. *For $2 \le J \le I-3$ the estimate*

$$\|y - y_h\|_{H^1(\Omega_J)} \le c\left(h d_J^{2-\mu-\alpha} |y|_{W^{2,\infty}_\alpha(\Omega_J'')} + d_J^{-1} \|y - y_h\|_{L^2(\Omega_J')}\right)$$

   is valid for $\alpha \in \mathbb{R}$.

2. Let $1 < q \le \infty$. Then for $J \ge I-2$ the inequality

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(c_I^5 h^{(2-\alpha)/\mu}|y|_{W^{2,\infty}_\alpha(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)$$

holds true for $-2/q < \alpha < 2-2/q$.

Proof. The proof relies on local finite element error estimates stated in [9] and on the interpolation error estimates given in Lemma 3.7. For $J = 0,\dots,I$ we get from Theorem 3.4 of [9]

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(\|y-I_hy\|_{H^1(\Omega'_J)} + d_J^{-1}\|y-I_hy\|_{L^2(\Omega'_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right),$$

where the constant c does not depend on c_I. In case that $2 \le J \le I-3$ one gets with Lemma 3.7 and Remark 3.8

$$\begin{aligned}\|y-y_h\|_{H^1(\Omega_J)} \le c\Big(& h d_J^{2-\mu-\alpha}|y|_{W^{2,\infty}_\alpha(\Omega''_J)} + h^2 d_J^{2-2\mu-\alpha}|y|_{W^{2,\infty}_\alpha(\Omega''_J)}\\ &+ d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big).\end{aligned}$$

Since $hd_J^{-\mu} \le hd_I^{-\mu} = c_I^{-\mu} \le 1$ we arrive at

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(hd_J^{2-\mu-\alpha}|y|_{W^{2,\infty}_\alpha(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right).$$

This is the first inequality of the assertion. For $J \ge I-2$ we proceed in an analogous way. But now we use the interpolation error estimates from Lemma 3.7, having regard to Remark 3.8, which are stated there for domains close to or at the corner. Let $\theta_l := \max\{0,(3-l)(1-\mu)-\alpha\}$ for $l = 0,1$. By this we obtain

$$\begin{aligned}\|y-y_h\|_{H^1(\Omega_J)} &\le c\Big(c_I^{\theta_1+1}h^{(2-\alpha)/\mu}|y|_{W^{2,\infty}_\alpha(\Omega''_J)} + c_I^{\theta_0+1}d_J^{-1}h^{(3-\alpha)/\mu}|y|_{W^{2,\infty}_\alpha(\Omega''_J)}\\ &\qquad + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big)\\ &\le c\left((c_I^{\theta_1+1}+c_I^{\theta_0})h^{(2-\alpha)/\mu}|y|_{W^{2,\infty}_\alpha(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)\\ &\le c\left(c_I^5 h^{(2-\alpha)/\mu}|y|_{W^{2,\infty}_\alpha(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right),\end{aligned}$$

where we used $d_J^{-1}h^{1/\mu} \le d_I^{-1}h^{1/\mu} = c_I^{-1}$, $\theta_1 \le 4$ and $\theta_0 \le 5$ in the last steps. □

Lemma 3.10. *Let $1 < q \le \infty$. Then for $y \in W^{2,\infty}_\gamma(\Omega_R)$, $\gamma \le 5/2 - 2\mu$ and $-2/q < \gamma < 2-2/q$ the inequality*

$$\|(r+d_I)^{-1/2}(y-y_h)\|_{L^2(\Omega_{R/8})} \le c\left(h^2|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)}\right)$$

holds.

Proof. We define the weight function $\sigma = r + d_I$. Furthermore, let χ be the characteristic function, which is equal to one in $\Omega_{R/8}$ and equal to zero in $\Omega_R \setminus \mathrm{cl}(\Omega_{R/8})$. Next, we introduce the boundary value problem

$$\begin{aligned} -\Delta w + w &= \sigma^{-1}(y - y_h)\,\chi \qquad && \text{in } \Omega_R, \\ \partial_n w &= 0 && \text{on } \partial\Omega_R \end{aligned}$$

with its weak formulation

$$a_{\Omega_R}(\varphi, w) = (\sigma^{-1}(y - y_h)\chi, \varphi)_{L^2(\Omega_R)} \quad \forall \varphi \in H^1(\Omega_R), \tag{3.20}$$

where the bilinear form $a_{\Omega_R} : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{R}$ is defined by

$$a_{\Omega_R}(\varphi, w) = \int_{\Omega_R} (\nabla\varphi \cdot \nabla w + \varphi w)\, \mathrm{d}x.$$

Since $r(r + d_I)^{-2} \le r^{-1}$ and $(y - y_h) \in H^1(\Omega)$ we can conclude using Lemma 2.1

$$\begin{aligned} \|\sigma^{-1}(y - y_h)\chi\|_{W^{0,2}_{1/2}(\Omega_R)} &\le \|\sigma^{-1}(y - y_h)\|_{W^{0,2}_{1/2}(\Omega_R)} \le \|y - y_h\|_{W^{0,2}_{-1/2}(\Omega_R)} \\ &\le \|y - y_h\|_{H^1(\Omega_R)} \end{aligned}$$

or more precisely $\sigma^{-1}(y - y_h)\chi \in W^{0,2}_{1/2}(\Omega_R)$. Thus, we get according to Lemma 2.4 that the solution w belongs to $W^{2,2}_{1/2}(\Omega_R)$, since $1 > 1/2 > 1 - \lambda = 1 - \pi/\omega$ for every angle ω in $(0, 2\pi)$. Moreover, if we use the inequality $r < r + d_I$ we obtain the validity of the a priori estimate

$$\|w\|_{W^{2,2}_{1/2}(\Omega_R)} \le c\|\sigma^{-1}(y - y_h)\|_{W^{0,2}_{1/2}(\Omega_{R/8})} \le c\|\sigma^{-1/2}(y - y_h)\|_{L^2(\Omega_{R/8})}. \tag{3.21}$$

Using Lemma 2.1 we can also show that

$$\|w\|_{H^1(\Omega_R)} = \|w\|_{W^{1,2}_0(\Omega_R)} \le c\|w\|_{W^{2,2}_{1/2}(\Omega_R)} \le c\|\sigma^{-1/2}(y - y_h)\|_{L^2(\Omega_{R/8})}. \tag{3.22}$$

Now, let η be an infinitely differentiable function in $\mathrm{cl}(\Omega)$, which is equal to one in $\Omega_{R/8}$, $\operatorname{supp}\eta \subset \Omega_{R/4}$ and $\partial_n \eta = 0$ on $\partial\Omega_R$. By setting $\varphi = \eta v$ in (3.20) with some $v \in H^1(\Omega_R)$ one can show that $\tilde{w} = \eta w$ fulfills the equation

$$a_{\Omega_R}(v, \tilde{w}) = (\eta\sigma^{-1}(y - y_h)\chi - \Delta\eta w - 2\nabla\eta \cdot \nabla w, v)_{L^2(\Omega_R)} \quad \forall v \in H^1(\Omega_R).$$

By this we get

$$\begin{aligned} \|\sigma^{-1/2}(y - y_h)\|^2_{L^2(\Omega_{R/8})} &= (\eta\sigma^{-1}(y - y_h)\chi, y - y_h)_{L^2(\Omega_R)} \\ &= a_{\Omega_R}(y - y_h, \tilde{w}) + (\Delta\eta w, y - y_h)_{L^2(\Omega_R)} + 2(\nabla\eta \cdot \nabla w, y - y_h)_{L^2(\Omega_R)} \\ &\le a_{\Omega_R}(y - y_h, \tilde{w}) + \Big(\|\Delta\eta w\|_{L^2(\Omega_R)} + 2\|\nabla\eta \cdot \nabla w\|_{L^2(\Omega_R)}\Big)\|y - y_h\|_{L^2(\Omega_R)} \\ &\le a_{\Omega_R}(y - y_h, \tilde{w}) + c\|w\|_{H^1(\Omega_R)}\|y - y_h\|_{L^2(\Omega_R)} \\ &\le a_{\Omega_R}(y - y_h, \tilde{w}) + c\|\sigma^{-1/2}(y - y_h)\|_{L^2(\Omega_{R/8})}\|y - y_h\|_{L^2(\Omega_R)}, \end{aligned} \tag{3.23}$$

where we used the Cauchy-Schwarz inequality and (3.22) in the last steps. It remains to estimate the first term in (3.23). Since $\tilde{w}$ is equal to zero in $\Omega_R\setminus \mathrm{cl}(\Omega_{R/4})$ we can use the Galerkin orthogonality of $y-y_h$, i.e. $a_{\Omega_R}(y-y_h, I_h\tilde{w}) = a(y-y_h, I_h\tilde{w}) = 0$. This yields together with an application of the Cauchy-Schwarz inequality

$$\begin{aligned} a_{\Omega_R}(y-y_h,\tilde{w}) &= a_{\Omega_R}(y-y_h,\tilde{w}-I_h\tilde{w}) \\ &\le c\sum_{J=2}^{I}\|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)}. \end{aligned} \tag{3.24}$$

