Local Mesh Refinement for the Discretisation of Neumann Boundary Control Problems on Polyhedra

Thomas Apel* Johannes Pfefferer* Max Winkler*

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Abstract

This paper deals with optimal control problems constrained by linear elliptic partial differential equations. The case where the right-hand side of the Neumann boundary is controlled, is studied. The variational discretisation concept for these problems is applied and discretisation error estimates are derived. On polyhedral domains one has to deal with edge and corner singularities which reduce the convergence rate of the discrete solutions, i.e., one can not expect convergence order two for linear finite elements on quasi-uniform meshes in general. As a remedy a local mesh refinement strategy is presented and a priori bounds for the refinement parameters are derived such that convergence with optimal rate is guaranteed. As a by-product finite element error estimates in the $H^1(\Omega)$-, $L^2(\Omega)$- and $L^2(\Gamma)$-norm for the boundary value problem are obtained, where the latter one turned out to be the main challenge.

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*Institut für Mathematik und Bauinformatik, Universität der Bundeswehr München, 81549 Neubiberg, Germany

1 Introduction

Local mesh refinement for the numerical solution of boundary value problems is a well known technique to compensate lower convergence rates due to singularities contained in the solution of those problems, and has intensively been studied in the literature, see e. g. [9, 33]. On polyhedral domains several papers exploit *a priori* knowledge about singularities occurring at edges and corners [3, 7, 12, 25]. The error estimates proven for graded meshes can also be used to derive error bounds for certain discretisation concepts applied to optimal control problems. It is the purpose of this paper to derive such an *a priori* error estimate especially for boundary control problems governed by pure Neumann conditions. In contrast to distributed control problems, finite element error estimates on the boundary for the Neumann boundary value problem have to be proven. Using standard techniques, e. g. a trace theorem or the Aubin-Nitsche method, when deriving such error estimates will yield suboptimal convergence rates only. Thus, we extend the techniques from [6] to the three-dimensional case.

There are several publications which deal with error estimates for Neumann boundary control problems in two-dimensional polygonal domains. In most cases one is interested in the convergence rate of the discrete control variable in the $L^2(\Gamma)$-norm. For convex domains the results of Hinze/Matthes [21] imply the convergence rate $\max\{\min\{2, \pi/\omega\} - \varepsilon, 3/2\}$ on quasi-uniform meshes using the variational discretisation approach, where $\varepsilon > 0$ is an arbitrary real number and ω the largest interior angle at the corner points of the polygon. However, the proven convergence rate is suboptimal. This result was improved by Mateos/Rösch [26] who were able to show the convergence rate $\min\{2, 1 + \pi/(2\omega), 1/2 + \pi/\omega\} - \varepsilon$ for convex and even non-convex domains. This estimate is sharp for domains with $\omega < 90^\circ$ or $\omega > 180^\circ$. Apel/Pfefferer/Rösch [5] used a mesh grading technique which ensures convergence order 3/2 for non-convex domains. Later on, Apel/Pfefferer/Rösch [6] proved that one can expect a convergence rate 2 up to a logarithmic factor if $\omega < 120^\circ$. For larger interior angles they used mesh grading to retain this convergence rate. The aforementioned results are also established for the *postprocessing approach* in [5, 6, 26].

To our knowledge, there are no publications that deal with error estimates on the boundary of polyhedra so far. Solely in [23] optimal error estimates on quasi-uniform meshes for the normal derivative are proven. However, estimates in the domain have been studied for various mesh refinement strategies. For instance Apel/Sändig/Whiteman [7] considered an isotropic refinement strategy for polyhedra with the restriction that the same mesh refinement parameter is used for all singular points. A more general strategy which allows to use a different strength of refinement for each edge and corner has been investigated by Lubuma/Nicaise [25]. Advanced refinement strategies using anisotropic mesh grading at singular edges have been intensively studied by Apel/Sirch [8] for prismatic domains, where only edge singularities occur, and on general polyhedra by Apel/Lombardi/Winkler [3] and Băcuţă/Nistor/Zikatanov [12]. For all these approaches the required error estimates on the boundary still have to be proven.

The aim of this paper is to derive a bound for the parameters used in the isotropic refinement strategy from [7] in order to guarantee an optimal convergence rate in $H^1(\Omega)$-, $L^2(\Omega)$- and $L^2(\Gamma)$-norm. Using these estimates we are then able to prove an optimal convergence rate for the variational discretization approach [20] applied to a Neumann control problem.

Let Ω be a polyhedron with boundary Γ. We consider the following elliptic Neumann boundary control problem:

$$J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \tag{1.1}$$

subject to

$$-\Delta y + y = f \text{ in } \Omega, \qquad \partial_n y = u \text{ on } \Gamma, \tag{1.2}$$

$$u \in U_{ad} := \{u \in L^2(\Gamma)\colon a \le u \le b \text{ a.e. on } \Gamma\}, \tag{1.3}$$

with a given function $f \in L^2(\Omega)$, regularisation parameter $\alpha > 0$ and constant control bounds $a, b \in \mathbb{R}$. It is known that the state equation (1.2) possesses a unique solution $y \in H^{3/2+\varepsilon_{reg}}(\Omega)$ with some fixed $\varepsilon_{reg} \in (0, \frac{1}{2}]$ depending on the geometry of Ω [14, Corollary 23.5]. We assume that the desired state is Hölder-continuous, i.e. $y_d \in C^{0,\sigma}(\overline{\Omega})$, with a Hölder exponent $\sigma \in (0,1)$.

The analysis of the continuous problem (1.1)–(1.3) is well developed (cf. [22, Section 1.5],[36, Section 2.5]). There exists a unique solution $\bar{u} \in L^2(\Gamma)$, and a necessary and sufficient optimality condition is given by

$$\begin{aligned} -\Delta\bar{y} + \bar{y} &= f \quad \text{in } \Omega & -\Delta\bar{p} + \bar{p} &= \bar{y} - y_d \quad \text{in } \Omega,\\ \partial_n\bar{y} &= \bar{u} \quad \text{on } \Gamma & \partial_n\bar{p} &= 0 \qquad\quad \text{on } \Gamma,\\ \int_\Gamma (\bar{p}(x) + \alpha\bar{u}(x))(u(x)-\bar{u}(x))\mathrm{d}s_x &\ge 0 & &\forall u \in U_{ad}, \end{aligned} \tag{1.4}$$

where $\bar{y}$ and $\bar{p}$ denote the state and adjoint state related to $\bar{u}$. Note that the boundary value problems in (1.4) have to be understood in the weak sense. The variational inequality is equivalent to the projection formula

$$\bar{u} = \Pi_{ad}\left(-\alpha^{-1}\bar{p}|_\Gamma\right), \tag{1.5}$$

where $\bar{p}|_\Gamma$ denotes the trace of $\bar{p}$ on Γ and

$$[\Pi_{ad}\, v](x) := \max\{a(x), \min\{b(x), v(x)\}\} \quad \text{a.e. on } \Gamma$$

denotes the $L^2(\Gamma)$-projection onto the set of admissible controls.

The paper is structured as follows: In Section 2 we consider the variational discretisation approach applied to the problem (1.1)–(1.3). An analogue to Formula (1.5) will give a relation between the discrete adjoint state and the discrete control. We refer to the books of Tröltzsch [36] and Hinze et al. [22] for a detailed discussion of optimality conditions and discretisation approaches for optimal control problems. The regularity of the solution of the state equation (1.2) is discussed in Section 3. There, weighted Sobolev spaces are introduced that allow to acquire the singular parts in the solution more accurately. Finally, a modified shift-theorem in these spaces is presented. Section 4 approaches the finite element error estimates for the state equation required to show the estimates for the optimal control problem (1.1)–(1.3). More precisely, we derive estimates in $L^2(\Omega)$-, $H^1(\Omega)$- and $L^2(\Gamma)$-norm, where the latter one is the first main result of this paper. The second main result – an optimal error estimate for the Neumann control problem – is presented in Section 5 which is confirmed numerically in Section 6.

2 Discretisation of the optimal control problem

For the discretisation of our optimal control problem (1.1)–(1.3) a couple of approaches exist. In this paper we will consider the variational discretisation approach [20] only. This approach employs discretization of the state and adjoint state variable but not of the control. The latter one is discretised implicitly by means of a discrete version of the projection formula (1.5).

In the following, the boundary value problems in (1.4) are considered in the weak sense. Hence, we define the bilinear form

$$a(v,w) := \int_\Omega [\nabla v(x)\cdot\nabla w(x) + v(x)w(x)]\,\mathrm{d}x$$

and the inner products $(\cdot,\cdot)_\Gamma$ in $L^2(\Gamma)$ and $(\cdot,\cdot)_\Omega$ in $L^2(\Omega)$, respectively. The weak form of the state equation (1.2) is then given by

$$a(y,v) = (f,v)_\Omega + (u,v)_\Gamma \qquad \forall v\in V := H^1(\Omega). \tag{2.1}$$

It is well-known that the Lax-Milgram lemma states the existence and uniqueness of a solution in $H^1(\Omega)$.

We are going to discretise problem (2.1) with continuous and piecewise linear finite elements. To this end, let $\{\mathcal{T}_h\}_{h<1}$ be a family of conforming tetrahedral meshes of Ω which are assumed to be *shape regular*. The mesh parameter h denotes the maximal diameter of all elements in $\mathcal{T}_h$, i. e.

$$h := \max_{T\in\mathcal{T}_h} \operatorname{diam} T.$$

The ansatz space for state and adjoint state is defined by

$$V_h := \left\{ v_h \in C(\overline{\Omega})\colon v_h|_T \in \mathcal{P}_1(T) \ \text{ for all } T\in\mathcal{T}_h \right\}. \tag{2.2}$$

Discretising the optimality system (1.4) yields now
Find $(\bar y_h, \bar u_h, \bar p_h) \in V_h\times U_{ad}\times V_h$*:*

$$\begin{aligned} a(\bar y_h, v_h) &= (f,v_h)_\Omega + (\bar u_h, v_h)_\Gamma && \forall v_h\in V_h,\\ a(v_h,\bar p_h) &= (\bar y_h - y_d, v_h)_\Omega && \forall v_h \in V_h,\\ \bar u_h &= \Pi_{ad}(-\alpha^{-1}\bar p_h|_\Gamma). \end{aligned} \tag{2.3}$$

This is a finite-dimensional system which possesses a unique solution and can be solved e. g. with the primal-dual active set strategy, the semi-smooth Newton method or the projected gradient algorithm [22, 36].

In the following, $S\colon L^2(\Gamma)\to L^2(\Omega)$ stands for the control-to-state mapping which maps a given control $u\in L^2(\Gamma)$ to the solution of the state equation (1.2) and we write $y = Su$. The operator $S_h\colon L^2(\Gamma)\to V_h\subset L^2(\Omega)$ stands for its discrete version defined by $S_h u := y_h$. Moreover, let S^* and S_h^* denote the adjoint operators to S and S_h, respectively. This allows us to write

$$\bar p|_\Gamma := S^*(S\bar u - y_d) \qquad \text{and} \qquad \bar p_h|_\Gamma := S_h^*(S_h\bar u_h - y_d). \tag{2.4}$$

The following convergence result for the variational discretisation concept is proven in [21]:

Lemma 1. *Let $\bar{u}$ be the solution of the continuous optimal control problem (1.1)–(1.3) and $\bar{u}_h$ the solution of the discrete problem given by*

$$J_h(u) := J(S_h u, u) \to \min! \qquad s.\,t. \qquad u \in U_{ad}. \tag{2.5}$$

Then, (2.3) forms a necessary optimality system for (2.5), and there exists a constant $c > 0$ such that

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq c \left(\|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} \right). \tag{2.6}$$

It remains to derive error estimates for the two terms on the right-hand side of (2.6) which will be done in Section 4. For the first term it is well-known that the convergence rate 2 is achieved on convex polyhedra [19] and we will see that this rate can be achieved with appropriate mesh refinement also for non-convex domains. The second error term in (2.6) is the finite element error for the adjoint equation evaluated on the boundary Γ only. Estimates of this kind are rather less standard and require very technical proofs. This will be considered in the second part of Section 4.

Remark 2. *Another often-used discretisation approach is the post-processing concept [29, 26]. The idea is to approximate state and adjoint state as usual by linear finite elements and the control variable by piecewise constant functions. The obtained solution $\bar{u}_h$ of the finite-dimensional problem can be improved by an application of the projection formula $\tilde{u}_h := \Pi_{ad}(-\alpha^{-1}\bar{p}_h)$. Using the new estimates of the present paper it should be possible to improve the convergence order for this concept either. Since several other terms have to be estimated, a complete proof would exceed the scope of this paper and is subject of a forthcoming one. In two space-dimensions optimal error estimates can be found in [6] which rely on the optimal finite element error estimates in $L^2(\Gamma)$ as well.*

3 Regularity of weak solutions

Let us first specify some notation. Throughout the paper, Ω is an open bounded polyhedron with corners $x^{(j)}$, $j \in \mathcal{C} := \{1, \ldots, d'\}$, edges M_k, $k \in \mathcal{E} := \{1, \ldots, d\}$, and plane faces F_ℓ, $\ell \in \mathcal{F} := \{1, \ldots, d^*\}$. Furthermore, let $X_j := \{k \colon x^{(j)} \in \overline{M}_k\}$ be the index set of all edges M_k having an endpoint in $x^{(j)}$. The boundary of Ω is denoted by Γ.

The aim of this section is to collect some regularity results for the solution of the boundary value problem

$$-\Delta y + y = f \quad \text{in } \Omega, \qquad\qquad \partial_n y = g \quad \text{on } \Gamma. \tag{3.1}$$

It is well known that if the input data satisfy $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, the solution possesses the regularity $y \in H^2(\Omega)$ provided that the boundary of Ω is smooth or convex and polyhedral, but this is not the case for domains with reentrant corners and edges. As a remedy one can use weighted Sobolev spaces, because these spaces contain weights that compensate the singularities that occur at corners and edges. Later, our assumptions upon the finite element mesh and the weights will depend on the singular exponents of the occurring singularities. Therefore, we associate to each edge M_k such an exponent λ_k^e and to each corner $x^{(j)}$ an exponent λ_j^c, and summarise them to vectors $\vec{\lambda}^e \in \mathbb{R}^d$ and $\vec{\lambda}^c \in \mathbb{R}^{d'}$, respectively. In case of the differential operator $-\Delta + I$ and pure Neumann boundary conditions the edge singular exponents λ_k^e are explicitly given by $\lambda_k^e := \pi/\omega_k$ (cf. [18, Section 2.5]), where ω_k

denotes the interior angle between the two faces intersecting at M_k. It is known that the occurring singularities can be described in cylinder coordinates (r_k, φ_k, z_k) where the z_k-axis coincides with M_k and $\varphi_k = 0$ on one of the faces:

$$S_k^e(r_k, \varphi_k, z_k) := r_k^{\lambda_k^e} \cos(\lambda_k^e \varphi_k).$$

Obviously, the singular functions are independent of the z_k-direction. However, the corner singularities are described in spherical coordinates $(\rho_j, \varphi_j, \theta_j)$ around $x^{(j)}$ and have the structure [18, Theorem 2.6.3]

$$S_j^c(\rho_j, \varphi_j, \theta_j) := \rho_j^{\lambda_j^c} F_j(\varphi_j, \theta_j), \qquad \text{where } \lambda_j^c := -1/2 + \sqrt{\lambda_j^{c,(2)} + 1/4}.$$

Here, $\lambda_j^{c,(2)}$ is the second-smallest eigenvalue (the first one is always zero for the Neumann problem) and F_j the corresponding eigenfunction of the Laplace-Beltrami operator $-\Delta_{\mathcal{G}_j}$, where $\mathcal{G}_j$ denotes the intersection of the polyhedral cone corresponding to the corner $x^{(j)}$ and the unit sphere centred at $x^{(j)}$. We thus have to assume that the corners have at least distance two from each other which can be achieved with appropriate scaling. More precisely, $\left(\lambda_j^{c,(2)}, F_j\right)$ solves the eigenvalue problem

$$-\Delta_{\mathcal{G}_j} F = \lambda F \quad \text{in} \quad \mathcal{G}_j, \qquad \partial_n F = 0 \quad \text{on} \quad \partial\mathcal{G}_j. \tag{3.2}$$

For further details we refer to [18, Section 2.6].

