

Optimal Control of a Linear Unsteady Fluid–Structure Interaction Problem

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Abstract In this paper, we consider optimal control problems governed by linear unsteady fluid–structure interaction problems. Based on a novel symmetric monolithic formulation, we derive optimality systems and provide regularity results for optimal solutions. The proposed formulation allows for natural application of gradient-based optimization algorithms and for space–time finite element discretizations.

Keywords Unsteady fluid–structure interaction · Optimal control · Adjoint equation · Optimality conditions

Mathematics Subject Classification 35B65 · 35M33 · 49K20 · 74F10

1 Introduction

Fluid–structure interaction (FSI) problems have been extensively studied from theoretical and numerical point of view in the last decade. More and more applications leading to optimal control, shape optimization, and parameter estimation of FSI are regarded recently. At the same time, efficient gradient-based algorithms for solving

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optimization problems governed by elliptic, parabolic, or hyperbolic equations are developed and deeply analyzed in the literature. All these algorithms are based on optimality systems containing appropriate adjoint equations, and building necessary optimality conditions for considered problems. However, especially in the context of optimal control problems for *unsteady* FSI problems, no optimality systems based on rigorous analysis are available in the literature.

Optimization problems for unsteady FSI configurations are studied, e.g., in [1–6], where different optimization algorithms are used, which are not based on adjoint equations. In [7,8], adjoint equations are derived for one-dimensional FSI configurations and in [9] for a stationary FSI problem. In [4,10,11], the authors discretize the FSI problem in space and time and solve in each time step a minimization problem by a gradient-based algorithm, in order to estimate Young's modulus. In [12], the authors derive formally necessary optimality conditions for an optimal control problem of a nonlinear time-dependent FSI configuration using shape derivatives. Further results on optimal feedback control of FSI can be found in [13–15], where corresponding Riccati equations are derived. In [16], the authors apply reduced basis methods for a shape optimization problem in the context of arterial blood flow.

In this paper, we formulate a model optimal control problem governed by a unsteady linear FSI problem, establish necessary optimality conditions, and analyze the regularity of the optimal solutions. To this end, we propose a novel *symmetric* monolithic formulation for the linear FSI problem. This formulation leads to an adjoint equation with the same structure as the considered linear FSI problem, which allows for a unified analytical and numerical treatment of the state and the adjoint systems.

One of the main issues in the analysis, as well as in the numerical solution of FSI problems, is the incorporation of coupling conditions between the equations describing the behavior of the fluid and of the structure, respectively. A correct treatment of such conditions for the adjoint system is indispensable for a precise description of the information transport across the interface between the fluid and the structure and, as a consequence, for an accurate evaluation of the derivatives required in gradient-based optimization algorithms. In the framework suggested in this paper, the coupling conditions in the adjoint systems have exactly the same structure as for the state system. This is advantageous not only from the theoretical point of view, but especially allows to use the same discretization schemes and the same practical solution algorithms for both the state and the adjoint systems.

The fact that the coupling conditions are directly incorporated in the variational formulation allows for a natural usage of Galerkin finite element discretizations in space and time. This is advantageous particularly for optimal control problems, since the two approaches optimize-then-discretize, i.e., the discretization of the optimality system from continuous level, and discretize-then-optimize, i.e., discretization of the state equation and subsequent construction of the optimality system on the discrete level, lead to the same discretization scheme; see, e.g., [17]. We refer, e.g., to [18–21] for a priori numerical analysis and adaptivity for Galerkin discretizations of parabolic optimal control problems. The application of these techniques to the FSI problem under consideration is a topic for future work.

Usually in FSI models, the Navier–Stokes equations are coupled with a nonlinear hyperbolic equation. As the solid motion involves large stress-induced displacements,

the fluid domain is not stationary. To solve the resulting system, the Navier–Stokes equations can be transformed to a reference domain; see, e.g., [22, 23]. This leads to a highly nonlinear coupled system. As our goal is to focus on the treatment of coupling conditions, we regard the linear FSI problem, where the Stokes equations are coupled with a linear wave equation on a domain with fixed interface; see a detailed description in Sect. 2. Although such a linear system neglects several problem-relevant phenomena, we believe that our results provide an important step toward tackling optimal control problems for nonlinear FSI models. Such linear FSI configurations have been already analyzed in [24–29], wherein the authors prove existence and regularity results. By introducing a damping term in the wave equation or in the coupling condition, uniform stability results are shown in [26, 29–34] independently of the geometry. A further class of nonlinear FSI models with a fixed interface for fluid flow motion around a smooth elastic object is analyzed in [35–37], wherein [36, 37] the authors even prove the existence of strong solutions for smooth initial conditions. A similar result can be found in [26, 38].

The rest of the paper is organized as follows. In Sect. 2, we present the considered problem and state the used notation in Sect. 2.1. After a discussion of possible monolithic formulations in Sect. 2.2, we recall known existence and regularity results from the literature for the employed linear model in Sect. 2.3. In Sect. 2.4, we introduce a novel symmetric monolithic formulation and adapt the results from Sect. 2.3 to it. Section 3 is devoted to the analysis of the optimal control problem. After the description of two model configurations, we discuss the existence of the optimal solution in Sect. 3.1. In Sect. 3.2, regularity results for the adjoint system are derived, which are indispensable to prove the main result, the optimality system, in Sect. 3.3.