Remember that $\tilde{w}-I_h\tilde{w}\equiv 0$ in Ω_J for $J=0,1$. Now each term on the right hand side of (3.24) is estimated separately. We distinguish between $2\le J\le I-3$ and $J=I,I-1,I-2$ as it has already been done in the previous lemmas. We get for $2\le J\le I-3$ with Lemma 3.9

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(hd_J^{2-\mu-\gamma}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)$$

and with Lemma 3.7 with $\alpha=1/2$

$$\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \le chd_J^{1/2-\mu}|\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)},$$

By means of these two estimates one can conclude for $2\le J\le I-3$

$$\begin{aligned} &\|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \\ &\quad\le c\left(h^2d_J^{5/2-2\mu-\gamma}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + hd_J^{-\mu}\|d_J^{-1/2}\left(y-y_h\right)\|_{L^2(\Omega'_J)}\right)|\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)} \\ &\quad\le c\left(h^2d_J^{5/2-2\mu-\gamma}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{-\mu}\|d_J^{-1/2}\left(y-y_h\right)\|_{L^2(\Omega'_J)}\right)|\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)}, \end{aligned} \tag{3.25}$$

where we used $hd_J^{-\mu}\le hd_I^{-\mu}=c_I^{-\mu}$. For $J=I,I-1,I-2$ we get from Lemma 3.9 for $1<q\le\infty$ and $-2/q<\gamma<2-2/q$

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(c_I^5h^{(2-\gamma)/\mu}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)$$

and from Lemma 3.7 with $\alpha=1/2$

$$\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \le cc_I^{\max\{0,1/2-\mu\}}h^{1/2\mu}|\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)}.$$

We can combine the last two estimates to arrive at

$$\begin{aligned} \|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} &\le c\Big(c_I^{11/2}h^{(5/2-\gamma)/\mu}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} \\ &\quad + c_I^{\max\{0,1/2-\mu\}}(h^{1/\mu}d_J^{-1})^{1/2}\|d_J^{-1/2}\left(y-y_h\right)\|_{L^2(\Omega'_J)}\Big)|\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)} \\ &\le c\Big(c_I^{11/2}h^{(5/2-\gamma)/\mu}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} \\ &\quad + c_I^{\max\{-1/2,-\mu\}}\|d_J^{-1/2}\left(y-y_h\right)\|_{L^2(\Omega'_J)}\Big)|\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)}, \end{aligned} \tag{3.26}$$

where we used $\max\{0, 1/2-\mu\} < 1/2$ and $h^{1/\mu} d_J^{-1} \le h^{1/\mu} d_I^{-1} = c_I^{-1}$. Let $\theta := \max\{-1/2, -\mu\}$. Inserting the inequalities (3.25) and (3.26) into (3.24) yields

$$\begin{aligned}
&a_{\Omega_R}(y-y_h, \tilde{w} - I_h\tilde{w}) \\
&\quad \le c \sum_{J=2}^{I-3} \left(h^2 d_J^{5/2-2\mu-\gamma} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{-\mu} \| d_J^{-1/2} (y-y_h) \|_{L^2(\Omega'_J)} \right) |\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)} \\
&\quad + c \sum_{J=I-2}^{I} \left(c_I^{11/2} h^{(5/2-\gamma)/\mu} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{\theta} \| d_J^{-1/2} (y-y_h) \|_{L^2(\Omega'_J)} \right) |\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)}.
\end{aligned}$$

If we additionally set $\gamma \le 5/2 - 2\mu$ we can conclude using $c_I^{-\mu} < c_I^\theta$ and $d_J^{-1} \le c\sigma^{-1}$

$$\begin{aligned}
a_{\Omega_R}(y-y_h, &\tilde{w} - I_h\tilde{w}) \\
&\le c \sum_{J=2}^{I} \left(c_I^{11/2} h^2 |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{\theta} \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega'_J)} \right) |\tilde{w}|_{W^{2,2}_{1/2}(\Omega'_J)}.
\end{aligned}$$

Now we get with $\sum_{J=2}^{I} 1 \sim |\ln h|$ and the discrete Cauchy-Schwarz inequality

$$\begin{aligned}
&a_{\Omega_R}(y-y_h, \tilde{w} - I_h\tilde{w}) \\
&\quad \le c \left(c_I^{11/2} h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{\theta} \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_R)} \right) |\tilde{w}|_{W^{2,2}_{1/2}(\Omega_R)}.
\end{aligned}$$

Since $|\tilde{w}|_{W^{2,2}_{1/2}(\Omega_R)} \le c\|w\|_{W^{2,2}_{1/2}(\Omega_R)}$ we can apply the a priori estimate (3.21), which yields

$$\begin{aligned}
&a_{\Omega_R}(y-y_h, \tilde{w} - I_h\tilde{w}) \\
&\le c \left(c_I^{11/2} h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{\theta} \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_R)} \right) \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_{R/8})}
\end{aligned} \tag{3.27}$$

By inserting (3.27) into (3.23) and dividing by $\| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_{R/8})}$ we obtain

$$\begin{aligned}
&\| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_{R/8})} \\
&\quad \le c \left(c_I^{11/2} h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{\theta} \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)} \right) \\
&\quad \le c \left(c_I^{11/2} h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{\theta} \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_{R/8})} + c_I^{\theta} \|y-y_h\|_{L^2(\Omega_R)} \right),
\end{aligned}$$

where we used $\sigma^{-1/2} = (r+d_I)^{-1/2} \le r^{-1/2} \le (R/8)^{-1/2} \le c$ if $r \ge R/8$. Finally, we get

$$\left(1 - cc_I^\theta\right) \| \sigma^{-1/2} (y-y_h) \|_{L^2(\Omega_{R/8})} \le c \left(c_I^{11/2} h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{\theta} \|y-y_h\|_{L^2(\Omega_R)} \right).$$

If one has chosen the parameter k in (3.3) large enough such that

$$cc_I^\theta = cc_I^{\max\{-1/2,-\mu\}} \le c\left(c_2 2^k\right)^{\max\{-1/2,-\mu\}} < 1,$$

then the desired result follows. □

In the remainder of this section the constant c_I is hidden in the generic constant c.

Lemma 3.11. *For $v_h \in V_h(\Omega)$ and $1 \le p \le \infty$ there exists a constant $c > 0$ such that*

$$\|v_h\|_{L^p(\partial\Omega_J^\pm)} \le ch^{-1/p} d_J^{-(1-\mu)/p} \|v_h\|_{L^p(\Omega'_J)} \quad \text{for } 1 \le J \le I-2,$$
$$\|v_h\|_{L^p(\partial\Omega_J^\pm)} \le ch^{-1/(p\mu)} \|v_h\|_{L^p(\Omega'_J)} \quad \text{for } J = I-1, I.$$

Proof. Let $E \subset \partial\Omega_J^{\pm\prime}$, and let $T \subset \Omega'_J$ be the corresponding triangle. By an affine change of variables to the reference edge $\hat{E}$ and reference triangle $\hat{T}$, respectively, using the continuity of $\hat{v}_h$ on $\mathrm{cl}(\hat{T})$ and the norm equivalence in finite dimensional spaces we obtain

$$\|v_h\|_{L^p(E)} \le ch_T^{1/p} \|\hat{v}_h\|_{L^p(\hat{E})} \le ch_T^{1/p} \|\hat{v}_h\|_{L^\infty(\hat{E})} \le ch_T^{1/p} \|\hat{v}_h\|_{L^\infty(\hat{T})}$$
$$\le ch_T^{1/p} \|\hat{v}_h\|_{L^p(\hat{T})} \le ch_T^{-1/p} \|v_h\|_{L^p(T)}.$$

Now we can sum up to get

$$\|v_h\|^p_{L^p(\partial\Omega_J^\pm)} \le \sum_{E \subset \partial\Omega_J^{\pm\prime}} \|v_h\|^p_{L^p(E)} \le c \sum_{T \subset \Omega'_J} \left(h_T^{-1} \|v_h\|^p_{L^p(T)} \right)$$
$$\le c \min_{T \subset \Omega'_J} h_T^{-1} \sum_{T \subset \Omega'_J} \|v_h\|^p_{L^p(T)}.$$

One can conclude the desired result with Lemma 3.6. □

Lemma 3.12. *Let $0 \le \delta \le 1/2$ and $1 < q \le \infty$. Then for $y \in W^{2,\infty}_\gamma(\Omega_R)$, $\gamma \le 2+\delta-2\mu$ and $-2/q < \gamma < 2-2/q$ the estimate*

$$\|y - y_h\|_{L^2(\Gamma^\pm_{R/16})} \le c \left(h^2 |\ln h|^{1+\delta} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y - y_h\|_{L^2(\Omega_R)} \right)$$

is valid.