Remark 3. *The eigenvalue problem (3.2) can be computed exactly for special cases only. We want to mention Stephan and Whiteman [35] who derived* $\lambda_j^{c,(2)} = 40/9$ *(*$\Rightarrow \lambda_j^c = 5/3$*) for the three-dimensional L-shape domain which is considered in the numerical experiments in Section 6.*

In general, the eigenvalue $\lambda_j^{c,(2)}$ *has to be computed approximately. Walden and Kellogg [38] and Beagles and Whiteman [10] present a way to solve (3.2) numerically using a finite difference method combined with Rayleigh quotient minimisation for the discrete eigenvalue problem. In the latter reference this technique was applied to the "Fichera domain" which denotes a domain around a corner at the intersection of three mutually orthogonal planes. In the numerical experiments of this reference the exponent* $\lambda_j^c \in (0.4335, 0.4576)$ *was computed approximately for the Dirichlet problem. However, the exponent* λ_j^c *depends on the type of boundary condition. For a pure Neumann boundary we have computed* $\lambda_j^c \approx 0.84$ *with the software package CoCoS [30, 31].*

We introduce now weighted Sobolev spaces and use the definition of Maz'ya and Rossmann [28]. Let U_j, $j \in \mathcal{C}$, be a covering of Ω with

$$\bigcup_{j=1}^{d'} U_j \supset \bar{\Omega} \qquad \text{and} \qquad \bar{U}_j \cap \bar{M}_k = \emptyset \text{ if } k \notin X_j.$$

By $r_k(x)$ we denote the distance of x to the edge M_k and by $\rho_j(x)$ the distance to the corner $x^{(j)}$. For some given weights $\vec{\beta} \in \mathbb{R}^{d'}, \vec{\delta} \in \mathbb{R}^d$ and a non-negative integer $\ell \in \mathbb{N}_0$ we define the space $W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)$ as the closure of $C_0^\infty(\bar{\Omega} \backslash \{x^{(1)}, \ldots, x^{(d')}\})$ with respect to the norm

$$\|v\|_{W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)} := \left(\sum_{j=1}^{d'} \int_{\Omega \cap U_j} \sum_{|\alpha| \le \ell} \rho_j(x)^{p(\beta_j - \ell + |\alpha|)} \prod_{k \in X_j} \left(\frac{r_k}{\rho_j}(x) \right)^{p\delta_k} |D^\alpha v(x)|^p \mathrm{d}x \right)^{1/p}, \tag{3.3}$$

if $p \in [1, \infty)$, and

$$\|v\|_{W^{\ell,\infty}_{\vec\beta,\vec\delta}(\Omega)} := \sum_{|\alpha|\le\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in\Omega\cap U_j} \rho_j(x)^{\beta_j-\ell+|\alpha|} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{\delta_k} |D^\alpha v(x)|. \tag{3.4}$$

if $p = \infty$. Corresponding semi-norms of $W^{\ell,p}_{\vec\beta,\vec\delta}(\Omega)$ are given by

$$|v|_{W^{\ell,p}_{\vec\beta,\vec\delta}(\Omega)} := \left(\sum_{j=1}^{d'} \int_{\Omega\cap U_j} \sum_{|\alpha|=\ell} \rho_j(x)^{p\beta_j} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{p\delta_k} |D^\alpha v(x)|^p \mathrm{d}x \right)^{1/p},$$

$$|v|_{W^{\ell,\infty}_{\vec\beta,\vec\delta}(\Omega)} := \sum_{|\alpha|=\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in\Omega\cap U_j} \rho_j(x)^{\beta_j} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{\delta_k} |D^\alpha v(x)|.$$

The related trace spaces $W^{\ell-1/p,p}_{\vec\beta,\vec\delta}(\Gamma)$ are induced by the natural norm

$$\|v\|_{W^{\ell-1/p,p}_{\vec\beta,\vec\delta}(\Gamma)} := \inf\{\|u\|_{W^{\ell,p}_{\vec\beta,\vec\delta}(\Omega)} : u|_{\Gamma\setminus\{x^{(1)},\ldots,x^{(d')}\}} = v\}. \tag{3.5}$$

In this paper we will frequently use the weighted Sobolev spaces $W^{\ell,p}_{\vec\beta,\vec\delta}(\mathcal{G})$ on some subset $\mathcal{G} \subset \Omega$. These are defined analogously except that the weights contained in the norm definition are still related to the corners and edges of Ω. Note that the weights related to corners and edges far away from $\mathcal{G}$ may be omitted.

The regularity of the weak solution of problem (3.1) in weighted Sobolev spaces is proven in [1, 13, 28].

Theorem 4. *Let be given some functions $f \in W^{0,p}_{\vec\beta,\vec\delta}(\Omega)$ and $g \in W^{1-1/p,p}_{\vec\beta,\vec\delta}(\Gamma)$ with $p \in (1, \infty)$. Assume that the edge and corner weights $\vec\delta$ and $\vec\beta$ satisfy*

$$\begin{aligned} &2 - 2/p - \min\{2, \lambda^e_k\} < \delta_k < 2 - 2/p && \textit{for all } k \in \mathcal{E}, \\ &2 - 3/p - \min\{1, \lambda^c_j\} < \beta_j < 3 - 3/p && \textit{for all } j \in \mathcal{C}. \end{aligned}$$

Then, the weak solution $y \in H^1(\Omega)$ of the boundary value problem (3.1) satisfies

$$D^\alpha y \in W^{1,p}_{\vec\beta,\vec\delta}(\Omega) \qquad \forall |\alpha| = 1.$$

Moreover, if $\vec\beta, \vec\delta \ge 0$, the a priori *estimate*

$$\sum_{|\alpha|=1} \|D^\alpha y\|_{W^{1,p}_{\vec\beta,\vec\delta}(\Omega)} + \|y\|_{L^p(\Omega)} \le c \left(\|f\|_{W^{0,2}_{\vec\beta,\vec\delta}(\Omega)} + \|g\|_{W^{1-1/p,p}_{\vec\beta,\vec\delta}(\Gamma)} \right) \tag{3.6}$$

holds.

Proof. The desired assertion is stated in Theorem 8.1.10 of [28] under the additional assumption that $\lambda = -1$ and $\lambda = 0$ are the only eigenvalues of the problem

$$-\Delta_{\mathcal{G}_j} v = \lambda(\lambda + 1)v \text{ in } \mathcal{G}_j, \qquad \partial_n v = 0 \text{ on } \partial\mathcal{G}_j,$$

that are contained in the strip $-1 \leq Re\,\lambda \leq 0$. Note, that this eigenvalue problem is the same as (3.2) when inserting the definition of λ_j^c (compare also [24, Equation (2.3.3)]). That this strip indeed contains only the eigenvalues 0 and -1 in our situation, and, that algebraic and geometric multiplicity are equal, has been discussed in [24, Section 2.3.1].

It remains to prove the *a priori* estimate (3.6) which is not directly stated in [28], but in the following, we outline how this estimate can be concluded. To this end, introduce the space

$$\mathcal{H} := \left\{ v \in L^p(\Omega) \colon D^\alpha v \in W^{1,p}_{\vec{\beta},\vec{\delta}} \quad \forall |\alpha| = 1 \right\}$$

with the naturally induced norm as stated on the left-hand side of (3.6), and the operator

$$\mathcal{A} := \begin{pmatrix} -\Delta + I \\ \partial_n \end{pmatrix} : \mathcal{H} \to W^{0,2}_{\vec{\beta},\vec{\delta}}(\Omega) \times W^{1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma).$$

It is easy to observe that the operator $\mathcal{A}$ is linear and bounded since the estimates

$$\begin{aligned} \|-\Delta u\|_{W^{0,p}_{\vec{\beta},\vec{\delta}}(\Omega)} &\leq c|u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(\Omega)}, \\ \|u\|_{W^{0,p}_{\vec{\beta},\vec{\delta}}(\Omega)} &\leq c\|u\|_{L^p(\Omega)}, \\ \|\partial_n u\|_{W^{1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Omega)} &\leq c \sum_{|\alpha|=1} \|D^\alpha u\|_{W^{1,p}_{\vec{\beta},\vec{\delta}}(\Omega)}, \end{aligned}$$

hold for arbitrary $u \in \mathcal{H}$. More precisely, the first estimate follows directly from the norm definition (3.3), the second one from a trivial embedding taking into account that $\vec{\beta}, \vec{\delta} \geq 0$, and the third one from the definition of the trace space (3.5). We also confirm that $\mathcal{A}$ is bijective which is equivalent to the existence and uniqueness of a solution in $\mathcal{H}$ and follows from the first part of this theorem and the Lax-Milgram Lemma. From the *bounded inverse theorem* [16, Theorem 3.7] we conclude that the inverse mapping $\mathcal{A}^{-1}$ is also continuous which is equivalent to (3.6). $\square$

The above theorem excludes $p = \infty$, but this case is needed since we have to exploit regularity in $W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega)$ in order to obtain optimal error estimates on the boundary. To overcome this issue we apply regularity results in weighted Hölder spaces.

Theorem 5. *Let be given a function $f \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$ and let $g \equiv 0$. Assume that the weights $\vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\delta} \in \mathbb{R}^d$ satisfy*

$$\begin{aligned} &\delta_k \geq 0, && 2 - \lambda_k^e < \delta_k < 2 && \text{for all } k \in \mathcal{E}, \\ &\beta_j \geq 0, && 2 - \lambda_j^c < \beta_j && \text{for all } j \in \mathcal{C}. \end{aligned}$$

Then, the weak solution $y \in H^1(\Omega)$ of (3.1) satisfies

$$D^\alpha y \in W^{1,\infty}_{\vec{\beta},\vec{\delta}}(\Omega) \qquad \forall |\alpha| = 1.$$

Proof. Let us first introduce the weighted Hölder spaces defined in [28, Section 8.2]. We denote by $U_{j,k} := \{x \in U_j \cap \Omega \colon r_k < 3\rho_j(x)/2\}$ for $k \in X_j$ a covering of U_j. Furthermore, a Hölder exponent $\sigma \in (0,1)$, a non-negative integer $\ell \in \mathbb{N}_0$ and some weights $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with $\delta_k \geq 0$ ($k \in \mathcal{E}$) are given. To each edge we associate the integer $m_k := [\delta_k - \sigma] + 1$.

The weighted Hölder space $C^{\ell,\sigma}_{\vec\beta,\vec\delta}(\Omega)$ denotes the space of ℓ times continuously differentiable functions on $\tilde\Omega := \Omega \setminus (\bigcup_{k\in\mathcal{E}} \overline{M}_k)$ with finite norm

$$\begin{aligned} \|u\|_{C^{\ell,\sigma}_{\vec\beta,\vec\delta}(\Omega)} :=& \sum_{j=1}^{d'} \sum_{|\alpha|\le\ell} \sup_{x\in U_j} \rho_j(x)^{\beta_j-\ell-\sigma+|\alpha|} \prod_{k\in X_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\max\{0,\delta_k-\ell-\sigma+|\alpha|\}} |(D^\alpha u)(x)| \\ &+ \sum_{j=1}^{d'} \sum_{k\in X_j} \sum_{|\alpha|=\ell-m_k} \sup_{\substack{x,y\in U_{j,k}\\ |x-y|<\rho_j(x)/2}} \rho_j(x)^{\beta_j-\delta_k} \frac{|(D^\alpha u)(x)-(D^\alpha u)(y)|}{|x-y|^{m_k+\sigma-\delta_k}} \\ &+ \sum_{j=1}^{d'} \sum_{|\alpha|=\ell} \sup_{\substack{x,y\in U_j\\ |x-y|<\rho_j(x)/2}} \rho_j^{\beta_j}(x) \prod_{k\in X_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\delta_k} \frac{|(D^\alpha u)(x)-(D^\alpha u)(y)|}{|y-x|^\sigma}. \end{aligned} \tag{3.7}$$

We introduce weights $\vec\beta' := \vec\beta + \sigma$ and $\vec\delta' := \vec\delta + \sigma$ and observe that the inequalities

$$\|f\|_{C^{0,\sigma}_{\vec\beta',\vec\delta'}(\Omega)} \le c\|f\|_{C^{0,\sigma}_{\vec\sigma,\vec 0}(\Omega)} \le c\|f\|_{C^{0,\sigma}(\Omega)}$$

hold, where the first inequality is a consequence of the embedding theorem [28, Lemma 8.2.1] which holds for arbitrary $\beta_j' \ge \sigma$ $(j\in\mathcal{C})$ and $\delta_k' \ge 0$ $(k\in\mathcal{E})$, and the second one can be confirmed when inserting $\beta_j = \sigma$ and $\delta_k = 0$ in the definition of the norm (3.7).

The assumptions upon $\vec\beta$ and $\vec\delta$ imply that

$$2-\lambda_k^e < \delta_k' - \sigma < 2,\ k\in\mathcal{E}, \quad \text{and} \quad 2-\lambda_j^c < \beta_j' - \sigma,\ j\in\mathcal{C}, \tag{3.8}$$

and with the regularity result from [27, Theorem 5.1 and Remark 5.1] we obtain $D^\alpha y \in C^{1,\sigma}_{\vec\beta',\vec\delta'}(\Omega)$ for all $|\alpha|=1$. It remains to show that

$$\|D^\alpha u\|_{W^{1,\infty}_{\vec\beta,\vec\delta}(\Omega)} \le c\|D^\alpha u\|_{C^{1,\sigma}_{\vec\beta',\vec\delta'}(\Omega)} \qquad \forall|\alpha|=1.$$

It suffices to bound the $W^{1,\infty}_{\vec\beta,\vec\delta}(\Omega)$-norm by the first row in the norm definition (3.7). Obviously, when inserting $\beta_j = \beta_j' - \sigma$, the corner weights coincide. Inserting $\delta_k = \delta_k' - \sigma \ge 0$ yields for the edge weights

$$\left(\frac{r_k}{\rho_j}\right)^{\delta_k} = \left(\frac{r_k}{\rho_j}\right)^{\delta_k'-\sigma} \le c\left(\frac{r_k}{\rho_j}\right)^{\max\{0,\delta_k'-\sigma-1+|\alpha|\}},$$

where we exploited $\delta_k' - \sigma \ge \max\{0, \delta_k'-\sigma-1+|\alpha|\}$ for all $|\alpha|\le 1$. Consequently, we have shown (3.8) and the assertion can be concluded. □

In order to derive finite element error estimates we have to apply several embeddings which are summarized in the following lemma. A proof can e. g. be found in Lemma 8.1.1 and Lemma 8.1.2 of [28].

Lemma 6. *Let be $\mathcal{G}\subset\Omega$. The following embeddings hold:*

1. *Let $1<q<p\le\infty$. Assume that the weights satisfy $\beta_j + 3/p < \beta_j' + 3/q$ for $j\in\mathcal{C}$, and $0<\delta_k+2/p<\delta_k'+2/q$ for $k\in\mathcal{E}$. Then the continuous embedding*

$$W^{\ell,p}_{\vec\beta,\vec\delta}(\mathcal{G}) \hookrightarrow W^{\ell,q}_{\vec\beta',\vec\delta'}(\mathcal{G})$$

holds.

2. *Assume that $p \in [1,\infty)$, $\beta_j \le 1+\beta_j'$ for $j \in \mathcal{C}$, and $\delta_k \le 1+\delta_k'$, such as $\delta_k, \delta_k' > -2/p$ for $k \in \mathcal{E}$. Then the continuous embedding*

$$W^{\ell+1,p}_{\vec\beta,\vec\delta}(\mathcal{G}) \hookrightarrow W^{\ell,p}_{\vec\beta',\vec\delta'}(\mathcal{G})$$

holds. In case of $\beta_j < 1+\beta_j'$, $j \in \mathcal{C}$, and $\delta_k < 1+\delta_k'$, $k \in \mathcal{E}$, the embedding is even compact.

Remark 7. *In [28] the case $p=\infty$ was excluded for the first part of Lemma 6. However, by consideration of the Hölder inequality one can easily confirm the validity of this assertion for $p=\infty$.*

4 Finite element error estimates for the boundary value problem

In this section, we will derive finite element error estimates for the boundary value problem

$$-\Delta y + y = f \quad \text{in} \quad \Omega, \qquad \partial_n y = g \quad \text{on} \quad \Gamma. \tag{4.1}$$

Let us recall, e. g. from [7], the family of graded meshes we are going to investigate. Denote the distance of $T \in \mathcal{T}_h$ to the edge M_k by $r_{k,T} := \operatorname{dist}(T, M_k)$ and the minimum distance to all singular points by $r_T := \min_{k\in\mathcal{E}} r_{k,T}$. For a given mesh grading parameter $\mu \in (0,1]$ and refinement radius $R>0$ the triangulation has to satisfy the condition

$$h_T \sim \begin{cases} h^{1/\mu}, & \text{if} \;\; r_T = 0,\\ h r_T^{1-\mu}, & \text{if} \;\; 0 < r_T < R,\\ h, & \text{if} \;\; R \le r_T, \end{cases} \tag{4.2}$$

for all $T \in \mathcal{T}_h$. This is a natural refinement condition which ensures that adjacent elements have approximately the same size. In case of a quasi-uniform mesh we simply set $\mu = 1$. The smaller this parameter is, the stronger the mesh is refined locally. If a mesh is refined according to (4.2), the number of elements contained in the triangulation is of order $\mathcal{O}(h^{-3})$ unless $\mu < 1/3$, compare also the discussions of Apel et al. [7].