Proof. Note that $\Gamma^\pm_{R/16} = \bigcup_{J=4}^I \partial\Omega_J^\pm$. It holds for $J = I-1, I$

$$\|y - y_h\|_{L^2(\partial\Omega_J^\pm)} \le \|y - I_h y\|_{L^2(\partial\Omega_J^\pm)} + \|I_h y - y_h\|_{L^2(\partial\Omega_J^\pm)}$$
$$\le cd_J^{1/2} \|y - I_h y\|_{L^\infty(\partial\Omega_J^\pm)} + \|I_h y - y_h\|_{L^2(\partial\Omega_J^\pm)},$$

where we have used $|\partial\Omega_J^\pm| \sim d_J$. The continuity of $y - I_h y$ on $\mathrm{cl}(\Omega_J)$ and Lemma 3.11 with $p = 2$ yields

$$\|y - y_h\|_{L^2(\partial\Omega_J^\pm)} \le cd_J^{1/2} \|y - I_h y\|_{L^\infty(\Omega_J)} + ch^{-1/(2\mu)} \|I_h y - y_h\|_{L^2(\Omega'_J)}.$$

Since $d_J \sim h^{1/\mu}$ for $J = I-1, I$ and $|\Omega'_J| \sim d_J^2$ we can proceed with

$$\|y - y_h\|_{L^2(\partial\Omega_J^\pm)}$$
$$\le cd_J^{1/2} \|y - I_h y\|_{L^\infty(\Omega_J)} + cd_J^{-1/2} \|y - I_h y\|_{L^2(\Omega'_J)} + cd_J^{-1/2} \|y - y_h\|_{L^2(\Omega'_J)}$$
$$\le cd_J^{1/2} \|y - I_h y\|_{L^\infty(\Omega'_J)} + cd_J^{-1/2} \|y - y_h\|_{L^2(\Omega'_J)}. \tag{3.28}$$

Next we consider the case $4 \le J \le I-2$. Again we use $|\partial\Omega_J^\pm| \sim d_J$ and the continuity of $y - I_h y$ on $\mathrm{cl}(\Omega_J)$. Thus we can write

$$\|y-y_h\|_{L^2(\partial\Omega_J^\pm)} \le c d_J^{1/2} \|y-y_h\|_{L^\infty(\partial\Omega_J^\pm)} \le c d_J^{1/2} \|y-y_h\|_{L^\infty(\Omega_J)}.$$

Since each subdomain Ω'_J has a positive distance to the corner for $4 \le J \le I-2$, we can use Theorem 10.1 in [30] with $s=0$ to get

$$\|y-y_h\|_{L^2(\partial\Omega_J^\pm)} \le c d_J^{1/2} |\ln h| \|y - I_h y\|_{L^\infty(\Omega'_J)} + c d_J^{-1/2} \|y-y_h\|_{L^2(\Omega'_J)}. \tag{3.29}$$

This is essentially Corollary 5.1 of [26] where the authors have already inserted an interpolation error estimate. In Example 10.1 of [30] the author proved that this result is also applicable for the domains Ω'_J, i.e. for domains which abut on the boundary but contain no corner point. Let $\delta \in [0, 1/2]$. Using (3.28) and (3.29) we arrive at

$$\begin{aligned}
&\|y-y_h\|_{L^2(\Gamma^\pm_{R/16})} = \left(\sum_{J=4}^{I} \|y-y_h\|^2_{L^2(\partial\Omega_J^\pm)} \right)^{1/2} \\
&\le c \left(\sum_{J=4}^{I} \left(d_J^{1/2} |\ln h| \|y - I_h y\|_{L^\infty(\Omega'_J)} + d_J^{-1/2} \|y-y_h\|_{L^2(\Omega'_J)} \right)^2 \right)^{1/2} \\
&\le c|\ln h| \max_{4\le J\le I} \left(d_J^\delta \|y - I_h y\|_{L^\infty(\Omega'_J)} \right) \left(\sum_{J=4}^{I} d_J^{1-2\delta} \right)^{1/2} + c \left(\sum_{J=4}^{I} \|d_J^{-1/2}(y-y_h)\|^2_{L^2(\Omega'_J)} \right)^{1/2}.
\end{aligned}$$

An application of the discrete Hölder inequality yields

$$\left(\sum_{J=4}^{I} d_J^{1-2\delta} \right)^{1/2} \le \left(\sum_{J=4}^{I} d_J \right)^{(1-2\delta)/2} \left(\sum_{J=4}^{I} 1 \right)^{\delta} \le c|\ln h|^\delta,$$

where we have used $\sum_{J=4}^{I} d_J \sim |\Gamma^\pm_{R/16}|$ and $\sum_{J=4}^{I} 1 \sim |\ln h|$ in the last step. Thus, we obtain

$$\begin{aligned}
\|y-y_h\|_{L^2(\Gamma^\pm_{R/16})} \le{}& c|\ln h|^{1+\delta} \max_{4\le J\le I} \left(d_J^\delta \|y - I_h y\|_{L^\infty(\Omega'_J)} \right) \\
&+ c\|(r+d_I)^{-1/2}(y-y_h)\|_{L^2(\Omega_{R/8})}.
\end{aligned}$$

Let $1 < q \le \infty$, $-2/q < \gamma < 2 - 2/q$ and $\gamma \le 2 + \delta - 2\mu \le 5/2 - 2\mu$. Then we get with Lemma 3.7 and Lemma 3.10

$$\begin{aligned}
\|y-y_h\|_{L^2(\Gamma^\pm_{R/16})} \le{}& ch^2 |\ln h|^{1+\delta} \max_{4\le J\le I} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} \\
&+ c\left(h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)} \right),
\end{aligned}$$

which ends the proof. □

Now we are able to prove Theorem 3.2.

Proof. We split the error on the boundary into the already introduced boundary parts,

$$\|y - y_h\|_{L^2(\Gamma)} \leq c \left(\sum_{j=1}^{m} \|y - y_h\|_{L^2(\Gamma^{\pm}_{R_j/16})} + \|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \right). \tag{3.30}$$

For each boundary part $\Gamma^{\pm}_{R_j/16}$, $j = 1, \dots, m$, we get from Lemma 3.12

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R_j/16})} \leq c \left(h^2 |\ln h|^{1+\delta} |y|_{W^{2,\infty}_{\gamma_j}(\Omega_{R_j})} + \|y - y_h\|_{L^2(\Omega_{R_j})} \right), \tag{3.31}$$

provided that $0 \leq \delta \leq 1/2$, $1 < q_j \leq \infty$, $\gamma_j \leq 2 + \delta - 2\mu_j$ and $-2/q_j < \gamma_j < 2 - 2/q_j$. If we set $\mu_j \geq \delta/2$ we can choose q_j such that $1 \leq 2/(2\mu_j - \delta) < q_j < \infty$. By this we get that (3.31) is valid for $0 \leq \gamma_j \leq 2 + \delta - 2\mu_j$ with some arbitrary $\delta \in [0, 1/2]$, i.e., it holds

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R_j/16})} \leq c \left(h^2 |\ln h|^{1+\delta} |y|_{W^{2,\infty}_{2+\delta-2\mu_j}(\Omega_{R_j})} + \|y - y_h\|_{L^2(\Omega_{R_j})} \right). \tag{3.32}$$

for every $\delta \in [0, 1/2]$. Next, we estimate the last term in the right hand side of (3.30). We can conclude from the embedding $L^\infty(\Gamma^0) \hookrightarrow L^2(\Gamma^0)$ and the fact that $y - y_h$ is a continuous function on $\mathrm{cl}(\tilde{\Omega}^0)$

$$\|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \leq c\|y - y_h\|_{L^\infty(\tilde{\Gamma}^0)} \leq c\|y - y_h\|_{L^\infty(\tilde{\Omega}^0)}.$$

Next we use Theorem 10.1 in [30] with $s = 0$ to get

$$\|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \leq c \left(|\ln h| \|y - I_h y\|_{L^\infty(\check{\Omega}^0)} + \|y - y_h\|_{L^2(\check{\Omega}^0)} \right),$$

Compare the proof of Lemma 3.12 for the applicability of this theorem in that case. Since the domain $\check{\Omega}^0 \subset \Omega^0$ has a constant, positive distance to the corner, we can conclude using standard interpolation theory

$$\|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \leq c \left(h^2 |\ln h| |y|_{W^{2,\infty}(\Omega^0)} + \|y - y_h\|_{L^2(\check{\Omega}^0)} \right). \tag{3.33}$$

Combining the inequalities (3.30), (3.32) and (3.33) we obtain

$$\|y - y_h\|_{L^2(\Gamma)} \leq c \left(h^2 |\ln h|^{1+\delta} |y|_{W^{2,\infty}_{\vec{2}+\vec{\delta}-2\vec{\mu}}(\Omega)} + \|y - y_h\|_{L^2(\Omega)} \right). \tag{3.34}$$

Using Lemma 2.6 we can conclude for some arbitrary $\sigma \in (0, 1)$

$$\|y\|_{W^{2,\infty}_{\vec{2}+\vec{\delta}-2\vec{\mu}}(\Omega)} \leq c\|f\|_{C^{0,\sigma}(\bar{\Omega})}, \tag{3.35}$$

if $\mu_j > \delta/2$ and $\max(0, 2 - \lambda_j) < 2 + \delta - 2\mu_j$ or $0 \leq 2 + \delta - 2\mu_j$ in case that $2 - \lambda_j < 0$. The latter condition is fulfilled for any $\mu_j \in (0, 1]$. The former one is equivalent to $\mu_j < \delta/2 + \lambda_j/2$ if $\mu_j \in (\delta/2, 1]$. It follows from Lemma 3.1 that

$$\|y - y_h\|_{L^2(\Omega)} \leq ch^2 \|f\|_{W^{0,2}_{\vec{1}-\vec{\mu}}(\Omega)} \leq ch^2 \|f\|_{L^2(\Omega)}, \tag{3.36}$$

if $\mu_j < \lambda_j$. Finally, the inequalities (3.34), (3.35) and (3.36) yield together with the embedding $C^{0,\sigma}(\bar{\Omega}) \hookrightarrow L^2(\Omega)$ the desired result. □