Remark 8. *The reader will already find $L^2(\Omega)$-error estimates for the refinement strategy (4.2) in [7], but this reference deals only with Dirichlet and certain mixed problems. The techniques used in this reference do not apply in our case since usually for the Dirichlet problem different weighted Sobolev spaces are employed to describe the regularity, see e.g. [28] for more sophisticated discussions.*

The first step is to derive some local interpolation error estimates that are used later to prove the convergence order of the finite element method both in the domain and on the boundary. We will use two different interpolation operators. For approximation errors in $L^\infty(\Omega)$, the usual Lagrange interpolant $I_h\colon C(\bar\Omega) \to V_h$ is the preferred operator due to its stability in $L^\infty(\Omega)$. However, the Lagrange interpolant is not always sufficient for our purposes. Thus, at some places we apply the quasi-interpolant $Z_h\colon W^{1,p}(\Omega) \to V_h$,

$p \in [1, \infty]$, as originally introduced by Scott and Zhang [34], which is defined even for non-smooth functions in Sobolev spaces $W^{1,p}(\Omega)$ by

$$[Z_h v](x) := \sum_{i=1}^{n} a_i \varphi_i(x).$$

Here, the functions φ_i denote the nodal basis functions. More precisely, when $\{x^i\}_{i=1}^n$ are the nodes of the triangulation $\mathcal{T}_h$, we have $\varphi_i(x_j) = \delta_{i,j}$ for all $i, j = 1, \ldots, n$. The difference to the Lagrange interpolant is the choice of the coefficients a_i which are defined by $a_i := (\Pi_{\sigma_i} v)(x^i)$ with

$$\Pi_{\sigma_i} \colon L^2(\sigma_i) \to \mathcal{P}_1(\sigma_i), \qquad (v - \Pi_{\sigma_i} v, w_h)_{L^2(\sigma_i)} = 0 \quad \forall w_h \in \mathcal{P}_1(\sigma_i).$$

It remains to specify the subsets σ_i. We will follow the choice used already in [34]:

- If x^i is an interior node, we choose $\sigma_i = T$ with some $T \in \mathcal{T}_h$ containing x^i.
- If x^i is a boundary node, we choose $\sigma_i = F \subset \Gamma$ where $F \ni x^i$ is a face of some $T \in \mathcal{T}_h$.

An essential advantage of Z_h over the usual Lagrange interpolant is the better stability property which is proved in [34]:

Lemma 9. *For some $T \in \mathcal{T}_h$ let S_T denote the patch of elements around T, i.e.*

$$S_T := \operatorname{int} \bigcup_{\substack{T' \in \mathcal{T}_h \\ \bar{T}' \cap \bar{T} \neq \emptyset}} \overline{T'}.$$

Then, for any integer $\ell \geq 1$, $p, q \in [1, \infty]$, and non-negative integer m, the following stability estimate holds:

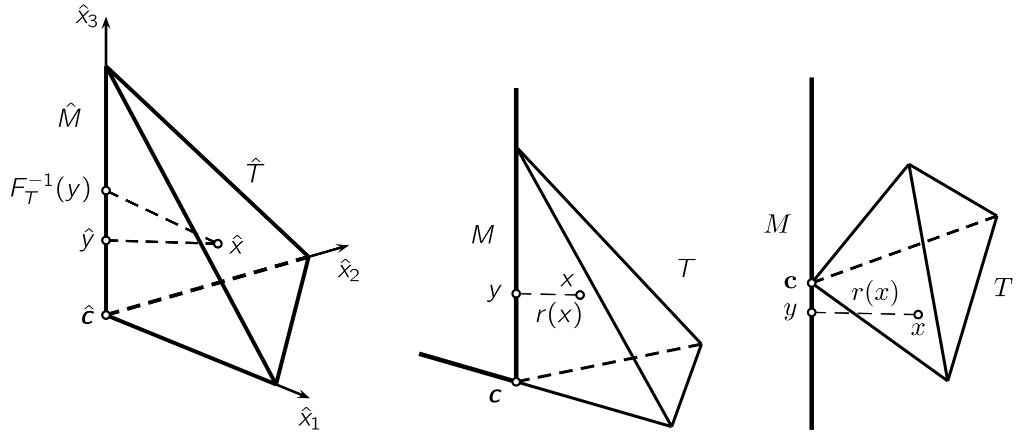
$$\|Z_h v\|_{W^{m,q}(T)} \leq c|T|^{1/q - 1/p} h_T^{-m} \sum_{j=0}^{\ell} h_T^j |v|_{W^{j,p}(S_T)} \qquad \forall v \in W^{\ell,p}(S_T). \tag{4.3}$$

Let $\hat{T}$ be the standard reference tetrahedron having vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (see Figure 1a). We denote by F_T the affine linear transformation from $\hat{T}$ to a world element $T \in \mathcal{T}_h$. The weighted Sobolev spaces on a reference setting are defined analogous to $W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)$ with slight modifications of the weight functions that we define by

$$\hat{\rho} := |\hat{x}|, \qquad \hat{r}(\hat{x}) := \operatorname{dist}(\hat{x}, \hat{M}),$$

where $\hat{M} := \{t\mathbf{e}_3 \colon t \in (0,1)\}$ is the reference edge. For simplification we assume throughout this paper that each element T touches at most one corner of Ω and has at most one edge which is contained in an edge of Ω. If an element violates this condition it has to be bisected appropriately. Analogous to (3.3) we define a norm of $W^{\ell,p}_{\beta,\delta}(\hat{T})$ with β, $\delta \in \mathbb{R}$ by

$$\|\hat{u}\|_{W^{\ell,p}_{\beta,\delta}(\hat{T})} := \left(\sum_{|\alpha| \leq \ell} \int_{\hat{T}} \hat{\rho}(\hat{x})^{p(\beta - \ell + |\alpha|)} \left(\frac{\hat{r}}{\hat{\rho}}(\hat{x}) \right)^{p\delta} |\hat{D}^\alpha \hat{u}(\hat{x})|^p \mathrm{d}\hat{x} \right)^{1/p}.$$



(a) Reference element

(b) Element touching an edge and a corner

(c) Element touching an edge in a single point

Figure 1: The reference element $\hat{T}$ and the different positions of the original element T.

Let us briefly discuss the relation between the weights in the reference setting and the original weights. In the following $\hat{x} \in \hat{T}$ is arbitrary and we set $x = F_T(\hat{x})$. For the case illustrated in Figure 1b that one edge of T is contained in an edge M of Ω, we define the points

$$y = \operatorname*{arg\,min}_{z \in M} |x - z|, \qquad \hat{y} = \operatorname*{arg\,min}_{\hat{z} \in \hat{M}} |\hat{x} - \hat{z}|.$$

Note that y and $F_T(\hat{y})$ are in general different points. For the transformation from $\hat{T}$ to T we have to exploit the property

$$r(x) = |x - y| \sim h_T|\hat{x} - F_T^{-1}(y)| \sim h_T|\hat{x} - \hat{y}| = h_T\hat{r}(\hat{x}), \tag{4.4}$$

where the second equivalence holds due to the assumed *shape regularity* of $\mathcal{T}_h$. Moreover, if T touches also the corner $\mathbf{c} := x^{(j)}$, we observe that

$$\rho(x) = |x - \mathbf{c}| \sim h_T|\hat{x} - \hat{\mathbf{c}}| = h_T\hat{\rho}(\hat{x}), \tag{4.5}$$

since $\mathbf{c} = F_T(\hat{\mathbf{c}})$. The case illustrated in Figure 1c where T touches the edge M only in a single point $\mathbf{c}$ is treated slightly different. Here, we get the property

$$r(x) = |x - y| \sim |x - \mathbf{c}| \sim h_T|\hat{x} - \hat{\mathbf{c}}| = h_T\hat{\rho}(\hat{x}), \tag{4.6}$$

and the edge weight becomes a corner weight in the reference setting. Note, that we did not consider weights related to corners and edges that are not touched by T, since these weights are not needed for our analysis.

In order to prove interpolation error estimates, the key step is the application of the Bramble-Hilbert Lemma on a reference element and we will need the following version in weighted Sobolev spaces.

Lemma 10. *Let be given some function $v \in W^{\ell,q}_{\beta,\delta}(\hat{T})$ with $q \in (1,\infty)$, a positive integer ℓ, and weights β, $\delta \in \mathbb{R}$ satisfying*

$$\beta < 4 - 3/q, \qquad -2/p < \delta < 3 - 2/q. \tag{4.7}$$

Then, a polynomial $p \in \mathcal{P}_{\ell-1}(\hat{T})$ of order $\ell - 1$ and some $c > 0$ exist such that

$$\|v - p\|_{W^{\ell,q}_{\beta,\delta}(\hat{T})} \leq c|v|_{W^{\ell,q}_{\beta,\delta}(\hat{T})}$$

holds.

Proof. It suffices to show that the norm equivalence

$$\|u\|_{W^{\ell,q}_{\beta,\delta}(\hat{T})} \sim |u|_{W^{\ell,q}_{\beta,\delta}(\hat{T})} + \sum_{|\alpha|\leq \ell-1} \left| \int_{\hat{T}} D^\alpha u(x) \mathrm{d}x \right|. \tag{4.8}$$

holds and to insert $u = v - p$ with some $p \in \mathcal{P}_{\ell-1}(\hat{T})$ such that the second part on the right-hand side vanishes. In Lemma 2.2 of [8] the equivalence (4.8) has been shown for a slightly different space. However, the key steps of the proof therein are the embeddings

$$W^{\ell,q}_{\beta,\delta}(\hat{T}) \overset{c}{\hookrightarrow} W^{\ell-1,q}_{\beta,\delta}(\hat{T}) \quad \text{and} \quad W^{1,q}_{\beta,\delta}(\hat{T}) \hookrightarrow W^{1,1}_{1,1}(\hat{T}) \hookrightarrow L^1(\hat{T}). \tag{4.9}$$

The first embedding is stated in part two of Lemma 6 and holds under the assumption $-2/q < \delta$. The second embedding in (4.9) is also a consequence of Lemma 6 and holds under the assumption (4.7). $\square$

The local interpolation error estimates depend on the position of the element $T \in \mathcal{T}_h$ wherefore we introduce the quantities

$$\rho_{j,T} := \operatorname{dist}(x^{(j)}, T) \quad j \in \mathcal{C}, \qquad r_{k,T} := \operatorname{dist}(M_k, T) \quad k \in \mathcal{E}, \qquad r_T := \min_{k\in\mathcal{E}} r_{k,T}.$$

Lemma 11. *Let $T \in \mathcal{T}_h$, $T \subset U_j$ for some $j \in \mathcal{C}$, and $u \in W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)$ with some $p \in (6/5, \infty]$ be given. Assume that the weight exponents satisfy $0 \leq \beta_j < 5/2 - 3/p$, and $0 \leq \delta_k < 5/3 - 2/p$ for all $k \in X_j$. Then, for $\ell = 0, 1$, the following interpolation error estimates hold:*

$$|u - Z_h u|_{H^\ell(T)} \leq$$
$$c h_T^{2-\ell} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \cdot \begin{cases} \rho_{j,T}^{-\beta_j} \prod\limits_{k\in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k}, & \text{if } \rho_{j,S_T} > 0,\ r_{k,S_T} > 0\ (\forall k \in X_j), \\ h_T^{-\delta_k} \rho_{j,T}^{\delta_k - \beta_j}, & \text{if } \rho_{j,S_T} > 0,\ r_{k,S_T} = 0, \\ h_T^{-\beta_j}, & \text{if } \rho_{j,S_T} = 0. \end{cases} \tag{4.10}$$

Furthermore, let $\kappa := \max\{\max_{k\in\mathcal{E}} \delta_k, \max_{j\in\mathcal{C}} \beta_j\}$ denote the largest weight. Then, the estimate above simplifies to

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \cdot \begin{cases} r_T^{-\kappa}, & \text{if } r_{S_T} > 0, \\ h_T^{-\kappa}, & \text{if } r_{S_T} = 0. \end{cases} \tag{4.11}$$

Proof. The estimate (4.10) in case of $r_{S_T} > 0$ follows from a standard interpolation error estimate since the regularity $u \in W^{2,p}(S_T)$ can be exploited. We may introduce the weights afterwards and obtain

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell} |T|^{1/2-1/p} \rho_{j,T}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)}. \tag{4.12}$$

Let us consider the case that S_T touches a singular point. For arbitrary first-order polynomials $w \in \mathcal{P}_1(S_T)$ we have $[Z_h w]|_T = w|_T$ and with the triangle inequality we get the decomposition

$$|u - Z_h u|_{H^\ell(T)} \le |u - w|_{H^\ell(T)} + |Z_h(u - w)|_{H^\ell(T)}. \tag{4.13}$$

We first consider the second part of (4.13). Using an inverse inequality, the transformation to the reference element and the stability of Z_h in $H^1(S_{\hat{T}})$ (compare Lemma 9) we get

$$\begin{aligned} |Z_h(u - w)|_{H^\ell(T)} &\le c h_T^{-\ell} \|Z_h(u - w)\|_{L^2(T)} \\ &\le c h_T^{-\ell} |T|^{1/2} \|\hat{Z}_h(\hat{u} - \hat{w})\|_{L^2(\hat{T})} \\ &\le c h_T^{-\ell} |T|^{1/2} \|\hat{u} - \hat{w}\|_{H^1(S_{\hat{T}})}. \end{aligned}$$

Here, $S_{\hat{T}}$ denotes the patch S_T transformed to the reference setting via the affine linear mapping F_T^{-1}. Note, that the patch $S_{\hat{T}}$ has diameter $d(S_{\hat{T}}) = \mathcal{O}(1)$ and contains a ball of radius $\rho(S_{\hat{T}}) = \mathcal{O}(1)$. For the first part of (4.13) the transformation to the reference element and the embedding $H^1(\hat{T}) \hookrightarrow L^2(\hat{T})$ yield

$$|u - w|_{H^\ell(T)} \le c h_T^{-\ell} |T|^{1/2} |\hat{u} - \hat{w}|_{H^\ell(\hat{T})} \le c h_T^{-\ell} |T|^{1/2} \|\hat{u} - \hat{w}\|_{H^1(S_{\hat{T}})}.$$

Using also the embedding $W^{2,6/5}(S_{\hat{T}}) \hookrightarrow H^1(S_{\hat{T}})$, the estimate (4.13) simplifies to

$$|u - Z_h u|_{H^\ell(T)} \le c h_T^{-\ell} |T|^{1/2} \|\hat{u} - \hat{w}\|_{W^{2,6/5}(S_{\hat{T}})}. \tag{4.14}$$

Now, we employ a Deny-Lions type argument, e. g. the version from Theorem 3.2 in [17] where the estimate depends only on $d(S_{\hat{T}})$ and $\rho(S_{\hat{T}})$, as well as the transformation back to S_T, and arrive at

$$|u - Z_h u|_{H^\ell(T)} \le c h_T^{-\ell} |T|^{1/2} |\hat{u}|_{W^{2,6/5}(S_{\hat{T}})} \le c h_T^{2-\ell} |T|^{1/2 - 5/6} |u|_{W^{2,6/5}(S_T)}, \tag{4.15}$$

where $|S_T| \sim |T|$ was exploited in the last step. Henceforth, we have to distinguish among the cases whether S_T touches a corner or only a single edge.

We first consider the case that S_T touches the edge M_k for some $k \in X_j$, but is away from the corners. The Hölder inequality with $q := 5p/6$ and $1/q + 1/q' = 1$ yields

$$\begin{aligned} \|v\|_{W^{0,6/5}(S_T)}^{6/5} &= \int_{S_T} r_k(x)^{6\delta_k/5} |v(x)|^{6/5} r_k(x)^{-6\delta_k/5} \mathrm{d}x \\ &\le \left(\int_{S_T} r_k(x)^{p\delta_k} |v(x)|^p \mathrm{d}x \right)^{6/(5p)} \left(\int_{S_T} r_k(x)^{-q' 6\delta_k/5} \mathrm{d}x \right)^{1/q'}. \end{aligned} \tag{4.16}$$

The second integral can be integrated exactly in cylinder coordinates (r_k, φ_k, z_k) around M_k and is bounded if $2 - q' 6\delta_k/5 > 0$. This condition is equivalent to $\delta_k < 5/3 - 2/p$ when inserting the definition of q and q'. As S_T is contained in a cylindrical sector around M_k having length and radius proportional h_T there exist constants $c_i > 0$, $i \in \{1, 2, 3\}$, such that

$$\begin{aligned} \left(\int_{S_T} r_k(x)^{-q' 6\delta_k/5} \mathrm{d}x \right)^{5/(6q')} &\le c \left(\int_{c_1}^{c_1 + c_2 h_T} \int_0^{c_3 h_T} r_k^{1 - q' 6\delta_k/5} \mathrm{d}r_k \mathrm{d}z_k \right)^{5/(6q')} \\ &\le c h_T^{-\delta_k} |T|^{5/6 - 1/p}, \end{aligned} \tag{4.17}$$

where we used $1/q' = 1 - 6/(5p)$ and $|T| \sim h_T^3$ in the last step. Inserting (4.16) with (4.17) into (4.15) leads to

$$|u - Z_h u|_{H^\ell(T)} \le c h_T^{2-\ell-\delta_k} |T|^{1/2-1/p} \rho_{j,T}^{\delta_k - \beta_j} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(S_T)}, \tag{4.18}$$

where we already inserted the remaining weights and exploited that S_T is away from the corner.

Let now S_T contain also the corner $\mathbf{c} := x^{(j)}$. Analogous to (4.16) we derive an embedding into an appropriate weighted Sobolev space and obtain using the Hölder inequality

$$\begin{aligned}
\|v\|^{6/5}_{W^{0,6/5}(S_T)} &= \int_{S_T} \rho_j(x)^{6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{6\delta_k/5} |v(x)|^{6/5} \rho_j(x)^{-6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{-6\delta_k/5} \mathrm{d}x \\
&\le c \left(\int_{S_T} \rho_j(x)^{p\beta_j} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{p\delta_k} |v(x)|^p \mathrm{d}x \right)^{6/(5p)} \\
&\quad \times \left(\int_{S_T} \rho_j(x)^{-q'6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{-q'6\delta_k/5} \mathrm{d}x \right)^{1/q'}.
\end{aligned} \tag{4.19}$$

We introduce spherical coordinates $(\rho, \varphi_k, \vartheta_k)$, which are centred at $\mathbf{c}$ and coincide with the edge M_k for $\vartheta_k = 0$. This definition implies that $r_k/\rho_j = \sin(\vartheta_k)$. The integrals over ϑ_k are bounded by a constant independent of h_T under the condition $-q'6\delta_k/5 > -2$ which is implied by $-2/p < \delta_k < 5/3 - 2/p$ for all $k \in X_j$. Hence, a constant $c_1 > 0$ exists such that the second integral in (4.19) can be simplified to

$$\begin{aligned}
\left(\int_{S_T} \rho_j(x)^{-q'6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{-q'6\delta_k/5} \mathrm{d}x \right)^{6/(5q')} &\le c \left(\int_0^{c_1 h_T} \rho^{2-q'6\beta_j/5} \mathrm{d}\rho \right)^{6/(5q')} \\
&\le c h_T^{-\beta_j} |T|^{5/6-1/p},
\end{aligned} \tag{4.20}$$

provided that $-q'6\beta_j/5 > -3$ which is equivalent to the assumption $\beta_j < 5/2 - 3/p$. As a consequence, we get from (4.15) using (4.19) and (4.20) the estimate

$$|u - Z_h u|_{H^\ell(T)} \le c h_T^{2-\ell-\beta_j} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(S_T)}, \qquad \text{if } \rho_{j,S_T} = 0,$$

and estimate (4.10) is proved completely.

Let us now investigate how the estimate (4.11) can be deduced from (4.10). The factors containing $r_{k,T}$ and $\rho_{j,T}$ obviously depend on the position of T. Therefore we introduce the following definitions. We denote the interior angle between the edges M_k and M_l, $k, l \in X_j$, by $\alpha_{k,l}$ and write $\alpha_j := \frac{1}{4} \min_{k,l\in X_j} \alpha_{k,l}$ for the quarter of the minimal angle between all edges having an endpoint in $x^{(j)}$. We define some cones $C_k^{\alpha_j}$, $k \in X_j$, also illustrated in Figure 2, by

$$C_k^{\alpha_j} := \{x \in U_j \cap \Omega \colon r_k(x)/\rho_j(x) \le \sin \alpha_j\}.$$

Outside of this cone, the angular distance $r_k(x)/\rho_j(x)$ is then bounded from below by a constant depending only on the angles α_j, but not on $\mathcal{T}_h$. If $T \cap C_k^{\alpha_j} = \emptyset$ for all $k \in X_j$, the angular distances to all edges are bounded from below, i. e.

$$\prod_{k\in X_j}\left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k}\le c.$$

Together with $\rho_{j,T}^{-\beta_j}\le r_T^{-\kappa}$ the estimate (4.12) leads to (4.11). Otherwise, if $T\cap C_k^{\alpha_j}\ne\emptyset$ for some $k\in X_j$, we have $r_T=r_{k,T}$, and since the angular distances to the other edges are bounded from below we get from (4.12) the estimate

$$|u-Z_hu|_{H^\ell(T)}\le ch_T^{2-\ell}|T|^{1/2-1/p}\rho_{j,T}^{\delta_k-\beta_j}r_{k,T}^{-\delta_k}|u|_{W^{2,p}_{\vec\beta,\vec\delta}(S_T)}. \tag{4.21}$$

Let us discuss the terms depending on $r_{k,T}$ and $\rho_{j,T}$. In case of $\delta_k\ge\beta_j$ we have $\rho_{j,T}^{\delta_k-\beta_j}\le c$. Otherwise, we exploit $r_{k,T}\le\rho_{j,T}$ and arrive at $\rho_{j,T}^{\delta_k-\beta_j}r_{k,T}^{-\delta_k}\le cr_T^{-\beta_j}$. Hence, for both cases estimate (4.11) follows from (4.21).

To deduce estimate (4.11) from (4.10) for $r_{S_T}=0$, the same technique can be applied. In case of $\rho_{j,S_T}>0$ we merely exploit that $\rho_{j,T}\ge ch_T$. □

Remark 12. *Let us recall the embeddings* (4.14) *and* (4.16) *(if $\rho_{j,S_T}>0$) or* (4.19) *(if $\rho_{j,S_T}=0$) used in the proof of Lemma 11. Generally speaking one can say that we exploited*

$$W^{2,p}_{\vec\beta,\vec\delta}(S_T)\hookrightarrow W^{2,6/5}(S_T)\hookrightarrow H^1(S_T),$$

which holds in particular for $\delta_k<5/3-2/p$. In the regularity result of Theorem 4 we demanded $\delta_k>2-2/p-\lambda_k^e$, and we can always find a weight δ_k satisfying both conditions when $\lambda_k^e>1/3$ which is indeed always the case. If we had used the Lagrange interpolant I_h which promises stability only in $L^\infty(T)$, we would have used the embeddings

$$W^{2,p}_{\vec\beta,\vec\delta}(T)\hookrightarrow W^{2,3/2}(T)\hookrightarrow L^\infty(T),$$

which hold only for $\delta_k<4/3-2/p$. One easily confirms that this condition and the assumptions of Theorem 4 can only be satisfied when $\lambda_k^e>2/3$ which is not the case for domains with edges having interior angle larger or equal 270°.

Using the local interpolation error estimates derived in Lemma 11 one can prove convergence rates for the finite element approximation of problem (4.1) in $H^1(\Omega)$ and $L^2(\Omega)$.

Theorem 13. *Let Ω be decomposed into a family of triangulations $\mathcal{T}_h$ satisfying the condition* (4.2)*. Let the mesh refinement parameter μ and the weights $\vec\beta\in[0,1)^{d'}$ and $\vec\delta\in[0,2/3)^d$ satisfy the inequalities*

$$\begin{aligned}1-\lambda_k^e&<\delta_k\le 1-\mu, &&\forall k\in\mathcal{E},\\ 1/2-\lambda_j^c&<\beta_j\le 1-\mu, &&\forall j\in\mathcal{C}.\end{aligned}$$

Then, for arbitrary input data $f\in W^{0,2}_{\vec\beta,\vec\delta}(\Omega)$, $g\in W^{1/2,2}_{\vec\beta,\vec\delta}(\Gamma)$ the a priori *error estimate*

$$\|y-y_h\|_{H^\ell(\Omega)}\le ch^{2-\ell}|y|_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)}\le ch^{2-\ell}\left(\|f\|_{W^{0,2}_{\vec\beta,\vec\delta}(\Omega)}+\|g\|_{W^{1/2,2}_{\vec\beta,\vec\delta}(\Omega)}\right). \tag{4.22}$$

holds for $\ell=0,1$.

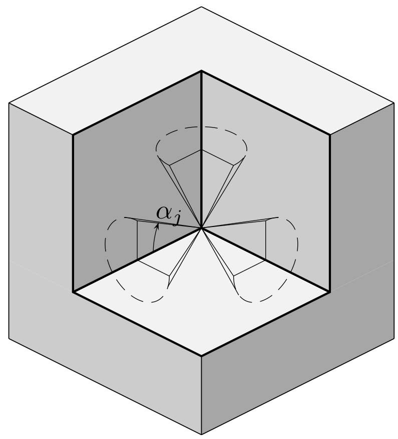


Figure 2: Definition of the cones $C_k^{\alpha_j}$ at a reentrant corner

Proof. As a consequence of Cea's Lemma and the decomposition of the domain Ω we obtain

$$\|y - y_h\|^2_{H^1(\Omega)} \le \sum_{T\in\mathcal{T}_h} \|y - Z_h y\|^2_{H^1(T)}. \tag{4.23}$$

It remains to insert the interpolation error estimates from Lemma 11 and to adjust the grading parameter such that the desired convergence rate is obtained. In case of $r_{S_T} = 0$ we have $h_T = h^{1/\mu}$ and with $\mu \le 1 - \kappa$ we get

$$\|y - Z_h y\|_{H^1(T)} \le c h^{(1-\kappa)/\mu} |y|_{W^{2,2}_{\vec\beta,\vec\delta}(S_T)} \le c h |y|_{W^{2,2}_{\vec\beta,\vec\delta}(S_T)}.$$

Otherwise, if $r_{S_T} > 0$ the mesh condition yields $h_T = h r_T^{1-\mu}$ and consequently

$$\|y - Z_h y\|_{H^1(T)} \le c h r_T^{1-\mu-\kappa} |y|_{W^{2,2}_{\vec\beta,\vec\delta}(S_T)} \le c h |y|_{W^{2,2}_{\vec\beta,\vec\delta}(S_T)}.$$

Inserting these local estimates into (4.23) leads to

$$\|y - y_h\|_{H^1(\Omega)} \le c h |y|_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)} \le c h \left(\|f\|_{W^{0,2}_{\vec\beta,\vec\delta}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\beta,\vec\delta}(\Omega)} \right). \tag{4.24}$$

Here, we also applied the regularity result of Theorem 4 whose conditions are satisfied under our assumptions upon $\vec\beta$ and $\vec\delta$. Since $\vec\beta \ge 0$ and $\vec\delta \ge 0$, hence $L^2(\Omega) \hookrightarrow W^{0,2}_{\vec\beta,\vec\delta}(\Omega)$, we finally obtain the estimate in the $L^2(\Omega)$-norm by the Aubin-Nitsche method. □

If we assume slightly better regularity of the input data in classical Sobolev spaces we get by the embeddings from Lemma 6 the following simplified version of Theorem 13.

Corollary 14. *Let be $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Then, the error estimate*

$$\|y - y_h\|_{H^\ell(\Omega)} \le c h^{2-\ell}$$

holds, provided that one of the following assumptions holds:

1. *The family of triangulations $\mathcal{T}_h$ is quasi-uniform (i.e. $\mu = 1$) and the singular exponents satisfy $\lambda_k^e > 1$, $\lambda_j^c > 1/2$ for all $k \in \mathcal{E}$ and $j \in \mathcal{C}$.*

2. The family of triangulations $\mathcal{T}_h$ is refined according to (4.2) with refinement parameter

$$\mu < \min\{\min_{k\in\mathcal{E}} \lambda_k^e, \min_{j\in\mathcal{C}} 1/2 + \lambda_j^c\}.$$

In the remainder of this section, the finite element error on the boundary Γ is investigated. The initial step of the convergence proof is an appropriate decomposition of Γ. In order to extract those parts of the domain which are under influence of singularities we define the sets

$$\begin{aligned} \Omega_R &:= \{x\colon 0 < r(x) < 1\} \cap \Omega, & \Gamma_R &:= \partial\Omega_R \cap \Gamma, \\ \hat{\Omega}_R &:= \{x\colon 0 < r(x) < \frac{1}{2}\} \cap \Omega, & \hat{\Gamma}_R &:= \partial\hat{\Omega}_R \cap \Gamma. \end{aligned} \tag{4.25}$$

Remember that $r(\cdot) := \min_{k\in\mathcal{E}} r_k(\cdot)$ stands for the minimum distance to the singular points. The boundary part which is not influenced by singularities is denoted by $\hat{\Gamma}_0 := \Gamma\backslash\hat{\Gamma}_R$.

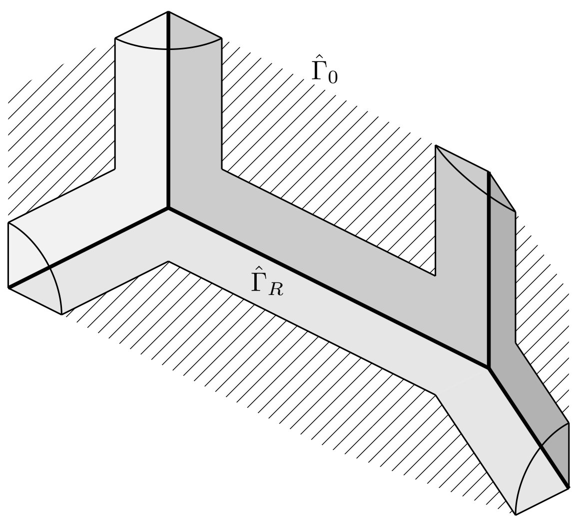


Figure 3: Decomposition of the boundary into $\hat{\Gamma}_R$ and $\hat{\Gamma}_0$.

For technical reasons we introduce a dyadic decomposition of the domain Ω_R. Therefore, let $d_i := 2^{-i}$, $i = 0, \ldots, I$ and let $c_I \geq 1$ be a constant independent of h such that $d_I = c_I h^{1/\mu}$ and hence $I \sim |\ln h|$. The constant c_I will be specified at the end of the proof of Theorem 17 and will play an important role. As illustrated in Figure 4 we define the sets

$$\Omega_i := \begin{cases} \{x\colon d_{i+1} < r(x) < d_i\} \cap \Omega_R, & \text{for } \ i = 0, 1, \ldots, I-1, \\ \{x\colon 0 < r(x) < d_I\} \cap \Omega_R, & \text{for } \ i = I, \end{cases}$$

which form a decomposition of Ω_R and $\hat{\Omega}_R$, i.e.

$$\Omega_R = \operatorname{int} \bigcup_{i=0}^{I} \bar{\Omega}_i, \qquad \hat{\Omega}_R = \operatorname{int} \bigcup_{i=1}^{I} \bar{\Omega}_i.$$

A decomposition of the boundary part Γ_R is then given by

$$\Gamma_i := \partial\Omega_i \cap \Gamma, \qquad i = 0, \ldots, I. \tag{4.26}$$

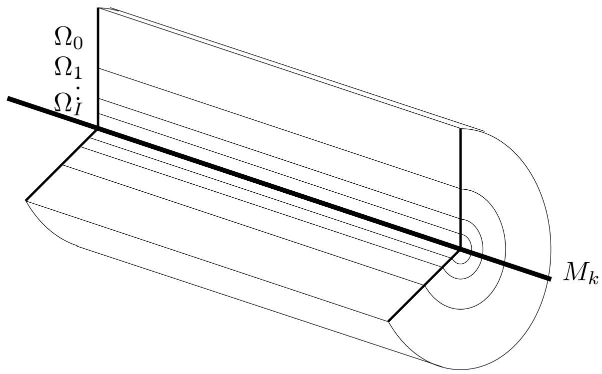


Figure 4: Dyadic decomposition of Ω_R along an edge.