4 The continuous optimal control problem

In this section we state the continuous optimality system for problem (1.2) and describe the regularity of its solution in weighted Sobolev spaces. The exposition follows those in [18, 1]. The solution operator $S : L^2(\Gamma) \to L^2(\Omega)$ which associates a state $y = Su$ to a control u via (1.3) has already been introduced in Section 1. We denote with $S^* : L^2(\Omega) \to L^2(\Gamma)$ the adjoint operator of S. One has

$$(Su, z)_{L^2(\Omega)} = (u, S^*z)_{L^2(\Gamma)} \quad \forall u \in L^2(\Gamma),\ z \in L^2(\Omega).$$

Furthermore, we define the operator $P : L^2(\Omega) \to H^1(\Omega)$ by $Pz := p$ where p is the solution of

$$\begin{aligned} -\Delta p + p &= z \quad \text{in } \Omega, \\ \partial_n p &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{4.1}$$

The operators S^* and P are related by $S^*z = (Pz)_{|\Gamma} = p_{|\Gamma}$. We can also associate an adjoint state to every control u by $P(Su - y_d)$. Finally, we define the projection

$$\Pi_{[a,b]} f(x) := \max(a, \min(b, f(x)))$$

and the set of admissible controls

$$U_{ad} := \{u \in L^2(\Gamma) : a \le u \le b \text{ a.e. on } \Gamma\}.$$

Theorem 4.1. *The optimal control problem* (1.2) *has a unique solution* $\bar{u} \in L^2(\Gamma)$*. Let* $\bar{y} = S\bar{u}$ *and* $\bar{p} = P(S\bar{u} - y_d)$ *be the state and adjoint state associated with* $\bar{u}$*. Then the variational inequality*

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \ge 0 \qquad \forall u \in U_{ad} \tag{4.2}$$

is satisfied, which can be expressed equivalently by

$$\bar{u}(x) = \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}(x)\right) \quad \text{for a.a. } x \in \Gamma. \tag{4.3}$$

Moreover, let $y_d \in C^{0,\sigma}(\bar{\Omega})$ *with some* $\sigma \in (0,1)$ *and let* β_j *and* γ_j *satisfy the conditions*

$$1 > \beta_j > \max(0, 1 - \lambda_j) \quad \text{or} \quad \beta_j = 0 \text{ and } 1 - \lambda_j < 0, \tag{4.4}$$

$$2 > \gamma_j > \max(0, 2 - \lambda_j) \quad \text{or} \quad \gamma_j = 0 \text{ and } 2 - \lambda_j < 0, \tag{4.5}$$

for $j = 1, \ldots, m$*. Then* $\bar{y}$ *belongs to* $W^{2,2}_{\vec{\beta}}(\Omega)$*,* $\bar{p}$ *to* $W^{2,\infty}_{\vec{\gamma}}(\Omega)$ *and* $\bar{p}_{|\Gamma}$ *to* $W^{2,\infty}_{\vec{\gamma}}(\Gamma)$*. Furthermore, the estimate*

$$\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} \le c\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

is valid.

Proof. Since the optimal control problem is linear quadratic and strictly convex, the existence and the uniqueness of an solution $\bar{u} \in L^2(\Gamma)$ is an immediate result, compare also [7, 6, 18]. The variational inequality (4.2) represents the necessary optimality condition which is also sufficient due to the strict convexity. The proof of the equivalence between the variational inequality (4.2) and the projection formula (4.3) can be found e.g. in [7]. To prove the assertion on the regularity and the a priori estimates we start with the optimal control $\bar{u}$ in $L^2(\Gamma)$. This implies according to [15] or Theorem 2.1 in [7] that $\bar{y}$ belongs to $H^{3/2}(\Omega)$ and

$$\|\bar{y}\|_{H^{3/2}(\Omega)} \leq c\|\bar{u}\|_{L^2(\Gamma)}. \tag{4.6}$$

Furthermore, one can conclude for any $\epsilon \in (0, 1/2] \cap (0, \sigma]$ with the Sobolev embedding theorem

$$\|\bar{y}\|_{C^{0,\epsilon}(\bar{\Omega})} \leq c\|\bar{y}\|_{H^{3/2}(\Omega)}. \tag{4.7}$$

Based on this we get using the results of Lemma 2.6 that $\bar{p}$ admits the estimate

$$\|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} \leq c\|y - y_d\|_{C^{0,\epsilon}(\bar{\Omega})} \leq c\left(\|y\|_{C^{0,\epsilon}(\bar{\Omega})} + \|y_d\|_{C^{0,\epsilon}(\bar{\Omega})}\right), \tag{4.8}$$

provided that the condition (4.5) is satisfied. Due to the Lax-Milgram Theorem and a standard trace theorem one also gets that

$$\|\bar{p}\|_{H^{1/2}(\Gamma)} \leq c\|\bar{p}\|_{H^1(\Omega)} \leq c\|y - y_d\|_{L^2(\Omega)} \leq c\left(\|y\|_{C^{0,\epsilon}(\bar{\Omega})} + \|y_d\|_{C^{0,\epsilon}(\bar{\Omega})}\right). \tag{4.9}$$

Since the optimal control $\bar{u}$ is related to the optimal adjoint state $\bar{p}$ via the projection formula (4.3) we obtain $\bar{u} \in H^{1/2}(\Gamma)$ and

$$\begin{aligned}
\|\bar{u}\|_{H^{1/2}(\Gamma)} &\leq \|\bar{u}\|_{L^2(\Gamma)} + \left(\int_\Gamma \int_\Gamma \frac{|\bar{u}(x_1) - \bar{u}(x_2)|^2}{|x_1 - x_2|^2}\, \mathrm{d}s_{x_1}\, \mathrm{d}s_{x_2}\right)^{1/2} \\
&= \|\bar{u}\|_{L^2(\Gamma)} + \left(\int_\Gamma \int_\Gamma \frac{|\Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}(x_1)\right) - \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}(x_2)\right)|^2}{|x_1 - x_2|^2}\, \mathrm{d}s_{x_1}\, \mathrm{d}s_{x_2}\right)^{1/2} \\
&\leq \|\bar{u}\|_{L^2(\Gamma)} + c\left(\int_\Gamma \int_\Gamma \frac{|\bar{p}(x_1) - \bar{p}(x_2)|^2}{|x_1 - x_2|^2}\, \mathrm{d}s_{x_1}\, \mathrm{d}s_{x_2}\right)^{1/2}
\end{aligned} \tag{4.10}$$

The last step can easily be verified, if one distinguishes the nine cases $-\bar{p}(x_1)/\nu < a \wedge -\bar{p}(x_2)/\nu < a$, $-\bar{p}(x_1)/\nu < a \wedge a \leq -\bar{p}(x_2)/\nu \leq b$, $-\bar{p}(x_1)/\nu < a \wedge -\bar{p}(x_2)/\nu > b$, $a \leq -\bar{p}(x_1)/\nu \leq b \wedge -\bar{p}(x_2)/\nu < a$, etc. Using the embedding $H^{1/2}(\Gamma) \hookrightarrow W^{1/2,2}_{\vec{\beta}}(\Gamma)$, which is valid for $\vec{\beta} \geq \vec{0}$, yields together with Lemma 2.4

$$\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c\|\bar{u}\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \leq c\|\bar{u}\|_{H^{1/2}(\Gamma)}, \tag{4.11}$$

provided that the condition (4.4) is fulfilled. Finally, the estimates (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11) yield the desired result. □

Corollary 4.2. *Let $y_d \in C^{0,\sigma}(\bar{\Omega})$ with some $\sigma \in (0,1)$ and let β_j satisfy the condition*

$$1/2 > \beta_j > \max(0, 3/4 - \lambda_j/2) \quad \text{or} \quad \beta_j = 0 \text{ and } 3/4 - \lambda_j/2 < 0 \tag{4.12}$$

for $j = 1, \dots, m$. Then the optimal adjoint state $\bar{p}|_\Gamma = S^*(S\bar{u} - y_d)$ belongs to $W^{2,2}_{2\vec{\beta}}(\Gamma) \hookrightarrow W^{1,\infty}_{\vec{\beta}}(\Gamma)$ and the a priori estimate

$$\|\bar{p}\|_{W^{1,\infty}_{\vec{\beta}}(\Gamma)} \leq c\|\bar{p}\|_{W^{2,2}_{2\vec{\beta}}(\Gamma)} \leq c\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \tag{4.13}$$

holds.