Note that the elements contained in Ω_i or intersecting Ω_i satisfy

$$h_T \sim \begin{cases} h d_i^{1-\mu}, & \text{for } \ i = 0, 1, \ldots, I-1, \\ h^{1/\mu}, & \text{for } \ i = I. \end{cases}$$

Furthermore, we will need the patches of Ω_i with its adjacent sets defined by

$$\Omega_i^{(k)} := \operatorname{int}\left(\bar{\Omega}_{\max\{0,i-k\}} \cup \ldots \cup \bar{\Omega}_i \cup \ldots \cup \bar{\Omega}_{\min\{I,i+k\}}\right), \quad k \in \mathbb{N},$$

and write $\Omega_i' := \Omega_i^{(1)}$, $\Omega_i'' := \Omega_i^{(2)}$.

The first step of the proof is to derive interpolation error estimates on the subdomains Ω_i. We will need estimates in the $H^\ell(\Omega_i)$-norm ($\ell = 0, 1$) as well as in the $L^\infty(\Omega_i)$-norm. In what follows, $\kappa := \max\{\max_{k\in\mathcal{E}} \delta_k, \max_{j\in\mathcal{C}} \beta_j\}$ denotes again the largest weight.

Lemma 15. *Let be given some function $u \in H^1(\Omega)$ such that $D^\alpha u \in W^{1,p}_{\vec{\beta},\vec{\delta}}(\Omega_i^{(k+1)})$ for all $|\alpha| = 1$, with $p \in [2, \infty]$, non-negative integer k, and weights satisfying*

$$\begin{aligned} 0 \le \delta_k < 5/3 - 2/p, &\quad k \in \mathcal{E}, \\ 0 \le \beta_j < 5/2 - 3/p, &\quad j \in \mathcal{C}. \end{aligned}$$

Then, for $\ell \in \{0, 1\}$, there holds

$$\|u - Z_h u\|_{H^\ell(\Omega_i^{(k)})} \le c \begin{cases} h^{2-\ell} d_i^{(2-\ell)(1-\mu)+1-2/p-\kappa} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(\Omega_i^{(k+1)})}, & \text{for } i = 0, 1, \ldots, I-k-2, \\ c_I^{\Theta_1+1-2/p} h^{(3-\ell-2/p-\kappa)/\mu} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(\Omega_i^{(k+1)})}, & \text{for } i = I-k-1, \ldots, I, \end{cases}$$

where $\Theta_1 := \max\{0, (7/2 - \ell - 3/p)(1-\mu) - \kappa\}$.

Moreover, if $D^\alpha u \in W^{1,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i^{(k+1)})$ for all $|\alpha| = 1$, with weights satisfying

$$\begin{aligned} 0 \le \delta_k < 5/3, &\quad k \in \mathcal{E}, \\ 0 \le \beta_j < 2, &\quad j \in \mathcal{C}, \end{aligned}$$

the estimate

$$\|u - I_h u\|_{L^\infty(\Omega_i^{(k)})} \le c \begin{cases} c h^2 d_i^{2(1-\mu)-\kappa} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i^{(k+1)})}, & \text{for } i = 0, 1, \ldots, I-k-2, \\ c_I^{\Theta_2} h^{(2-\kappa)/\mu} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i^{(k+1)})}, & \text{for } i = I-k-1, \ldots, I, \end{cases}$$

holds, where $\Theta_2 := \max\{0, 2(1-\mu) - \kappa\}$.

Proof. Without loss of generality we prove the assertion for $k = 0$. The same arguments can be applied in case of $k > 0$ either. Let us first derive the estimate in the $H^\ell(\Omega_i)$-norm for $i = 0, 1, \ldots, I-2$. By the discrete Hölder inequality we get

$$|u - Z_h u|^2_{H^\ell(\Omega_i)} \le \left(\sum_{T \cap \Omega_i \neq \emptyset} 1 \right)^{1-2/p} \left(\sum_{T \cap \Omega_i \neq \emptyset} |u - Z_h u|^p_{H^\ell(T)} \right)^{2/p}. \tag{4.27}$$

The number of elements contained in Ω_i can be estimated by

$$\sum_{T \cap \Omega_i \neq \emptyset} 1 \le c \frac{|\Omega_i|}{\min_{T \cap \Omega_i \neq \emptyset} |T|} \le c d_i^2 \max_{T \cap \Omega_i \neq \emptyset} |T|^{-1}. \tag{4.28}$$

In the proof of Lemma 11 we have already shown that the local interpolation error for T with $r_T > 0$ can be estimated by

$$|u - Z_h u|_{H^\ell(T)} \le c h_T^{2-\ell} |T|^{1/2-1/p} r_T^{-\kappa} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)}. \tag{4.29}$$

Inserting this together with the mesh condition $h_T \sim h d_i^{1-\mu}$ as well as $r_T \ge d_{i+1} = \frac{1}{2} d_i$ and (4.28) into (4.27) leads to

$$|u - Z_h u|^2_{H^\ell(\Omega_i)} \le c h^{2(2-\ell)} d_i^{2((2-\ell)(1-\mu)+1-2/p-\kappa)} |u|^2_{W^{2,p}_{\vec{\beta},\vec{\delta}}(\Omega_i')}.$$

Extracting the root yields the first assertion. Let us now consider the case $i = I-1, I$. The number of elements can be estimated by

$$\sum_{T \cap \Omega_i \neq \emptyset} 1 \le c d_I^2 |T_{min}|^{-1},$$

where T_{min} is an arbitrary element touching a singular edge having diameter $h_{T_{min}} \sim h^{1/\mu}$. Due to the mesh condition we have to distinguish between the cases whether T touches the singular points or not. If T is away from the singular edges/corners the estimate (4.29) can be applied again and using $|T| \sim h_T^3$, the mesh condition $h_T \sim h r_T^{1-\mu}$, as well as $h^{1/\mu} \le r_T \le d_i \sim c_I h^{1/\mu}$ we obtain

$$\begin{aligned} |u - Z_h u|_{H^\ell(T)} &\le c h^{2-\ell+3/2-3/p} r_T^{(2-\ell+3/2-3/p)(1-\mu)-\kappa} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \\ &\le c c_I^{\Theta_1} h^{(2-\ell-\kappa)/\mu} |T_{min}|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)}, \end{aligned} \tag{4.30}$$

where we used the property $|T_{min}| \sim h^{3/\mu}$ in the last step. With the local interpolation error estimate from Lemma 11 and the mesh criterion $h_T \sim h^{1/\mu}$ we get almost the same estimate for elements touching the singular points, namely

$$|u - Z_h u|_{H^\ell(T)} \le c h^{(2-\ell-\kappa)/\mu} |T_{min}|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)}. \tag{4.31}$$

The estimates (4.30) and (4.31) can be combined to

$$|u - Z_h u|_{H^\ell(T)} \le c c_I^{\Theta_1} h^{(2-\ell-\kappa)/\mu} |T_{min}|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \qquad \forall T \in \mathcal{T}_h \colon T \cap \Omega_i \neq \emptyset.$$

Next, we apply the Hölder inequality (4.27) and obtain

$$|u - Z_h u|_{H^\ell(\Omega_i)} \le c c_I^{\Theta_1} h^{(2-\ell-\kappa)/\mu} d_I^{(1-2/p)} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(\Omega_i')}$$
$$\le c c_I^{\Theta_1 + 1 - 2/p} h^{(3-\ell-2/p-\kappa)/\mu} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(\Omega_i')}.$$

In the remainder of this proof the $L^\infty(\Omega_i)$ error estimates are considered. Therefore, let T^* be the element where the maximum of $|u(x) - I_h u(x)|$ is attained. Recall the covering $\{U_j\}_{j=1}^{d'}$ of Ω introduced on page 6. Without loss of generality we may assume that $T^* \subset U_j$. If $T^* \cap \Omega_i \neq \emptyset$ for some $i = 0, 1, \ldots, I-2$, we have $u \in W^{2,\infty}(T^*)$, and thus we may apply a standard interpolation error estimate and introduce the weights afterwards. We obtain

$$\|u - I_h u\|_{L^\infty(\Omega_i)} \le \|u - I_h u\|_{L^\infty(T^*)} \le c h_{T^*}^2 \rho_{j,T}^{-\beta_k} \prod_{k \in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T^*)}$$
$$\le c h_{T^*}^2 r_{T^*}^{-\kappa} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T^*)} \le c h^2 d_i^{2(1-\mu)-\kappa} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(\Omega_i')}. \tag{4.32}$$

In the third step we used the technique already applied in the proof of Lemma 11, where all factors depending on $\rho_{j,T}$ and $r_{k,T}$ were simplified to $r_T^{-\kappa}$. The last step follows from the mesh condition $h_T \sim h d_i^{1-\mu}$ and from $r_{T^*} \sim d_i$.

For $i = I-1, I$ we have lower regularity on elements touching the singular points. Thus, standard interpolation error estimates cannot be applied. Analogous to the proof of Lemma 11 we introduce a polynomial $w \in \mathcal{P}_1(T^*)$, split the error into two parts and consider the approximation error on a reference element. With the embedding $W^{1,p}(\hat T) \hookrightarrow L^\infty(\hat T)$ for some $p > 3$, we obtain the estimate

$$\|u - I_h u\|_{L^\infty(T^*)} \le \|\hat u - \hat w\|_{L^\infty(\hat T)} + \|\hat I_h(\hat u - \hat w)\|_{L^\infty(\hat T)} \le c \|\hat u - \hat w\|_{W^{1,p}(\hat T)}.$$

First, we consider the case that T^* is away from the corner points but possesses an edge which is contained in the edge M_k, $k \in \mathcal{E}$ of Ω. We apply the embedding $W^{2,p}_{1,1}(\hat T) \hookrightarrow W^{1,p}(\hat T)$ from Lemma 6 and the Bramble-Hilbert type argument in weighted Sobolev spaces presented in Lemma 10, and obtain

$$\|u - I_h u\|_{L^\infty(T^*)} \le \|\hat u - \hat w\|_{W^{2,p}_{1,1}(\hat T)} \le c |\hat u|_{W^{2,p}_{1,1}(\hat T)}. \tag{4.33}$$

Under the assumption $\delta_k < 5/3$ and $p = 3 + \varepsilon$ with sufficiently small $\varepsilon > 0$ we get from Lemma 6 and the transformation from $\hat T$ to T^* using (4.4) the estimate

$$|\hat u|_{W^{2,p}_{1,1}(\hat T)} \le c |\hat u|_{W^{2,\infty}_{\delta_k,\delta_k}(\hat T)} \le c h_{T^*}^{2-\delta_k} \rho_{j,T^*}^{\delta_k - \beta_j} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T^*)}. \tag{4.34}$$

In the last step we already inserted the remaining weights and used the fact that $\rho_{j,T^*} > 0$. Furthermore, analogous to the proof of Lemma 11 we can show $h_{T^*}^{2-\delta_k} \rho_{j,T^*}^{\delta_k - \beta_j} \le c h_{T^*}^{2-\kappa}$, and conclude from (4.34) and (4.33)

$$\|u - I_h u\|_{L^\infty(T^*)} \le c h_{T^*}^{2-\kappa} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T^*)}. \tag{4.35}$$

If T^* touches the edge M_k only in a single point we obtain exactly the same estimate by replacing in (4.33) and (4.34) the space $W^{2,p}_{1,1}(\hat T)$ by $W^{2,p}_{1,0}(\hat T)$ and $W^{2,\infty}_{\delta_k,\delta_k}(\hat T)$ by $W^{2,\infty}_{\delta_k,0}(\hat T)$, and by using the property (4.6) instead of (4.4).

Let now T^* touch additionally the corner $x^{(j)}$ and let an edge of T^* be contained in M_k, $k \in X_j$. The other edges M_ℓ, $\ell \in X_j \setminus \{k\}$, meeting in $x^{(j)}$ can be neglected, as T touches them only in $x^{(j)}$. From (4.6) and (4.6) we conclude for these edges $1 = \hat{\rho}/\hat{\rho} \sim r_\ell/\rho_j$. We consider again (4.33), employ the embedding from Lemma 6 with $\beta_j < 2$ and $\delta_k < 5/3$, and obtain using (4.4) and (4.5) the estimate

$$|\hat{u}|_{W^{2,p}_{1,1}(\hat{T})} \leq c|\hat{u}|_{W^{2,\infty}_{\beta_j,\delta_k}(\hat{T})} \leq ch_{T^*}^{2-\beta_j}|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T^*)}$$

In the last step we merely inserted the remaining weights which are bounded on T^*. After insertion into (4.33) we arrive again at (4.35).

In conclusion, we have shown that if T^* touches the singular points the estimate

$$\|u - I_h u\|_{L^\infty(T^*)} \leq ch^{(2-\kappa)/\mu}|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T^*)} \tag{4.36}$$

holds, and otherwise, we obtain from (4.32) with $h^{1/\mu} \leq r_{T^*} \leq d_i \sim c_I h^{1/\mu}$, $i = I-1, I$, and some computations

$$\|u - I_h u\|_{L^\infty(T^*)} \leq ch^2 r_{T^*}^{2(1-\mu)-\kappa}|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T^*)} \leq cc_I^{\Theta_2} h^{(2-\kappa)/\mu}|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T^*)}. \tag{4.37}$$

The estimates (4.36) and (4.37) imply the last assertion since $T^* \subset \Omega_i'$. □

We first show an initial error estimate on a single boundary strip Γ_i and will combine it to a global estimate in Theorem 17.

Lemma 16. *Let $y \in H^1(\Omega_R) \cap L^\infty(\Omega_R)$ and denote by y_h its Ritz projection, i. e.*

$$\int_{\Omega_R} (\nabla(y - y_h) \cdot \nabla v_h + (y - y_h)v_h) = 0 \qquad \forall v_h \in V_h.$$

Then, for all $i \in \{1, \ldots, I\}$ the local estimate

$$\|y - y_h\|_{L^2(\Gamma_i)} \leq c\left(d_i^{1/2}|\ln h| \inf_{\chi \in V_h} \|y - \chi\|_{L^\infty(\Omega_i')} + d_i^{-1/2}\|y - y_h\|_{L^2(\Omega_i')}\right) \tag{4.38}$$

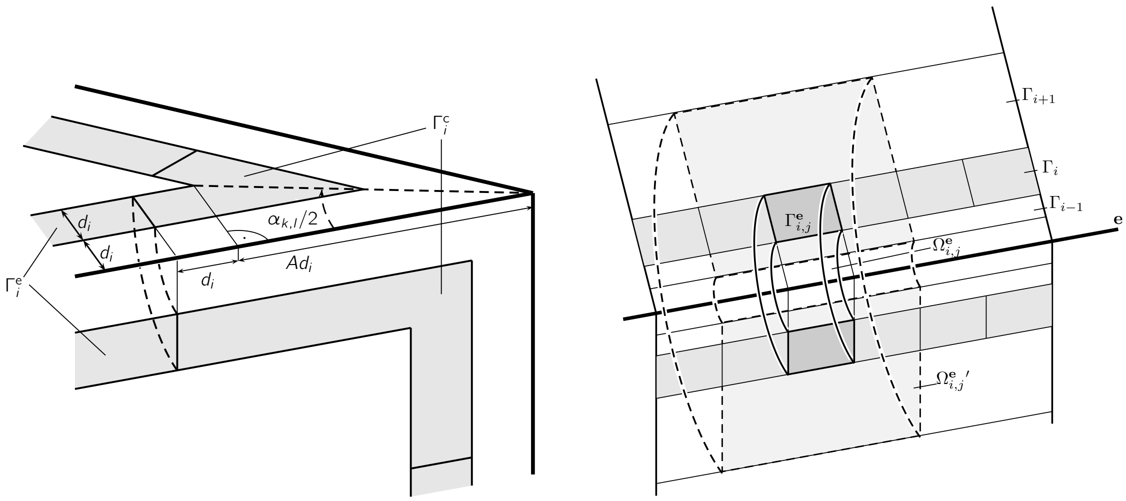
holds.