Proof. We begin by proving that the first inequality of (4.13) holds for $0 \leq \beta_j < 1/2$, $j = 1, \dots, m$. The Sobolev inequality yields with some $q \in (1, 2]$

$$\begin{aligned}
\|\bar{p}\|_{W^{1,\infty}_{\vec{\beta}}(\Gamma)} &= \|\bar{p}\|_{W^{1,\infty}(\Gamma^0)} + \sum_{j=1}^m \sum_{|\alpha| \leq 1} \|r_j^{\beta_j} \partial_t^\alpha \bar{p}\|_{L^\infty(\Gamma_j^\pm)} \\
&\leq \|\bar{p}\|_{W^{2,2}(\Gamma^0)} + c\sum_{j=1}^m \sum_{|\alpha| \leq 1} \|r_j^{\beta_j} \partial_t^\alpha \bar{p}\|_{W^{1,q}(\Gamma_j^\pm)}.
\end{aligned} \tag{4.14}$$

For $0 < \beta_j < 1/2$ we set $1/q = \max((\vec{2} - \vec{\beta})/2, (\vec{3} + 2\vec{\beta})/4)$. Thus, $1 < q < 2$. We get with the product rule and Lemma 2.1

$$\begin{aligned}
\sum_{|\alpha| \leq 1} \|r_j^{\beta_j} \partial_t^\alpha \bar{p}\|_{W^{1,q}(\Gamma_j^\pm)} &\leq c\left(\|\bar{p}\|_{W^{0,q}_{\beta_j - 1}(\Gamma_j^\pm)} + \|\bar{p}\|_{W^{1,q}_{\beta_j - 1}(\Gamma_j^\pm)} + \|\bar{p}\|_{W^{2,q}_{\beta_j}(\Gamma_j^\pm)}\right) \\
&\leq c\|\bar{p}\|_{W^{2,q}_{\beta_j}(\Gamma_j^\pm)} \leq c\|\bar{p}\|_{W^{2,2}_{2\beta_j}(\Gamma_j^\pm)}
\end{aligned} \tag{4.15}$$

In case that $\beta_j = 0$ we directly get this result from (4.14) with $q = 2$. In summary one obtains from the inequalities (4.14) and (4.15) for $0 \leq \beta_j < 1/2$ the validity of the first inequality of (4.13). Finally, if we set the weights β_j according to (4.12) and define the weights $\delta_j = \max(0, 7/8 + 3\beta_j/2 - \lambda_j/4)$ for $j = 1, \dots, m$ in a clever way, then we get using Lemma 2.1 and Theorem 4.1 the desired result

$$\|\bar{p}\|_{W^{2,2}_{2\vec{\beta}}(\Gamma)} \leq c\|\bar{p}\|_{W^{2,\infty}_{\vec{\delta}}(\Gamma)} \leq c\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

with some $\sigma \in (0,1)$. □

5 The discrete optimal control problem

In this section we fully discretize the optimal control problem (1.2). The state and the adjoint state will be discretized by piecewise linear and globally continuous functions. The control will be approximated by piecewise constant functions.

In Section 3 we have already introduced the graded triangulations $\mathcal{T}_h$ of Ω with its boundary triangulations $\mathcal{E}_h$. The space V_h has been defined by

$$V_h = \{y_h \in C(\bar{\Omega}) : y_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h\}.$$

Furthermore, we introduce the spaces

$$U_h := \{u_h \in L^\infty(\Gamma) : u_h|_G \in \mathcal{P}_0(G)\ \forall G \in \mathcal{E}_h\}$$

and

$$U_h^{ad} := U_h \cap U^{ad}.$$

The discrete variant of the state equation reads as follows: Find for each $u \in L^2(\Gamma)$ the unique element $y_h \in V_h$ satisfying

$$a(y_h, v_h) = (u, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h, \tag{5.1}$$

where the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined in (2.5). We denote with $S_h : L^2(\Gamma) \to L^2(\Omega)$ the discrete solution operator which maps a control u to $S_h u := y_h$ via (5.1). The fully discretized version of the optimal control problem can now be stated as

$$J_h(\bar{u}_h) = \min_{u_h \in U_h^{ad}} J_h(u_h), \tag{5.2}$$

with

$$J_h(u_h) := \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|_{L^2(\Gamma)}.$$

As in the continuous case we can deduce an optimality system for the discrete problem. For that purpose we also introduce the discrete version of the solution operator P which is denoted by $P_h : L^2(\Omega) \to H^1(\Omega)$ and defined by $P_h z = p_h$ with some function $z \in L^2(\Omega)$ and p_h being the unique element in V_h such that

$$a(v_h, p_h) = (z, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h.$$

Let $S_h^* : L^2(\Omega) \to L^2(\Gamma)$ denote the discretized version of the operator S^*. Then we can conclude that $S_h^* z = (P_h z)_{|\Gamma}$, since the adjoint of the discrete solution operator is equal to the discretization of the adjoint solution operator. Furthermore, we have that

$$(S_h u, z)_{L^2(\Omega)} = (u, S_h^* z)_{L^2(\Gamma)} \qquad \forall z \in L^2(\Omega),\ \forall u \in L^2(\Gamma).$$

Finally, the discrete adjoint state is the unique element $P_h(S_h u - y_d) \in V_h$.

Lemma 5.1. *The discrete optimal control problem* (5.2) *admits a unique solution* $\bar{u}_h$*. Let* $\bar{y}_h = S_h \bar{u}_h$ *and* $\bar{p}_h = P_h(S_h \bar{u}_h - y_d)$ *be the discrete state and discrete adjoint state associated with* $\bar{u}_h$*. Then the discrete variational inequality*

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \qquad \forall u_h \in U_{ad}^h \tag{5.3}$$

is satisfied.

Proof. This follows analogously to the continuous case. □

6 Results from numerical integration

For the subsequent discretization error analysis of the optimal control problem we need some results from numerical integration on the boundary. Remember, that the triangulation $\mathcal{T}_h$ of the domain Ω induces a segmentation $\mathcal{E}_h$ of the boundary Γ. We define the distance of the edge E to the corner $x^{(j)}$ by $r_{E,j} := \inf_{(x_1,x_2)\in E} |x - x^{(j)}|$ and the element size h_E by $h_E := \operatorname{diam} E$. According to (3.1) there holds

$$\begin{array}{ll} c_1 h^{1/\mu_j} \le h_E \le c_2 h^{1/\mu_j} & \text{for } r_{E,j} = 0, \\ c_1 h r_{E,j}^{1-\mu_j} \le h_E \le c_2 h r_{E,j}^{1-\mu_j} & \text{for } 0 < r_{E,j} \le R_j, \\ c_1 h \le h_E \le c_2 h & \text{for } r_{E,j} > R_j \end{array} \tag{6.1}$$

for $j = 1, \ldots, m$. Furthermore, for $j = 1, \ldots, m$ let $\mathcal{E}_{h,j}$ be the sub-triangulation of $\mathcal{E}_h$ such that $\bigcup_{E\in\mathcal{E}_{h,j}} \bar{E} \subset \Gamma_j^\pm$ and $E \cap \Gamma_j^\pm \ne E$ for all $E \notin \mathcal{E}_{h,j}$. We set $\mathcal{E}_{h,0} = \mathcal{E}_h \setminus \bigcup_{j=1}^n \mathcal{E}_{h,j}$.

Now, let S_E be the midpoint of the edge $E \in \mathcal{E}_h$. Then the projection operator R_h is defined as the 0-interpolator onto U_h, i.e.

$$R_h : C(\Gamma) \to U_h,\ f \to R_h f,$$

where

$$(R_h f)(x) := f(S_E) \quad \text{if } x \in E.$$

In the proof of Lemma 3.7 we have already derived interpolation error estimates in T. There we used the embedding $W^{2,q'}(\hat{T}) \hookrightarrow L^\infty(\hat{T})$ if $q' > 1$. Afterwards we applied the Deny-Lions Lemma and embeddings, which hold for weighted Sobolev spaces (cf. the steps between (3.13) and (3.16)). If we would do the same for boundary elements E, we would get a too restrictive condition on the mesh grading parameters μ_j. For that reason we use Lemma 2.2 in the proof of the following lemma about the approximation properties of the operator R_h.

Lemma 6.1. *Let $1/4 < \mu_j \le 1$. Then the estimate*

$$\left| \int_E (f - R_h f) ds \right| \le c h^2 |E|^{1/2} |f|_{W^{2,2}_{2(1-\mu_j)}(E)}$$

holds true for any element $E \in \mathcal{E}_{h,j}$ and every function $f \in W^{2,2}_{2(1-\mu_j)}(E)$.