Proof. Let us first show the assertion for $i \in 1, \ldots, I-2$. For technical reasons we decompose the domain Ω_i as follows. Let (x_k, y_k, z_k), $k \in \mathcal{E}$, denote Cartesian coordinate systems having origin in some corner $\mathbf{c} = x^{(j)}$, $j \in \mathcal{C}$ with $k \in X_j$, such that the z_k-axes coincide with the edges M_k. Moreover, define $\alpha^j_{min} := \min_{k,l \in X_j} \alpha_{k,l}$ where $\alpha_{k,l}$ is the angle between the edges M_k and M_l, and introduce the set

$$\Omega_i^{\mathbf{c}} := \bigcup_{k \in X_j} \{x \in \Omega_i \colon z_k(x) < (1 + A)\, d_i\}, \quad \text{where} \quad A := 2\cot\frac{\alpha^j_{min}}{2} \sim 1,$$

with outer boundary $\Gamma_i^{\mathbf{c}} := \partial\Omega_i^{\mathbf{c}} \cap \Gamma$ (compare Figure 5a). It is easy to confirm that $|\Gamma_i^{\mathbf{c}}| \sim d_i^2$. To obtain the desired estimate on $\Gamma_i^{\mathbf{c}}$ we apply the Hölder inequality and a trace theorem and obtain

$$\|y - y_h\|_{L^2(\Gamma_i^{\mathbf{c}})} \leq d_i\|y - y_h\|_{L^\infty(\Gamma_i^{\mathbf{c}})} \leq d_i\|y - y_h\|_{L^\infty(\Omega_i^{\mathbf{c}})}. \tag{4.39}$$



(a) Definition of $\Omega_i^{\mathbf{e}}$ and $\Omega_i^{\mathbf{c}}$

(b) Definition of $\Omega_{i,j}^{\mathbf{e}}$ and its patch

Figure 5: Illustration of the domains defined in the proof of Lemma 16

Now we can apply the local maximum norm estimate of Theorem 10.1 and Example 10.1 in [37], which reads in our situation

$$\|y - y_h\|_{L^\infty(\Omega_i^{\mathbf{c}})} \le c \left(|\ln h| \inf_{\chi \in V_h} \|y - \chi\|_{L^\infty(\Omega_i')} + d^{-3/2} \|y - y_h\|_{L^2(\Omega_i')} \right), \tag{4.40}$$

with $d := \operatorname{dist}(\partial\Omega_i^{\mathbf{c}} \setminus \Gamma,\ \partial\Omega_i' \setminus \Gamma)$. Due to our construction we find that $d \sim d_i$ and inserting (4.40) into (4.39) yields

$$\|y - y_h\|_{L^2(\Gamma_i^{\mathbf{c}})} \le c \left(d_i |\ln h| \inf_{\chi \in V_h} \|y - \chi\|_{L^\infty(\Omega_i')} + d_i^{-1/2} \|y - y_h\|_{L^2(\Omega_i')} \right). \tag{4.41}$$

It remains to derive this estimate on that part of Ω_i which excludes neighbourhoods of the corners. We fix an edge $\mathbf{e} := M_k$ having length L and endpoints $x^{(j)}$, $x^{(j')}$, introduce the interval

$$Z := ((1 + A)\, d_i, L - (1 + A)\, d_i)$$

and define the domain

$$\Omega_i^{\mathbf{e}} := \{x \in \Omega_i \colon z_k(x) \in Z\},$$

having outer boundary $\Gamma_i^{\mathbf{e}} := \partial\Omega_i^{\mathbf{e}} \cap \Gamma$ which is illustrated in Figure 5a. However, the measure of $\Gamma_i^{\mathbf{e}}$ is only of order d_i and a direct application of (4.39)–(4.41) would yield a worse estimate. Hence we introduce a further decomposition and split the interval Z into $N \sim d_i^{-1}$ subintervals Z_j such that $|Z_j| \sim d_i$. Then, $\Omega_i^{\mathbf{e}}$ can also be expressed as the union of the sets

$$\Omega_{i,j}^{\mathbf{e}} := \{x \in \Omega_i \colon z_k(x) \in Z_j\}, \qquad i = 1, \ldots, N. \tag{4.42}$$

Again, we write $\Gamma_{i,j}^{\mathbf{e}} := \partial\Omega_{i,j}^{\mathbf{e}} \cap \Gamma$ and confirm that the desired property $|\Gamma_{i,j}^{\mathbf{e}}| \sim d_i^2$ holds. To apply the local maximum norm estimate (4.40) in a reasonable way we also have to define patches

$$\Omega_{i,j}^{\mathbf{e}}{}' := \left\{x \in \Omega_i' \colon z_k(x) \in \operatorname{int}\left(\overline{Z_{j-1}} \cup \overline{Z_j} \cup \overline{Z_{j+1}}\right)\right\},$$

where we set

$$Z_0 := (Ad_i,\ (1+A)\, d_i)\,,$$
$$Z_{N+1} := (L-(1+A)\, d_i,\ L-Ad_i)\,.$$

Due to this construction we have the property

$$\mathrm{dist}(\partial\Omega_{i,j}^{\mathbf{e}}{}' \setminus \Gamma,\ \partial\Omega_{i,j}^{\mathbf{e}} \setminus \Gamma) \sim d_i. \tag{4.43}$$

Exploiting the decomposition (4.42), the Hölder inequality, a trace theorem and the local maximum norm estimate (4.40) with property (4.43) leads to

$$\begin{aligned}
\|y-y_h\|_{L^2(\Gamma_i^{\mathbf{e}})}^2 &\le \sum_{j=1}^N d_i^2 \|y-y_h\|_{L^\infty(\Omega_{i,j}^{\mathbf{e}})}^2 \\
&\le c\sum_{j=1}^N \left(d_i^2 |\ln h|^2 \|y-\chi\|_{L^\infty(\Omega_{i,j}^{\mathbf{e}}{}')}^2 + d_i^{-1}\|y-y_h\|_{L^2(\Omega_{i,j}^{\mathbf{e}}{}')}^2 \right) \\
&\le c\left(d_i |\ln h|^2 \|y-\chi\|_{L^\infty(\Omega_i')}^2 + d_i^{-1}\|y-y_h\|_{L^2(\Omega_i')}^2 \right).
\end{aligned} \tag{4.44}$$

In the last step we exploited that $\sum_{i=1}^N 1 \sim d_i^{-1}$ and that $\cup_{j=1}^N \Omega_{i,j}{}' \subset \Omega_i'$. Finally we observe that

$$\Gamma_i = \bigcup_{j\in\mathcal{C}} \Gamma_i^{x^{(j)}} \cup \bigcup_{k\in\mathcal{M}} \Gamma_i^{M_k}$$

and together with (4.41) and (4.44) we arrive at the assertion.

It remains to show (4.38) also for $i = I-1, I$ which cannot be obtained with the same technique, since the local maximum norm estimate (4.40) is not applicable if Ω_i' contains the singular points. Therefore, we first insert an arbitrary element $\chi \in V_h$ as intermediate function and apply the triangle inequality which leads to

$$\|y-y_h\|_{L^2(\Gamma_i)} \le c\left(\|y-\chi\|_{L^2(\Gamma_i)} + \|\chi-y_h\|_{L^2(\Gamma_i)}\right). \tag{4.45}$$

Next, we apply the Hölder inequality and a trace theorem to get

$$\|y-\chi\|_{L^2(\Gamma_i)} \le c d_i^{1/2} \|y-\chi\|_{L^\infty(\Omega_i)}. \tag{4.46}$$

For the second part of (4.45) we exploit that $\chi - y_h$ is a function from a finite-dimensional space. Let $\mathcal{F}$ denote the set of faces F of elements $T \in \mathcal{T}_h$ with $F \subset \Gamma_i$. The faces have diameter $h_F \sim h_T$ if $F \subset \overline{T}$. Applying a trace theorem on a reference setting as well as a norm equivalence leads to

$$\begin{aligned}
\|\chi-y_h\|_{L^2(\Gamma_i)}^2 &\le \sum_{F\in\mathcal{F}} \|\chi-y_h\|_{L^2(F)}^2 \\
&\le c\sum_{F\in\mathcal{F}} h_F^2 \|\hat\chi-\hat y_h\|_{L^2(\hat F)}^2 \le c\sum_{T\cap\Omega_i\neq\emptyset} h_T^2\|\hat\chi-\hat y_h\|_{L^2(\hat T)}^2 \\
&\le c\sum_{T\cap\Omega_i\neq\emptyset} h_T^{-1}\|\chi-y_h\|_{L^2(T)}^2 \le ch^{-1/\mu}\|\chi-y_h\|_{L^2(\Omega_i')}^2.
\end{aligned} \tag{4.47}$$

Extracting the root and taking $d_i \sim h^{1/\mu}$ into account yields

$$\|\chi - y_h\|_{L^2(\Gamma_i)} \le c d_i^{-1/2} \|\chi - y_h\|_{L^2(\Omega_i')}.$$

With the triangle inequality and the Hölder inequality exploiting that $|\Omega_i'| \sim d_i^2$, we obtain

$$\|\chi - y_h\|_{L^2(\Gamma_i)} \le c\left(d_i^{1/2}\|y-\chi\|_{L^\infty(\Omega_i')} + d_i^{-1/2}\|y - y_h\|_{L^2(\Omega_i')}\right). \tag{4.48}$$

Inserting (4.46) and (4.48) into (4.45) implies the estimate (4.38) for $i = I-1, I$. $\square$

The next step of the proof is to derive a finite element error estimate on the boundary $\hat\Gamma_R$ which is under influence of corner and edge singularities. Therefore, we localise the solution y with a smooth cut-off function $\omega \in C^\infty(\Omega)$ satisfying

$$\omega|_{\hat\Omega_R} \equiv 1 \qquad \text{and} \qquad \operatorname{supp}\omega \subset \Omega_R, \tag{4.49}$$

and define $\tilde y := \omega y$. Let

$$V_h(\Omega_R) := \{v_h \in V_h \colon v_h \equiv 0 \text{ in } \Omega\setminus\Omega_R\}$$

denote the space of ansatz functions vanishing outside of Ω_R. This definition implies that $\operatorname{supp} v_h \subset \Omega_R$. In what follows, $\tilde y_h \in V_h(\Omega_R)$ denotes the Ritz-projection of $\tilde y$, i.e.

$$a(\tilde y - \tilde y_h, v_h) = 0, \quad \text{for all } v_h \in V_h(\Omega_R). \tag{4.50}$$

An error estimate for this Ritz-projection is considered in the following theorem:

Theorem 17. *Let $D^\alpha \tilde y \in W^{1,2}_{\vec\alpha,\vec\gamma}(\Omega_R) \cap W^{1,\infty}_{\vec\beta,\vec\delta}(\Omega_R)$ for all $|\alpha| = 1$, with weights satisfying*

$$\begin{aligned}
&0 \le \alpha_j < 1, && 0 \le \beta_j < 2, && \forall j \in \mathcal{C},\\
&0 \le \gamma_k < 2/3, && 0 \le \delta_k < 5/3, && \forall k \in \mathcal{E}.
\end{aligned}$$

Assume that the refinement condition (4.2) with parameter

$$\begin{aligned}
\mu &\le 1 - \kappa_2, && \textit{with} \quad \kappa_2 := \max\{\max_{j\in\mathcal{C}} \alpha_j, \max_{k\in\mathcal{E}} \gamma_k\},\\
\mu &\le \frac{5}{4} - \frac{\kappa_\infty}{2}, && \textit{with} \quad \kappa_\infty := \max\{\max_{j\in\mathcal{C}} \beta_j, \max_{k\in\mathcal{E}} \delta_k\}.
\end{aligned}$$

Then, for $\tilde y$ and $\tilde y_h$ defined in (4.50) the estimate

$$\|\tilde y - \tilde y_h\|_{L^2(\hat\Gamma_R)} \le c h^2 |\ln h|^{3/2}\left(|\tilde y|_{W^{2,2}_{\vec\alpha,\vec\gamma}(\Omega_R)} + |\tilde y|_{W^{2,\infty}_{\vec\beta,\vec\delta}(\Omega_R)}\right)$$

holds.

Proof. We consider the decomposition of the boundary $\hat\Gamma_R$ into the segments $\Gamma_i := \partial\Omega_i \cap \Gamma$ introduced in (4.26). In Lemma 16 we have already derived the local estimate

$$\|\tilde y - \tilde y_h\|_{L^2(\Gamma_i)} \le c\left(d_i^{1/2}|\ln h|\|\tilde y - I_h\tilde y\|_{L^\infty(\Omega_i')} + d_i^{-1/2}\|\tilde y - \tilde y_h\|_{L^2(\Omega_i')}\right) \tag{4.51}$$

on Γ_i for all $i = 1, \dots, I$. Now, we sum up all boundary parts Γ_i and incorporate the property $d_i = \frac{1}{2} d_{i-1} \geq \frac{1}{2}(d_I + r) := \gamma(r)$ on Γ_i which leads to

$$\|\tilde{y} - \tilde{y}_h\|^2_{L^2(\hat{\Gamma}_R)} \leq c \left(|\ln h|^2 \sum_{i=1}^{I} d_i \|\tilde{y} - I_h \tilde{y}\|^2_{L^\infty(\Omega_i')} + \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|^2_{L^2(\Omega_R)} \right). \tag{4.52}$$

For the first term on the right-hand side of (4.52) we exploit $I \sim |\ln h|$ and the local interpolation error estimate in $L^\infty(\Omega_i')$-norm from Lemma 15 which leads to

$$\sum_{i=1}^{I} d_i \|\tilde{y} - I_h \tilde{y}\|^2_{L^\infty(\Omega_i')} \leq c |\ln h| h^4 |\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)}. \tag{4.53}$$

Here, we already inserted the assumption $\mu \leq 5/4 - \kappa_\infty/2$. The second part requires an estimate for a weighted $L^2(\Omega_R)$ error. Therefore, we adopt the technique that was applied in the proof of Lemma 6.2 in [33] and use a duality argument. First, we decompose the error into

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_R)} \leq \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_R \setminus (\Omega_0 \cup \Omega_1))} + \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_0 \cup \Omega_1)}.$$

On the outermost rings $\Omega_0 \cup \Omega_1$ we exploit that $\gamma \sim 1$ and use Theorem 13 which leads to

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_0 \cup \Omega_1)} \leq c \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_R)} \leq c h^2 |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)}. \tag{4.54}$$

For the error on $\tilde{\Omega}_R := \Omega_R \setminus (\Omega_0 \cup \Omega_1)$ we apply the representation

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} = \sup_{\substack{g \in C_0^\infty(\tilde{\Omega}_R) \\ \|g\|_{L^2(\tilde{\Omega}_R)} = 1}} (\gamma^{-1/2}(\tilde{y} - \tilde{y}_h), g) \tag{4.55}$$

and consider the auxiliary problem

$$-\Delta w + w = \gamma^{-1/2} g \quad \text{in} \quad \Omega_R, \qquad\qquad \partial_n w = 0 \quad \text{on} \quad \partial\Omega_R. \tag{4.56}$$

From the weak formulation of (4.56) we can deduce

$$(\gamma^{-1/2}(\tilde{y} - \tilde{y}_h), g) = (\tilde{y} - \tilde{y}_h, \gamma^{-1/2} g) = a(\tilde{y} - \tilde{y}_h, w). \tag{4.57}$$

We introduce a further cut-off function $\eta \in C_0^\infty(\Omega_R)$ such that

$$\eta \equiv 1 \text{ on } \tilde{\Omega}_R := \Omega_R \setminus (\overline{\Omega}_0 \cup \overline{\Omega}_1), \qquad \text{and} \qquad \operatorname{supp} \eta \subset \hat{\Omega}_R,$$

and we make use of the decomposition $w = w_1 + w_2$ with $w_1 := \eta w$ and $w_2 := (1 - \eta) w$. The definition of w_1 implies that $Z_h w_1 \in V_h(\Omega_R)$ which allows us to apply the Galerkin orthogonality (4.50). This yields

$$\begin{aligned}
&a(\tilde{y} - \tilde{y}_h, w_1) = a(\tilde{y} - \tilde{y}_h, w_1 - Z_h w_1) \\
&\leq c \Bigg(\sum_{i=3}^{I-3} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i)} + \sum_{i=I-2}^{I} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i)} \\
&+ \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_1')} \|w_1 - Z_h w_1\|_{H^1(\Omega_1')} \Bigg).
\end{aligned} \tag{4.58}$$

It remains to estimate the three terms on the right-hand side. The cases $i = 3, \ldots, I-3$ and $i = I-2, \ldots, I$ such as $i = 0, 1, 2$ are considered separately.