Proof. First, we observe that the integral vanishes for any polynomial p of order one, hence

$$\begin{aligned} \left| \int_E (f - R_h f) ds \right| &= \left| \int_E (f - p - R_h(f - p)) ds \right| \\ &\le |E| \left(\|f - p\|_{L^\infty(E)} + \|R_h(f - p)\|_{L^\infty(E)} \right) \\ &\le c|E| \|f - p\|_{L^\infty(E)}. \end{aligned} \tag{6.2}$$

In the last step we used that R_h is a bounded operator from $L^\infty(E)$ to $L^\infty(E)$ with norm 1. Now we distinguish between edges E with $r_{E,j} > 0$ and $r_{E,j} = 0$. In the first case we use the embedding $W^{2,2}(\hat{E}) \hookrightarrow L^\infty(\hat{E})$ and the Deny-Lions Lemma [10] after the transformation to the reference edge $\hat{E}$. We obtain

$$\|f-p\|_{L^\infty(E)} = \|\hat{f}-\hat{p}\|_{L^\infty(\hat{E})} \le c\|\hat{f}-\hat{p}\|_{W^{2,2}(\hat{E})} \le c|\hat{f}|_{W^{2,2}(\hat{E})}. \tag{6.3}$$

The reverse transformation together with $h_E \sim hr_{E,j}^{1-\mu}$ yields

$$\left|\int_E (f-R_hf)ds\right| \le c|E|^{1/2}h_E^2|f|_{W^{2,2}(E)} \le c|E|^{1/2}h^2|f|_{W^{2,2}_{2(1-\mu_j)}(E)}.$$

In the second case, $r_{E,j}=0$, we use the embedding $W^{2,2}_{2(1-\mu_j)}(\hat{E}) \hookrightarrow W_1^{2,4/(5-4\mu_j)}(\hat{E}) \hookrightarrow W^{1,4/(5-4\mu_j)}(\hat{E}) \hookrightarrow L^\infty(\hat{E})$, which holds for $1/4 < \mu_j \le 1$ (cf. Lemma 2.1). By this we obtain using Lemma 2.2

$$\begin{aligned}
\|f-p\|_{L^\infty(E)} &= \|\hat{f}-\hat{p}\|_{L^\infty(\hat{E})} \le c\|\hat{f}-\hat{p}\|_{W^{2,2}_{2(1-\mu_j)}(\hat{E})} \\
&\le c\left(|\hat{f}-\hat{p}|_{W^{2,2}_{2(1-\mu_j)}(\hat{E})} + \sum_{\delta\le 1}\left|\int_{\hat{E}} \partial_t^\delta(\hat{f}-\hat{p})d\hat{s}\right|\right) \\
&= c\left(|\hat{f}|_{W^{2,2}_{2(1-\mu_j)}(\hat{E})} + \sum_{\delta\le 1}\left|\int_{\hat{E}} \partial_t^\delta(\hat{f}-\hat{p})d\hat{s}\right|\right),
\end{aligned} \tag{6.4}$$

since p is polynomial of order one. Next, we choose p such that the last term of (6.4) vanishes, which is possible without any restriction. This yields

$$\|f-p\|_{L^\infty(E)} \le c|\hat{f}|_{W^{2,2}_{2(1-\mu_j)}(\hat{E})} \le ch_E^{2\mu_j}|E|^{-1/2}|f|_{W^{2,2}_{2(1-\mu_j)}(E)}, \tag{6.5}$$

where we used $\hat{r}_j^{2(1-\mu_j)} \sim h_E^{-2(1-\mu_j)}r_j^{2(1-\mu_j)}$. The inequalities (6.2) and (6.5) yield with $h_E \sim h^{1/\mu_j}$ the assertion for $r_{E,j}=0$. □

Corollary 6.2. *Let $E \in \mathcal{E}_h$ and $f \in W^{2,2}(E)$. Then the estimate*

$$\left|\int_E (f-R_hf)ds\right| \le ch^2|E|^{1/2}|f|_{W^{2,2}(E)}$$

is valid.

Proof. This follows from (6.2) and (6.3) together with $h_E \le ch$ after the reverse transformation to the edge E. □

Lemma 6.3. *Let $0 < \mu_j \le 1$. Then the estimate*

$$\left|\int_E (f-R_hf)ds\right| \le ch|E||f|_{W^{1,\infty}_{1-\mu_j}(E)}$$

holds true for any element $E \in \mathcal{E}_{h,j}$ and every function $f \in W^{1,\infty}_{1-\mu_j}(E)$.

Proof. We get from (6.2)

$$\left|\int_E (f - R_h f)ds\right| \le c|E|\|f - p\|_{L^\infty(E)} \le c|E|\|\hat{f} - \hat{p}\|_{L^\infty(\hat{E})}$$
$$\le c|E|\|\hat{f} - \hat{p}\|_{W^{1,q}(\hat{E})} \le c|E||\hat{f}|_{W^{1,q}(\hat{E})} \tag{6.6}$$

where we used the embedding $W^{1,q}(\hat{E}) \hookrightarrow L^\infty(\hat{E})$ with some $q > 1$ and the Deny-Lions Lemma [10]. Now we consider the case that $r_{E,j} > 0$. After the reverse transformation to the edge E one gets with $h_E \sim h r_{E,j}^{1-\mu_j}$

$$\left|\int_E (f - R_h f)ds\right| \le c|E|^{1-1/q} h_E |f|_{W^{1,q}(E)} \le c|E| h r_{E,j}^{1-\mu_j} |f|_{W^{1,\infty}(E)} \le c|E|\, h\, |f|_{W^{1,\infty}_{1-\mu_j}(E)}.$$

In case that $r_{E,j} = 0$ we can conclude using (6.6) with $q = 2/(2 - \mu_j) > 1$ and the embedding $W^{1,\infty}_{1-\mu_j}(\hat{E}) \hookrightarrow W^{1,2/(2-\mu_j)}(\hat{E})$, which holds for $0 < \mu_j \le 1$,

$$\left|\int_E (f - R_h f)ds\right| \le c|E||\hat{f}|_{W^{1,\infty}_{1-\mu_j}(\hat{E})}.$$

Finally, the reverse transformation to the edge E together with $\hat{r}_j^{1-\mu_j} \sim h_E^{\mu_j - 1} r_j^{1-\mu_j}$ and $h_E \sim h^{1/\mu_j}$ yields

$$\left|\int_E (f - R_h f)ds\right| \le c|E| h_E h_E^{\mu_j - 1} |f|_{W^{1,\infty}_{1-\mu_j}(E)} \le c|E||h||f|_{W^{1,\infty}_{1-\mu_j}(E)}.$$

□

Corollary 6.4. *Let $E \in \mathcal{E}_h$ and $f \in W^{1,\infty}(E)$. Then the estimate*

$$\left|\int_E (f - R_h f)ds\right| \le ch|E||f|_{W^{1,\infty}(E)}$$

holds.

Proof. One gets this result from (6.6) with $q = \infty$ and $h_E \le ch$ after the reverse transformation to the edge E. □

7 Discretization error estimates for the postprocessing approach

Based on the results of the previous sections we first analyze in the following the fully discrete optimal control problem of Section 5 with respect to its discretization error. Afterwards we construct in a postprocessing step a new control which possesses better approximation properties. Let

$$K_1 := \bigcup_{E \in \mathcal{E}_h : \bar{u} \notin W^{2,2}_{2(\vec{1}-\vec{\mu})}(E)} E, \qquad K_2 := \bigcup_{E \in \mathcal{E}_h : \bar{u} \in W^{2,2}_{2(\vec{1}-\vec{\mu})}(E)} E.$$

Through the rest of this paper we make the following assumption on the measure of the set K_1.

Assumption 7.1. *We assume that* $\text{meas}(K_1) < ch$.

Remark 7.2. *This assumption is satisfied in many practical applications. For example it is fulfilled if the optimal control* $\bar{u}$ *has only a finite number of kinks due to the projection on the interval* $[a,b]$. *See Section 4 in [18] for a more sophisticated discussion on its validity.*

Let us define the L^2-projection of a function $f \in L^2(\Gamma)$ as the piecewise constant function in U_h that fulfills

$$Q_h f|_E \equiv \frac{1}{|E|} \int_E f(x) ds$$

on any element $E \in \mathcal{E}_h$. The following approximation property of the L^2-projection is proven in Corollary 4.8 of [1].

Corollary 7.3. *For any element* $E \in \mathcal{E}_h$ *and any functions* $f \in H^1(E)$ *and* $v \in H^s(E)$, $s = [0,1]$, *the estimate*

$$(f - Q_h f, v)_{L^2(E)} \le c h_T^{s+1} |f|_{H^1(E)} |v|_{H^s(E)}$$

is valid.

Lemma 7.4. *Let Assumption 7.1 be satisfied. Then the estimate*

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

is valid, provided that the mesh parameters $\vec{\mu}$ *are chosen such that* $\vec{1}/2 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$.

Proof. First, we introduce the function $S(\bar{u} - R_h\bar{u})$ and apply the triangle inequality. This yields

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \le \|(S_h - S)(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)}. \tag{7.1}$$

Applying the Nitsche method together with $\vec{\mu} < \vec{\lambda}$ (cf. the proof of Lemma 4.1 in [1]) we get for the first term in (7.1)

$$\begin{aligned} \|(S_h - S)(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} &\le ch\|S(\bar{u} - R_h\bar{u})\|_{H^1(\Omega)} \\ &\le ch\|\bar{u} - R_h\bar{u}\|_{L^2(\Gamma)} \le ch^2 |\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}, \end{aligned} \tag{7.2}$$

where we used the a priori estimate given by the Lax-Milgram Theorem, a standard interpolation error estimate and $h_T \le ch$ in the last steps. Note that we use $W^{1,2}_{\vec{0}}(\Gamma)$ instead of $H^1(\Gamma)$ since the former is defined piecewise. Let $z = S(\bar{u} - R_h\bar{u})$. Then we get for the second term in (7.1)

$$\begin{aligned} \|S(\bar{u} - R_h\bar{u})\|^2_{L^2(\Omega)} &= (S(\bar{u} - R_h\bar{u}), z)_{L^2(\Omega)} = (\bar{u} - R_h\bar{u}, S^* z)_{L^2(\Gamma)} \\ &= (\bar{u} - Q_h\bar{u}, S^* z)_{L^2(\Gamma)} + (Q_h\bar{u} - R_h\bar{u}, S^* z)_{L^2(\Gamma)}, \end{aligned} \tag{7.3}$$

where we introduced the intermediate function $Q_h\bar{u}$. Again, we estimate both terms in (7.3) separately. One obtains for the first term with Corollary 7.3 and $h_T \le ch$

$$(\bar{u} - Q_h\bar{u}, S^*z)_{L^2(\Gamma)} = \sum_{E\in\mathcal{E}_h} (\bar{u} - Q_h\bar{u}, S^*z)_{L^2(E)} \le c \sum_{E\in\mathcal{E}_h} h^2|\bar{u}|_{H^1(E)}|S^*z|_{H^1(E)}.$$