For the first part of the right-hand side of (4.58), i. e. $i = 3, \ldots, I-3$, we apply the local $H^1(\Omega_i)$ error estimate of Corollary 9.1 in [37] which reads in our situation

$$\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} \le c \left(|\tilde{y} - Z_h \tilde{y}|_{H^1(\Omega_i')} + d_i^{-1} \|\tilde{y} - Z_h \tilde{y}\|_{L^2(\Omega_i')} + d_i^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right). \quad (4.59)$$

Inserting the interpolation error estimates of Lemma 15 leads to the inequality

$$\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} \le c \left(h d_i^{2-\mu-\kappa_\infty} (1 + h d_i^{-\mu}) |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + d_i^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right). \quad (4.60)$$

Further, note that $h d_i^{-\mu} \le h d_I^{-\mu} = c_I^{-\mu} \le 1$ in case of $c_I > 1$. To estimate the term $\|w_1 - Z_h w_1\|_{H^1(\Omega_i)}$ for $i = 3, \ldots, I-3$ in (4.58) we directly apply the local $H^1(\Omega_i)$ interpolation error estimates of Lemma 15 with $p = 2$. Note that w_1 is an element of $W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)$ for arbitrary polyhedra, since the assumptions of Theorem 4 are always satisfied as $\lambda_k^e > 1/2$ $(k \in \mathcal{E})$ and $\lambda_j^c > 0$ $(j \in \mathcal{C})$. With $\kappa = 1/2$, Lemma 15 now implies

$$\|w_1 - Z_h w_1\|_{H^1(\Omega_i)} \le c h d_i^{1/2-\mu} |w_1|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} = c h d_i^{1/2-\mu} |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \quad (4.61)$$

The last step is a consequence of the fact that $\eta \equiv 1$ on $\tilde{\Omega}_R$ and hence $w_1 \equiv w$ on all Ω_i with $i = 3, \ldots, I-3$. Combining (4.60) and (4.61) yields for $i = 3, \ldots, I-3$ the estimate

$$\begin{aligned} &\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i)} \\ &\le c \left(h^2 d_i^{5/2-2\mu-\kappa_\infty} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + h d_i^{-1/2-\mu} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\le c \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + c_I^{-\mu} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned} \quad (4.62)$$

The last step follows from the assumption upon μ and the definition of the domains Ω_i, more precisely we exploited $d_i^{-\mu} < c_I^{-\mu} h^{-1}$. To obtain a similar estimate in case of $i = I-2, \ldots, I$ we have to insert the appropriate interpolation error estimates of Lemma 15 into (4.59) which implies

$$\begin{aligned} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} &\le c \left(c_I^{\Theta_1+1} h^{(2-\kappa_\infty)/\mu} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + d_i^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right), \\ \|w_1 - Z_h w_1\|_{H^1(\Omega_i)} &\le c c_I^{\max\{0,1/2-\mu\}} h^{1/2\mu} |w_1|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned}$$

In the following we neglect the factor c_I where not needed since it is independent of h. Combining both estimates leads to

$$\begin{aligned} &\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i)} \\ &\le c \left(h^{(5/2-\kappa_\infty)/\mu} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + c_I^{\max\{0,1/2-\mu\}} h^{1/2\mu} d_I^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right) |w_1|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\le c \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned} \quad (4.63)$$

The last step follows from the assumption upon μ and the fact that $d_i \sim d_I = c_I h^{1/\mu}$ for $i = I-2, \dots, I$. Moreover, we exploited $w \equiv w_1$ on Ω_i' for $i = I-2, \dots, I$.

For the last part of (4.58) we apply Theorem 13 for the finite element error (compare also (4.54)) and Lemma 15 for the interpolation error. The factors d_0, d_1 and d_2 are of order one and can thus be neglected. We then obtain

$$\begin{aligned} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_1')} \|w_1 - Z_h w_1\|_{H^1(\Omega_1')} &\le ch^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)} |w_1|_{W^{2,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_1'')} \\ &\le ch^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)} \left(|w|_{W^{2,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_1'')} + \|w\|_{H^1(\Omega_1'')} \right). \end{aligned} \tag{4.64}$$

The last step is a consequence of the Leibniz rule and the fact that $D^\alpha \eta \le c$ for all $|\alpha| \le 2$, as well as the property $1/16 \le r(x) \le 1$ on Ω_1'' which allows us to neglect the weights hidden in the norm definition.

We may now insert the estimates (4.62), (4.63) and (4.64) into (4.58) which leads to

$$\begin{aligned} &a(\tilde{y} - \tilde{y}_h, w_1) \\ &\le c \sum_{i=3}^{I} \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_i'')} + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_i')} \qquad (4.65) \\ &\quad + ch^2 \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)} \right) \left(|w|_{W^{2,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)} \right) \\ &\le c \left(h^2 |\ln h|^{1/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)} \right) + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \right) \\ &\times \left(|w|_{W^{2,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)} \right). \qquad (4.66) \end{aligned}$$

Next, we show that w possesses the regularity demanded by the right-hand side, which follows from Theorem 4 and the Lax-Milgram Lemma once we have shown that

$$\gamma^{-1/2} g \in W^{0,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_R) \cap (H^1(\Omega_R))^*. \tag{4.67}$$

For some fixed $x \in U_j$ define $\bar{k} \in X_j$ such that $r_{\bar{k}}(x) = r(x)$. The angular distance to the edges M_k with $k \in X_j \setminus \{\bar{k}\}$ is bounded from below, i.e. $r_k/\rho_j \ge c$ (compare also Figure 2). Consequently, we obtain

$$\begin{aligned} \gamma^{-1}(x) = (d_I + r(x))^{-1} \le r(x)^{-1} = r_{\bar{k}}(x)^{-1} = \rho_j(x)^{-1} \left(\frac{r_{\bar{k}}}{\rho_j}(x) \right)^{-1} \\ \le c\rho_j(x)^{-1} \prod_{k \in X_j} \left(\frac{r_k}{\rho_j}(x) \right)^{-1}, \end{aligned} \tag{4.68}$$

and directly conclude

$$\|\gamma^{-1/2} g\|_{W^{0,2}_{\vec{\mathbf{1}}/2,\vec{\mathbf{1}}/2}(\Omega_R)} \le c\|g\|_{L^2(\Omega_R)} \le c. \tag{4.69}$$

To show the boundedness in $(H^1(\Omega_R))^*$ we use the operator norm representation, the Cauchy-Schwarz inequality as well as the boundedness of g in $L^2(\Omega_R)$, and arrive at

$$\|\gamma^{-1/2} g\|_{(H^1(\Omega_R))^*} = \sup_{\varphi \in H^1(\Omega_R)} \frac{(g, \gamma^{-1/2}\varphi)_{\Omega_R}}{\|\varphi\|_{H^1(\Omega_R)}} \le c \sup_{\varphi \in H^1(\Omega_R)} \frac{\|\gamma^{-1/2}\varphi\|_{L^2(\Omega_R)}}{\|\varphi\|_{H^1(\Omega_R)}}. \tag{4.70}$$

Taking again (4.68) into account leads to

$$\|\gamma^{-1/2}\varphi\|_{L^2(\Omega_R)} \le c\|\varphi\|_{W^{0,2}_{-\vec{1}/2,-\vec{1}/2}(\Omega_R)} \le c\|\varphi\|_{H^1(\Omega_R)}, \tag{4.71}$$

where the second step is a consequence of the embedding $W^{1,2}_{\vec{0},\vec{0}}(\Omega_R) \hookrightarrow W^{0,2}_{-\vec{1}/2,-\vec{1}/2}(\Omega_R)$ (see Lemma 6) and the fact that the spaces $W^{1,2}_{\vec{0},\vec{0}}(\Omega_R)$ and $H^1(\Omega_R)$ are equivalent. Inserting (4.71) into (4.70) and taking also (4.69) into account yields (4.67), and consequently

$$|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)} \le c. \tag{4.72}$$

The estimate (4.65) then becomes

$$\begin{aligned} &a(\tilde{y}-\tilde{y}_h, w_1) \\ &\le c\left(h^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)}\right) + c_I^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)}\right). \end{aligned} \tag{4.73}$$

It remains to derive a similar estimate with w_2 instead of w_1. Therefore, we exploit that $w_2 \equiv 0$ on $\tilde{\Omega}_R$, and $\partial_n w_2 \equiv 0$ on $\partial\Omega_R$. Partial integration then yields

$$\begin{aligned} a(\tilde{y}-\tilde{y}_h, w_2) &= (\tilde{y}-\tilde{y}_h, -\Delta w_2) + (\tilde{y}-\tilde{y}_h, w_2) + (\tilde{y}-\tilde{y}_h, \partial_n w_2)_{\partial\Omega_R} \\ &\le \|\tilde{y}-\tilde{y}_h\|_{L^2(\Omega_R)}\|w_2\|_{H^2(\Omega_R\setminus\tilde{\Omega}_R)}. \end{aligned} \tag{4.74}$$

We exploit the property $D^\alpha\eta \le c$ for all $|\alpha| \le 2$ and the fact that $\Omega_R \setminus \tilde{\Omega}_R$ has positive distance to the singular points, and arrive at

$$\|w_2\|_{H^2(\Omega_R\setminus\tilde{\Omega}_R)} \le c\|w\|_{H^2(\Omega_R\setminus\tilde{\Omega}_R)} \le c\left(\|w\|_{H^1(\Omega_R)} + |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)}\right) \le c,$$

where the last estimate is another application of (4.72). Moreover, we insert the global estimate from Theorem 13 into (4.74) and get

$$a(\tilde{y}-\tilde{y}_h, w_2) \le ch^2|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)}. \tag{4.75}$$

With the equations (4.55) and (4.57), the decomposition $w = w_1 + w_2$, and the estimates (4.73) and (4.75), we get

$$\begin{aligned} &\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \\ &\le c\left(c_I^{\Theta_1+1}h^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)}\right) + c_I^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)}\right). \end{aligned} \tag{4.76}$$

We fix the generic constant c and choose c_I sufficiently large such that $cc_I^{\max\{-1/2,-\mu\}} \le 1/2$. This allows us to apply a kick-back argument and we consequently arrive at

$$\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \le ch^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)}\right).$$

Taking also (4.54) into account leads to

$$\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\Omega_R)} \le ch^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)}\right).$$

Inserting this together with (4.53) into (4.52) yields the assertion. □

Now we are able to prove the main result of this section.

Theorem 18. *Assume that $f \in C^{0,\sigma}(\overline{\Omega})$ and $g \equiv 0$ and that the triangulation $\mathcal{T}_h$ satisfies the condition (4.2). Moreover, let be given weights $\vec{\alpha} \in [0,1)^{d'}$, $\vec{\beta} \in [0,2)^{d'}$ and $\vec{\gamma} \in [0,2/3)^d$, $\vec{\delta} \in [0,5/3)^d$, which satisfy*

$$\begin{aligned} &\frac{1}{2} - \lambda_j^c < \alpha_j \leq 1 - \mu, && 2 - \lambda_j^c < \beta_j \leq \frac{5}{2} - 2\mu, && \forall j \in \mathcal{C}, \\ &1 - \lambda_k^e < \gamma_k \leq 1 - \mu, && 2 - \lambda_k^e < \delta_k \leq \frac{5}{2} - 2\mu, && \forall k \in \mathcal{E}. \end{aligned} \tag{4.77}$$

Let y denote the solution of the boundary value problem (4.1) and y_h its finite element approximation. Then some $c > 0$ exists such that

$$\|y-y_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2} \left(\sum_{|\alpha|=1} \|D^\alpha y\|_{W^{1,2}_{\vec{\alpha},\vec{\gamma}}(\Omega)} + \sum_{|\alpha|=1} \|D^\alpha y\|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(\Omega)} + \|y\|_{L^\infty(\Omega)} \right). \tag{4.78}$$

Proof. For technical reasons we introduce further subsets

$$\breve{\Omega}_R := \operatorname{int} \bigcup_{i=2}^I \overline{\Omega_i}, \quad \tilde{\Omega}_R := \operatorname{int} \bigcup_{i=3}^I \overline{\Omega_i}, \quad \breve{\Gamma}_R := \partial\breve{\Omega}_R \cap \Gamma, \quad \tilde{\Gamma}_R := \partial\tilde{\Omega}_R \cap \Gamma.$$

Note that we have the relation $\tilde{\Omega}_R \subset\subset \breve{\Omega}_R \subset\subset \hat{\Omega}_R \subset\subset \Omega_R \subset\subset \Omega$ in the sense of Wahlbin [37, Chapter 10]. Let ω be the cut-off function defined in (4.49). In order to apply Theorem 17 we insert the intermediate function $\tilde{y}_h$ and exploit that $\tilde{y} := \omega y$ coincides with y in $\hat{\Omega}_R$. This leads to

$$\|y - y_h\|_{L^2(\breve{\Gamma}_R)} \leq \|\tilde{y} - \tilde{y}_h\|_{L^2(\hat{\Gamma}_R)} + \|\tilde{y}_h - y_h\|_{L^2(\breve{\Gamma}_R)}. \tag{4.79}$$

For the first part we may now apply the result of Theorem 17 and obtain

$$\|\tilde{y} - \tilde{y}_h\|_{L^2(\hat{\Gamma}_R)} \leq ch^2|\ln h|^{3/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega_R)} \right). \tag{4.80}$$

Note that it is possible to construct a cut-off function ω satisfying (4.49) and $\|D^\alpha \omega\|_{L^\infty(\Omega_R)} \leq c$ with some $c > 0$ depending only on $|\alpha|$ and $\operatorname{dist}(\partial\hat{\Omega}_R \setminus \Gamma, \partial\Omega_R \setminus \Gamma) = 1/2$. Moreover, we have $D^\alpha \omega = 0$ on $\hat{\Omega}_R$ for $|\alpha| \geq 1$. The Leibniz rule then yields

$$\begin{aligned} |\tilde{y}|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(\Omega_R)} = |\omega y|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(\Omega_R)} &\leq c \left(|y|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(\Omega_R)} + \|y\|_{W^{1,p}(\Omega\setminus\hat{\Omega}_R)} \right) \\ &\leq c \left(\sum_{|\alpha|=1} \|D^\alpha y\|_{W^{1,p}_{\vec{\beta},\vec{\delta}}(\Omega)} + \|y\|_{L^p(\Omega)} \right), \quad p \in [1,\infty], \end{aligned} \tag{4.81}$$

where we also exploited that the weights are bounded within $\Omega \setminus \hat{\Omega}_R$.

Let us discuss the second part of (4.79). For the error of $\tilde{y}_h - y_h$ we first apply a trace theorem and exploit that $\tilde{y}_h - y_h$ is discrete harmonic on $\hat{\Omega}_R$. Hence, the discrete Caccioppoli-type estimate of Lemma 3.3 in [15] can be applied and we obtain

$$\|\tilde{y}_h - y_h\|_{L^2(\breve{\Gamma}_R)} \leq c\|\tilde{y}_h - y_h\|_{H^1(\breve{\Omega}_R)} \leq c\|\tilde{y}_h - y_h\|_{L^2(\hat{\Omega}_R)} \leq c \left(\|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_R)} + \|y - y_h\|_{L^2(\Omega)} \right).$$

Here we also used the properties $\operatorname{dist}(\partial\hat{\Omega}\backslash\Gamma, \partial\breve{\Omega}\backslash\Gamma) = 1/4 \leq c$ and $y = \tilde{y}$ on $\hat{\Omega}_R$. An application of Theorem 13 yields then

$$\|\tilde{y}_h - y_h\|_{L^2(\breve{\Gamma}_R)} \leq ch^2 \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega)} + |y|_{W^{2,2}_{\vec{\alpha},\vec{\gamma}}(\Omega)}\right) \leq ch^2 \left(\sum_{|\alpha|=1} \|D^\alpha y\|_{W^{1,2}_{\vec{\alpha},\vec{\gamma}}(\Omega)} + \|y\|_{L^2(\Omega)}\right), \tag{4.82}$$

where we reused the technique applied in (4.81) in the last step. Consequently we get from (4.79) the estimate

$$\|y-y_h\|_{L^2(\breve{\Gamma}_R)} \leq ch^2|\ln h|^{3/2} \left(\sum_{\alpha=1} \|D^\alpha y\|_{W^{1,2}_{\vec{\alpha},\vec{\gamma}}(\Omega_R)} + \sum_{\alpha=1} \|D^\alpha y\|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(\Omega)} + \|y\|_{L^\infty(\Omega)}\right). \tag{4.83}$$

Let us consider the error on the remaining part $\Gamma\backslash\breve{\Gamma}_R$ where we have no influence of the singularities. One can directly apply the trace theorem in L^∞-norm which yields

$$\|y - y_h\|_{L^2(\Gamma\backslash\breve{\Gamma}_R)} \leq c\|y - y_h\|_{L^\infty(\Gamma\backslash\breve{\Gamma}_R)} \leq c\|y - y_h\|_{L^\infty(\Omega\backslash\breve{\Omega}_R)}. \tag{4.84}$$

Now we may apply the local maximum norm estimate of Theorem 10.1 in [37] (compare also (4.40)) which yields

$$\|y - y_h\|_{L^\infty(\Omega\backslash\breve{\Omega}_R)} \leq c\left(|\ln h|\|y - I_h y\|_{L^\infty(\Omega\backslash\tilde{\Omega}_R)} + \|y - y_h\|_{L^2(\Omega\backslash\tilde{\Omega}_R)}\right). \tag{4.85}$$

The second part on the right-hand side of (4.85) has been estimated in Theorem 13 whose assumptions are obviously satisfied. The element where the maximum of $|y - I_h y|$ is acquired is denoted by T^*. An application of a standard $L^\infty(T^*)$ interpolation error estimate implies

$$\|y - I_h y\|_{L^\infty(\Omega\backslash\tilde{\Omega}_R)} \leq \|y - I_h y\|_{L^\infty(T^*)} \leq ch^2|y|_{W^{2,\infty}(T^*)}$$

and we may insert the weights which are bounded by a constant within $\Omega\backslash\tilde{\Omega}_R$. This estimate together with (4.85) and (4.84) yields

$$\|y - y_h\|_{L^2(\Gamma\backslash\breve{\Gamma}_R)} \leq ch^2|\ln h||y|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(\Omega)},$$

and together with (4.83) the desired estimate (4.78) follows. $\square$

From the above theorem we directly conclude the following observations.

Corollary 19. *Assume that $f \in C^{0,\sigma}(\overline{\Omega})$ for arbitrary $\sigma \in (0,1)$, and $g \equiv 0$. The error estimate*

$$\|y - y_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2}$$

holds, provided that one of the following assumptions is satisfied:

1. *The triangulation $\mathcal{T}_h$ is quasi-uniform (i. e. $\mu = 1$), and the singular exponents satisfy $\lambda_k^e > 3/2, \lambda_j^c > 3/2$ for all $k \in \mathcal{E}$ and $j \in \mathcal{C}$.*
2. *The triangulation $\mathcal{T}_h$ is refined according to (4.2) with parameter*

$$\mu < \frac{1}{4} + \frac{1}{2}\min\left\{\min_{k\in\mathcal{E}} \lambda_k^e, \min_{j\in\mathcal{C}} \lambda_k^c\right\}. \tag{4.86}$$

Remark 20. *One observes that the refinement condition necessary for an optimal convergence rate in $L^2(\Gamma)$-norm is a different one than for error estimates in the $L^2(\Omega)$-norm (see Corollary 14). However, on the one hand we have*

$$\mu < \frac{1}{4} + \frac{1}{2}\lambda_k^e < \lambda_k^e,$$

since $\lambda_k^e > 1/2$, and, on the other hand we have

$$\mu < \frac{1}{4} + \frac{1}{2}\lambda_j^c < \frac{1}{2} + \lambda_j^c,$$

since $\lambda_j^c > 0$. Thus, the mesh grading condition required for optimal error estimates on the boundary from Corollary 19 implies the condition required for the estimates in the domain from Corollary 14.

Remark 21. *Let us recall the fact that $\lambda_k^e > 1/2$ ($k \in \mathcal{E}$) and $\lambda_j^c > 0$ ($j \in \mathcal{C}$). If there are only singular edges but no corners, one can always find a refinement parameter $\mu > 1/2$ satisfying (4.86). However, it may happen that the corner singular exponent is very close to 0 and hence, we have to choose μ close to $1/4$. With our refinement strategy the number of elements contained in $\mathcal{T}_h$ is then not of order h^{-3}, and as a consequence we would not get optimal convergence with respect to the number of degrees of freedom #DOF. As a remedy, we can use another refinement strategy, for instance the one from [25], which allows a stronger refinement only towards the corner with parameter $\nu \in (0,1]$. In a vicinity of a corner we can choose ν arbitrarily small and still have $\#DOF = \mathcal{O}(h^{-3})$. Unfortunately, the extension of the proof of Theorem 18 to this advanced refinement strategy is not obvious.*

5 Error estimates for the optimal control problem

We are now in the position to formulate the second main result of this paper. Replacing the error norms in Lemma 1 with the estimates obtained in Corollary 14 and Corollary 19 yields:

Theorem 22. *Let $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$, and $f \in L^2(\Omega)$. By $(\bar{y}, \bar{u})$ and $(\bar{y}_h, \bar{u}_h)$ we denote the solutions of (1.1) and (2.5), respectively. Assume that the refinement criterion (4.2) is satisfied with parameter*

$$\mu < \frac{1}{4} + \frac{1}{2}\min\left\{\min_{k\in\mathcal{E}} \lambda_k^e, \min_{j\in\mathcal{C}} \lambda_j^c\right\}. \tag{5.1}$$

Then, the error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}$$

holds.

Proof. To show the desired estimate we insert the results from Corollary 14 and Corollary 19 into Lemma 1 and get

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2} \tag{5.2}$$

which holds true once we have shown

$$\|\bar{u}\|_{H^{1/2}(\Gamma)} + \|\bar{y} - \bar{y}_d\|_{C^{0,\sigma}(\overline{\Omega})} \le c. \tag{5.3}$$

As a consequence of the projection formula (1.5), we get

$$\|\bar{u}\|_{H^{1/2}(\Gamma)} \leq c \left(\|\bar{u}\|_{L^2(\Gamma)} + |\Pi_{ad}(-\alpha^{-1}\bar{p})|_{H^{1/2}(\Gamma)} \right). \tag{5.4}$$

For the second term on the right-hand side we exploit the definition of the $H^{1/2}(\Gamma)$-seminorm, distinguish among the subsets of Γ where $\bar{u}$ is active and inactive, and find analogous to [6, Theorem 4.1] that

$$|\Pi_{ad}(-\alpha^{-1}\bar{p})|_{H^{1/2}(\Gamma)} c \leq |-\alpha^{-1}\bar{p}|_{H^{1/2}(\Gamma)}.$$

Together with a trace theorem and the Lax-Milgram Lemma we get from (5.4)

$$\|\bar{u}\|_{H^{1/2}(\Gamma)} \leq c \left(\|\bar{u}\|_{L^2(\Gamma)} + \|\bar{y} - y_d\|_{L^2(\Omega)} \right) \leq c \left(\|f\|_{L^2(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{L^2(\Omega)} \right) \leq c. \tag{5.5}$$

In order to show the regularity of $\bar{y}$ in (5.3) we exploit the embedding

$$H^{3/2+\varepsilon_{reg}}(\Omega) \hookrightarrow C^{0,\sigma}(\overline{\Omega})$$

which holds for $\sigma \in (0, \varepsilon_{reg})$. The regularity $\bar{y} \in H^{3/2+\varepsilon_{reg}}(\Omega)$ with some $\varepsilon_{reg} \in (0, 1/2]$ is stated in [14, Corollary 23.5] and holds under the assumption $f \in L^2(\Omega)$ and $\bar{u} \in H^{1/2}(\Gamma)$, which has been discussed already in (5.5). Collecting up all regularity results leads to the assertion. □

Corollary 23. *If the conditions of Theorem 22 are satisfied the estimate*

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq ch^2 |\ln h|^{3/2}.$$

holds.

Proof. Using the representation (2.4) we get

$$\begin{aligned}\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} &= \|S^*(\bar{y} - y_d) - S_h^*(\bar{y}_h - y_d)\|_{L^2(\Gamma)} \\ &\leq c\|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Gamma)}.\end{aligned}$$

Applying the boundedness of S_h^* from $L^2(\Omega)$ to $L^2(\Gamma)$, and the estimates from Theorem 18 and Theorem 13, leads to the assertion. □

Example 24 (Pure edge singularities)**.** *The case of pure edge singularities is the easier case since the singular exponent $\lambda_k^e := \pi/\omega_k$ is explicitly known. Consequently, the assumption of Theorem 22 reads*

$$\mu < \frac{1}{4} + \frac{\pi}{2\omega_k},$$

which guarantees optimal convergence rates. If the angle tends to $360°$ the refinement parameter has to be chosen close to $1/2$. The L-shaped domain with interior angle of $270°$ has edge singular exponent $\lambda_k^e = 2/3$ and thus, the assumptions of Theorem 22 require

$$\mu < \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.$$

The singular exponents for the endpoints of the singular edge are given by $\lambda_j^c = 5/3$ which is proven in [35]. Due to $\lambda_j^c > \lambda_k^e$ no additional refinement towards the corner is necessary here. The three-dimensional L-shape will be considered in our experiments in Section 6.

Example 25 (Fichera corner)**.** *In Section 3 we have already discussed the singular behaviour at a Fichera corner and found that the corner singular exponent is approximately given by* $\lambda_j^c \approx 0.84$*. For the edges having an endpoint in this corner we have* $\lambda_k^e = 2/3$*. The assumptions of Theorem 22 then yield the condition*

$$\mu < \frac{1}{4} + \frac{1}{2}\min\left\{\lambda_j^c, \min_{k\in X_j}\lambda_k^e\right\} = \frac{1}{4} + \frac{\lambda_k^e}{2} = \frac{7}{12}.$$

Because of $\lambda_j^c > \lambda_k^e$*, the Fichera corner does not require an additional refinement in a vicinity of the corner. The criterion for the singular edges is stronger in this case.*

6 Numerical experiments

To confirm our theoretical investigations we constructed a simple benchmark example. We consider the problem

$$\min_{(y,u)\in L^2(\Omega)\times U_{ad}} \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} + \int_\Gamma y(x)g_2(x)\mathrm{d}s_x$$

subject to the constraints

$$-\Delta y + y = f \quad \text{in} \quad \Omega, \qquad \partial_n y = u + g_1 \quad \text{on} \quad \Gamma,$$

and $u \le 0.4$. Due to the additional term in the target functional the adjoint problem now reads

$$-\Delta p + p = y - y_d \text{ in } \Omega, \qquad \partial_n p = g_2 \text{ on } \Gamma.$$

This allows us to construct an example which has an analytic solution. Note that our theory can be simply extended to this modified problem.

In this example, Ω is the three-dimensional L-shape, see Figure 6. The exact solution given in cylinder coordinates (r,φ,z) with z-axis coinciding with the singular edge is set to

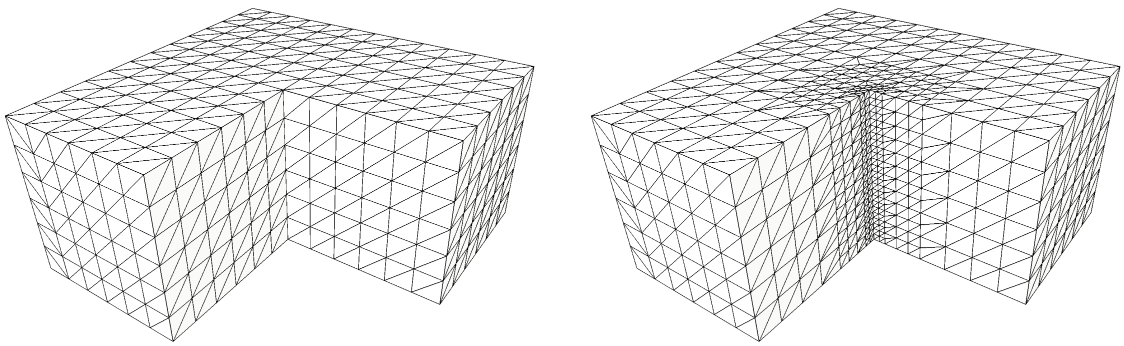
$$\bar{y} = (16z^4 - 32z^3 + 16z^2 + 1)r^{\lambda^e}\cos(\lambda^e\varphi).$$

Obviously, it contains the dominant part of the edge singularity if $\lambda^e := \pi/\omega = 2/3$. Furthermore, let $\bar{p} = \bar{y}$ be the optimal adjoint state and let $\bar{u}$ be defined by the projection formula. The input data f and y_d can be computed by means of state and adjoint equation and the functions $g_1, g_2 \in L^2(\Gamma)$ are introduced such that the boundary conditions of state and adjoint equation are satisfied. The singular behaviour on this domain has been discussed in Example 24.

We computed the solution of our model problem on the uniform mesh ($\mu = 1$), on a slightly refined mesh ($\mu = 0.7777$), on a mesh which guarantees optimal convergence for the finite element error in $L^2(\Omega)$-norm but not in $L^2(\Gamma)$-norm ($\mu = 0.6666$), and on a mesh satisfying the assumptions of the main result Theorem 22 ($\mu = 0.5$). The refinement routine works as follows. We marked all cells for those

$$h_T > h\left(\frac{\tilde{r}_T}{R}\right)^{1-\mu} \quad \text{with} \quad \tilde{r}_T := \sup_{x\in T}\sqrt{x_1^2 + x_2^2} \tag{6.1}$$

holds. The parameter R is the refinement radius, i. e. all elements which have distance less than R to the singular edge are at least refined once when $\mu < 1$, provided that all elements



(a) $\mu = 1$

(b) $\mu = 0.5$

Figure 6: Local refinement of a very coarse grid

within this radius have equal diameter (see e. g. Figure 6a). In the present experiment we used $R = 0.2$. All marked cells are then decomposed regularly, i.e. each tetrahedron is divided into eight pieces. To construct a conforming closure we adapt the strategy that has been studied in [11]. The marking and refinement procedure is repeated until there is no element satisfying (6.1) any more. This refinement strategy applied to a coarse mesh can be seen in Figure 6b.

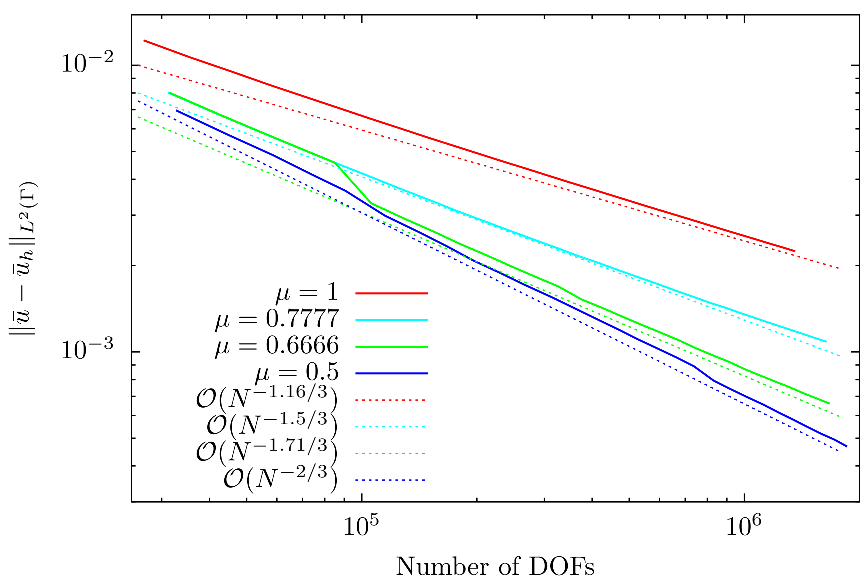


Figure 7: Error $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$, solid lines: measured error, dotted lines: expected behaviour

Figure 7 confirms that the measured error coincides with the theoretically predicted behaviour which is illustrated by the dotted lines. The results widely coincide with experiments for two-dimensional problems with the post-processing strategy in [6].

7 Conclusion and outlook

In the present paper we derived a convergence proof for the variational discretisation concept applied to a Neumann boundary control problem. We found that we can achieve the optimal convergence order of almost two on general polyhedra with sufficient mesh refinement. In comparison to distributed control problems, we now have to use stronger mesh refinement due to the error estimate on the boundary discussed in Theorem 18. The predicted behaviour was also confirmed by numerical experiments.

Subject of future research is to prove discretisation error estimates for other discretisation approaches like the post-processing technique [26]. Also in these convergence proofs the finite element error in the domain and on the boundary derived in Section 4 are involved.

The advantage of the presented refinement strategy is that it is easy to implement. However, it is actually not necessary to refine along the x_3-direction of a singular edge because one can show that the x_3-derivative of our solution has a higher regularity (compare Section 6.5.1 of [28]). As a consequence anisotropic finite elements are often used. Convergence analysis for the finite element method has been considered, e. g. in [2, 3, 4, 8]. The missing part is the proof of a finite element error estimate on the boundary for these meshes. The technique presented here is not directly applicable, since the maximum norm estimate (4.40) used in Lemma 16 is only valid for at least locally quasi-uniform meshes (cf. [32]). The same holds for the local $H^1(\Omega)$-estimate used in (4.59). An extension to anisotropic elements is subject of current research.

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