Next we apply the discrete Cauchy-Schwarz inequality, the trace theorem, the Sobolev inequality, Lemma 2.1 and the a priori estimate from Lemma 2.4. This yields

$$\begin{aligned}(\bar{u} - Q_h\bar{u}, S^*z)_{L^2(\Gamma)} &\le ch^2|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}|S^*z|_{W^{1,2}_{\vec{0}}(\Gamma)} \le ch^2|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}\|Pz\|_{H^{3/2+\epsilon}(\Omega)} \\ &\le ch^2|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}\|Pz\|_{W^{2,4/3+\epsilon}(\Omega)} \le ch^2|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}\|Pz\|_{W^{2,2}_{\vec{1}/2-\vec{\epsilon}}(\Omega)} \\ &\le ch^2|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}\|z\|_{W^{0,2}_{\vec{1}/2-\vec{\epsilon}}(\Omega)} \le ch^2|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)}\|z\|_{L^2(\Omega)}, \end{aligned} \tag{7.4}$$

which holds for $\vec{0} < \vec{\epsilon} < \vec{1}/2 - \max(0, \vec{1} - \vec{\lambda})$. For the second term in (7.3) we get with the Hölder inequality

$$\begin{aligned}(Q_h\bar{u} - R_h\bar{u}, S^*z)_{L^2(\Gamma)} &\le \|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)}\|S^*z\|_{L^\infty(\Gamma)} \\ &\le c\|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)}\|z\|_{L^2(\Omega)}, \end{aligned} \tag{7.5}$$

where we used the embedding $W^{1,2}_{\vec{0}}(\Gamma) \hookrightarrow L^\infty(\Gamma)$ and $|S^*z|_{W^{1,2}_{\vec{0}}(\Gamma)} \le c\|z\|_{L^2(\Omega)}$ as in (7.4). Since $R_h\bar{u}$ is constant on every element E we can continue with

$$\begin{aligned}\|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)} &= \|Q_h(\bar{u} - R_h\bar{u})\|_{L^1(\Gamma)} = \sum_{E\in\mathcal{E}_h}\left|\int_E (\bar{u} - R_h\bar{u})\,ds\right| \\ &= \sum_{j=0}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}}\left|\int_E (\bar{u} - R_h\bar{u})\,ds\right| + \sum_{j=0}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}}\left|\int_E (\bar{u} - R_h\bar{u})\,ds\right|. \end{aligned}$$

Using Lemmas 6.1 and 6.3 and Corollaries 6.2 and 6.4 we get for $\mu_j > 1/4$

$$\begin{aligned}\|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)} \le c &\left(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_1}} h|E||\bar{u}|_{W^{1,\infty}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} h|E||\bar{u}|_{W^{1,\infty}_{1-\mu_j}(E)} \right. \\ &\left. + \sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_2}} h^2|E|^{1/2}|\bar{u}|_{W^{2,2}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} h^2|E|^{1/2}|\bar{u}|_{W^{2,2}_{2(1-\mu_j)}(E)} \right) \\ &\le ch|K_1|\left(|\bar{u}|_{W^{1,\infty}(K_1\cap\Gamma^0)} + \sum_{j=1}^m |\bar{u}|_{W^{1,\infty}_{1-\mu_j}(K_1\cap\Gamma^\pm_j)}\right) \\ &+ ch^2|K_2|^{1/2}\left(|\bar{u}|_{W^{2,2}(K_2\cap\Gamma^0)} + \sum_{j=1}^m |\bar{u}|_{W^{2,2}_{2(1-\mu_j)}(K_2\cap\Gamma^\pm_j)}\right) \\ &\le ch^2\left(|\bar{u}|_{W^{1,\infty}_{\vec{1}-\vec{\mu}}(K_1)} + |\bar{u}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(K_2)}\right), \end{aligned} \tag{7.6}$$

where we used the discrete Cauchy-Schwarz inequality and Assumption 7.1. Collecting the results from the inequalities (7.1), (7.2), (7.3), (7.4), (7.5) and (7.6) yields

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \leq ch^2 \left(|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)} + |\bar{u}|_{W^{1,\infty}_{\vec{1}-\vec{\mu}}(K_1\cap\Gamma)} + |\bar{u}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(K_2\cap\Gamma)} \right). \tag{7.7}$$

Next, we take into account that $\bar{u}$ is given by the projection formula (4.3). We divide the boundary Γ into the boundary parts $\mathcal{I}$, where $\bar{u} = -\bar{p}/\nu$, and $\mathcal{A}$, where $\bar{u} = a$ or $\bar{u} = b$. We obtain for $\vec{\mu} > \vec{1}/2$

$$|\bar{u}|_{W^{1,2}_{\vec{0}}(\Gamma)} \leq |\bar{u}|_{W^{1,2}_{\vec{0}}(\mathcal{I})} + |\bar{u}|_{W^{1,2}_{\vec{0}}(\mathcal{A})} \leq c|\bar{p}|_{W^{1,2}_{\vec{0}}(\mathcal{I})} \leq c\|\bar{p}\|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(\Gamma)} \tag{7.8}$$

The last step holds due to the embedding $W^{2,2}_{2(\vec{1}-\vec{\mu})}(\Gamma) \hookrightarrow W^{2,2}_{\vec{1}}(\Gamma) \hookrightarrow W^{1,2}_{\vec{0}}(\Gamma)$, which is valid for $\vec{\mu} > \vec{1}/2$ (cf. Lemma 2.1). Analogously we get

$$\begin{aligned} |\bar{u}|_{W^{1,\infty}_{\vec{1}-\vec{\mu}}(K_1)} + |\bar{u}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(K_2)} &\leq c \left(|\bar{p}|_{W^{1,\infty}_{\vec{1}-\vec{\mu}}(K_1\cap\mathcal{I})} + |\bar{p}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(K_2\cap\mathcal{I})} \right) \\ &\leq c \left(|\bar{p}|_{W^{1,\infty}_{\vec{1}-\vec{\mu}}(\Gamma)} + |\bar{p}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(\Gamma)} \right). \end{aligned} \tag{7.9}$$

In summary one obtains from the inequalities (7.7), (7.8) and (7.9)

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \leq ch^2 \left(|\bar{p}|_{W^{1,\infty}_{\vec{1}-\vec{\mu}}(\Gamma)} + \|\bar{p}\|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(\Gamma)} \right).$$

The results of Corollary 4.2 imply for $\vec{1}/2 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right).$$

□

Lemma 7.5. *Let $v \in L^2(\Gamma)$ and $z \in L^2(\Omega)$. The discrete solution operators S_h and S_h^* admit for $\vec{0} < \vec{\mu} \leq \vec{1}$ the estimates*

$$\begin{aligned} \|S_h v\|_{L^2(\Omega)} &\leq c\|v\|_{L^2(\Gamma)}, \\ \|P_h z\|_{L^2(\Omega)} &\leq c\|z\|_{L^2(\Omega)}, \\ \|S_h^* z\|_{L^2(\Gamma)} &\leq c\|z\|_{L^2(\Omega)}. \end{aligned}$$

Proof. We prove the second and third inequality. The first one can be proven analogously. The coercivity of the bilinear form, the Cauchy-Schwarz inequality and the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ yield

$$\|P_h z\|^2_{H^1(\Omega)} \leq ca(P_h z, P_h z) = c(z, P_h z)_{L^2(\Omega)} \leq c\|z\|_{L^2(\Omega)}\|P_h z\|_{H^1(\Omega)}.$$

One gets the second inequality with the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and the third one with the trace theorem $\|S_h^* z\|_{L^2(\Gamma)} \leq c\|P_h z\|_{H^1(\Omega)}$. □

Lemma 7.6 (Supercloseness). *Let Assumption 7.1 and the condition $\vec{1}/2 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$ be fulfilled. Then the estimate*

$$\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

holds true.

Proof. To prove this Lemma we proceed similar to the proof of Lemma 5.2 of [1]. In Lemma 5.1 of [1] the validity of the inequality

$$\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)}$$

is stated. Inserting appropriate intermediate functions yields

$$\begin{aligned}\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le\ & (R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\ &+ (\bar{p} - S_h^*(S_hR_h\bar{u} - y_d), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\ &+ (S_h^*(S_hR_h\bar{u} - y_d) - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)}. \end{aligned} \tag{7.10}$$

We are going to estimate each term on the right hand side of (7.10) separately. Since $\bar{u}_h - R_h\bar{u}$ is constant on every boundary element E we obtain for the first term

$$\begin{aligned}(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} &= \sum_{E\in\mathcal{E}_h}\int_E (R_h\bar{p} - \bar{p})(\bar{u}_h - R_h\bar{u})ds \\ &= \sum_{j=1}^m\sum_{E\in\mathcal{E}_{h,j}} (\bar{u}_h - R_h\bar{u})_{|E}\int_E (R_h\bar{p} - \bar{p})ds,\end{aligned}$$

Using Lemma 6.1 and Corollary 6.2 we can conclude as in the proof of Lemma 7.4 for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\begin{aligned}&(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\ &\le ch^2\left(\sum_{E\in\mathcal{E}_{h,0}}|E|^{1/2}(\bar{u}_h - R_h\bar{u})_{|E}|\bar{p}|_{W^{2,2}(E)} + \sum_{j=1}^m\sum_{E\in\mathcal{E}_{h,j}}|E|^{1/2}(\bar{u}_h - R_h\bar{u})_{|E}|\bar{p}|_{W^{2,2}_{2(1-\mu_j)}(E)}\right) \\ &= ch^2\left(\sum_{E\in\mathcal{E}_{h,0}}\|\bar{u}_h - R_h\bar{u}\|_{L^2(E)}|\bar{p}|_{W^{2,2}(E)} + \sum_{j=1}^m\sum_{E\in\mathcal{E}_{h,j}}\|\bar{u}_h - R_h\bar{u}\|_{L^2(E)}|\bar{p}|_{W^{2,2}_{2(1-\mu_j)}(E)}\right) \\ &\le ch^2\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)}|\bar{p}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(\Gamma)} \\ &\le ch^2\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right).\end{aligned} \tag{7.11}$$

The last two inequalities hold with respect to the discrete Cauchy-Schwarz inequality and Corollary 4.2. For the second term in (7.10) we get with the Cauchy-Schwarz inequality

$$(\bar{p} - S_h^*(S_hR_h\bar{u} - y_d), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \le \|\bar{p} - S_h^*(S_hR_h\bar{u} - y_d)\|_{L^2(\Gamma)}\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)}.$$

We again introduce intermediate functions, apply the triangle inequality and Lemma 7.5. By this we get

$$\begin{aligned}
&\|\bar p - S_h^*(S_hR_h\bar u - y_d)\|_{L^2(\Gamma)} = \|S^*(\bar y - y_d) - S_h^*(S_hR_h\bar u - y_d))\|_{L^2(\Gamma)}\\
&\quad\le \|(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)\bar u\|_{L^2(\Gamma)} + \|S_h^*S_h(\bar u - R_h\bar u)\|_{L^2(\Gamma)}\\
&\quad\le \|(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Gamma)} + c\|(S - S_h)\bar u\|_{L^2(\Omega)} + c\|S_h(\bar u - R_h\bar u)\|_{L^2(\Gamma)}.
\end{aligned}$$

These three terms has been estimated in Theorem 3.2, Lemma 3.1, and Lemma 7.4. Thus, one obtains for $\vec{1}/2 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2 < \vec{\lambda}$ and some $\epsilon \in (0, 1/2] \cap (0, \sigma]$ with $\sigma \in (0, 1)$

$$\begin{aligned}
&(\bar p - S_h^*(S_hR_h\bar u - y_d), \bar u_h - R_h\bar u)_{L^2(\Gamma)}\\
&\le ch^2|\ln h|^{3/2}\left(\|\bar y - y_d\|_{C^{0,\epsilon}(\bar\Omega)} + \|\bar y\|_{W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right)\|\bar u_h - R_h\bar u\|_{L^2(\Gamma)}\\
&\le ch^2|\ln h|^{3/2}\left(\|\bar y\|_{W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right)\|\bar u_h - R_h\bar u\|_{L^2(\Gamma)}\\
&\le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right)\|\bar u_h - R_h\bar u\|_{L^2(\Gamma)} \qquad (7.12)
\end{aligned}$$

where we used the embedding $W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega) \hookrightarrow W^{2,2}_{\vec{1}/2}(\Omega) \hookrightarrow W^{2,2/(2-\epsilon)}(\Omega) \hookrightarrow C^{0,\epsilon}(\bar\Omega)$ (according to Lemma 2.1 and the Sobolev inequality) and Theorem 4.1. Having in mind the definition of $\bar p_h$ and S_h^* we get for the third term in (7.10)

$$\begin{aligned}
&(S_h^*(S_hR_h\bar u - y_d) - \bar p_h, \bar u_h - R_h\bar u)_{L^2(\Gamma)} = (S_h^*(S_h(R_h\bar u - \bar u_h)), \bar u_h - R_h\bar u)_{L^2(\Gamma)}\\
&\quad = (S_h(R_h\bar u - \bar u_h), S_h(\bar u_h - R_h\bar u))_{L^2(\Omega)} = -\|S_h(R_h\bar u - \bar u_h)\|^2_{L^2(\Omega)} \le 0. \qquad (7.13)
\end{aligned}$$

Finally the inequalities (7.10), (7.11), (7.12) and (7.13) imply the desired result. □

We define the projection $\tilde u_h$ of $\bar p_h$ by

$$\tilde u_h = \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar p_h\right).$$

This projection is piecewise linear and continuous, but this postprocessed control does not belong to the discrete admissible set in general. However one can prove that $\tilde u_h$ possesses superconvergence properties.

Theorem 7.7. *Let the assumption 7.1 be satisfied. Then the discretization error estimates*

$$\begin{aligned}
\|\bar y - \bar y_h\|_{L^2(\Omega)} &\le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right),\\
\|\bar p - \bar p_h\|_{L^2(\Gamma)} + \|\bar p - \bar p_h\|_{L^2(\Omega)} &\le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right),\\
\|\bar u - \tilde u_h\|_{L^2(\Gamma)} &\le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right)
\end{aligned}$$

hold, provided that the grading parameter μ fulfills the condition $\vec{1}/2 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$.

Proof. Let $\epsilon \in (0, 1/2] \cap (0, \sigma]$ with $\sigma \in (0, 1)$ and $\vec{1}/2 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$. Introducing intermediate functions, the triangle inequality and Lemma 7.5 yield

$$\begin{aligned}
\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S_h(R_h\bar{u} - \bar{u}_h)\|_{L^2(\Omega)} \\
&\leq \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + c\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}.
\end{aligned}$$

If we apply the results of Lemma 3.1, Lemma 7.4, Lemma 7.6 and Theorem 4.1 we obtain

$$\begin{aligned}
\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq ch^2|\ln h|^{3/2} \left(\|\bar{y}\|_{W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \\
&\leq ch^2|\ln h|^{3/2} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right). \qquad (7.14)
\end{aligned}$$

The error of the adjoint state on the boundary and in the domain can be estimated by

$$\begin{aligned}
\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &\leq \|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Gamma)} \\
&\quad + \|(P - P_h)(\bar{y} - y_d)\|_{L^2(\Omega)} + \|P_h(\bar{y} - y_h)\|_{L^2(\Omega)}, \\
&\leq \|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} + \|(P - P_h)(\bar{y} - y_d)\|_{L^2(\Omega)} \\
&\quad + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}.
\end{aligned}$$

where we used Lemma 7.5. Next, we apply Lemma 3.1, Theorem 3.2 and (7.14). By this we get

$$\begin{aligned}
&\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \\
&\quad \leq ch^2|\ln h|^{3/2} \left(\|\bar{y} - y_d\|_{C^{0,\epsilon}(\bar{\Omega})} + \|\bar{y} - y_d\|_{W^{0,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \\
&\quad \leq ch^2|\ln h|^{3/2} \left(\|\bar{y}\|_{C^{0,\epsilon}(\bar{\Omega})} + \|\bar{y}\|_{W^{0,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{W^{0,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)
\end{aligned}$$

The embeddings $W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega) \hookrightarrow W^{2,2}_{\vec{1}/2}(\Omega) \hookrightarrow W^{2,2/(2-\epsilon)}(\Omega) \hookrightarrow C^{0,\epsilon}(\bar{\Omega}) \hookrightarrow W^{0,2}_{\vec{1}-\vec{\mu}}(\Omega)$, which hold according to Lemma 2.1 and the Sobolev embedding theorem, yield together with Theorem 4.1

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq ch^2|\ln h|^{3/2} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right). \qquad (7.15)$$

Finally, we observe that the projection operator $\Pi_{[a,b]}$ is Lipschitz continuous (cf. also the proof of Theorem 4.1). This implies together with (7.15)

$$\begin{aligned}
\nu\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} &= \nu \left\| \Pi_{[a,b]} \left(-\frac{1}{\nu}\bar{p} \right) - \Pi_{[a,b]} \left(-\frac{1}{\nu}\bar{p}_h \right) \right\|_{L^2(\Gamma)} \leq c\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \\
&\leq ch^2|\ln h|^{3/2} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right),
\end{aligned}$$

which ends the proof. $\square$

Remark 7.8. *If one takes account of the finite element error estimate on the boundary given in Theorem 3.2, then the same discretization error estimates can be proven for the concept of variational discretizations proposed in [12] for distributed control problems and [6] for Neumann boundary control problems, compare also Section 7 in [18] and Section 6 in [1].*

Remark 7.9. *A numerical example which shows the proven convergence rates can be found in Section 7 of [1].*

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