2 A Linear FSI Problem

In this article, we will analyze the following linear-quadratic optimal control problem subject to a linear fluid–structure interaction problem:

$$\min J(q, u, v) := \frac{\gamma_f}{2} \int_I \|v - v_d\|_{L^2(\Omega_f)}^2 dt + \frac{\gamma_s}{2} \int_I \|u - u_d\|_{L^2(\Omega_s)}^2 dt + \frac{\alpha}{2} \|q\|_Q^2, \quad (1)$$

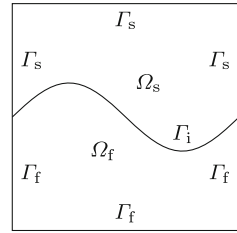
subject to the system of equations

$$\begin{aligned} \rho_f \partial_t v - \nu \Delta v - \nabla p &= B_f q && \text{in } \Omega_f \times I, \\ \operatorname{div} v &= 0 && \text{in } \Omega_f \times I, \\ \rho_s \partial_{tt} u - \mu \Delta u &= B_s q && \text{in } \Omega_s \times I, \end{aligned} \quad (2a)$$

the coupling and boundary conditions

$$\begin{aligned} v \partial_{n_f} v - p n_f + \mu \partial_{n_s} u &= 0 && \text{on } \Gamma_i \times I, \\ v &= \partial_t u && \text{on } \Gamma_i \times I, \\ v &= 0 && \text{on } \Gamma_f \times I, \\ u &= 0 && \text{on } \Gamma_s \times I, \end{aligned} \quad (2b)$$

Fig. 1 An exemplary domain Ω with fixed interface Γ_i



and the initial conditions and box constraints for the control

$$\begin{aligned}
 u(0) &= u_0, \quad \partial_t u(0) = u_1, \quad v(0) = v_0, \\
 q_a &\leq q \leq q_b.
 \end{aligned}
 \tag{2c}$$

Therefore, $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ is a domain separated in two disjoint Lipschitz sub-domains Ω_s and Ω_f with $\bar{\Omega} = \bar{\Omega}_s \cup \bar{\Omega}_f$, as presented in Fig. 1. Furthermore, $I = (0, T)$ is a given time interval and $\Gamma := \partial\Omega$ denotes the outer boundary, which is split into two parts $\Gamma_s := \Gamma \cap \bar{\Omega}_s$ and $\Gamma_f := \Gamma \cap \bar{\Omega}_f$, where we assume that Γ_s has positive measure in Γ . The interface between Ω_f and Ω_s is denoted by $\Gamma_i := \bar{\Omega}_f \cap \bar{\Omega}_s$. Moreover, n_s is the unit outward normal vector on Γ_i with respect to the region Ω_s and $n_f = -n_s$ is the unit outward normal vector with respect to the region Ω_f .

On Ω_f , the fluid is described via the Stokes equations and the structure on Ω_s behaves according to the linear wave equation, where ρ_f, ρ_s, ν , and μ are given material parameters. We note that all the results presented in this paper can be extended to the model problem, where the wave equations are replaced by the linear Lamé system.

The variables v and p denote the velocity and pressure of the fluid in Ω_f , and u denotes the structure displacement in Ω_s . At the interface Γ_i , the momentum has to be conserved. Therefore, we demand the directional derivatives to coincide on $\Gamma_i \times I$ (dynamic coupling condition): $\nu \partial_{n_f} v - p n_f = -\mu \partial_{n_s} u$. Additionally, the fluid is not allowed to enter the structure domain (kinematic coupling condition). For slow fluid flow, we can assume a no-slip condition, which implies that structure and fluid velocity have to be equal. Therefore, we demand $v = \partial_t u$ on the interface $\Gamma_i \times I$. At the outer boundaries Γ_f and Γ_s , we prescribe homogeneous Dirichlet boundary conditions.

The control q is going to be either time-dependent or distributed in space and controlling the volume force through the linear operators B_f and B_s ; see the two configurations in Sect. 3 for details. In addition, the control variable is subject to control constrains with the bounds $q_a, q_b \in \mathbb{R} \cup \{\pm\infty\}$ and $q_a < q_b$. The variables v_d and u_d are given desired states, and $\alpha > 0$ is a given regularization parameter. To enable observation on both or just on one sub-domain, the parameters $\gamma_f, \gamma_s \geq 0$ can be chosen appropriately.

Throughout, we assume the following properties of the initial data:

Assumption 1 The initial data u_0, u_1 , and v_0 satisfy

$$\begin{aligned}
 u_0 &\in H^1(\Omega_s)^d, \quad \Delta u_0 \in L^2(\Omega_s)^d, \quad u_0|_{\Gamma_s} = 0, \quad u_1 \in H^1(\Omega_s)^d, \quad u_1|_{\Gamma_s} = 0, \\
 v_0 &\in H^1(\Omega_f)^d, \quad \operatorname{div} v_0 = 0, \quad \Delta v_0 \in L^2(\Omega_f)^d, \quad v_0|_{\Gamma_f} = 0,
 \end{aligned}$$

and $v_0|_{\Gamma_i} = u_1|_{\Gamma_i}$. Furthermore, there exists a $p_0 \in H^1(\Omega_f)$ such that

$$(p_0 n_f - \nu \nabla v_0^T n_f)|_{\Gamma_i} = (\mu \nabla u_0^T n_s)|_{\Gamma_i}.$$

For brevity, we introduce

$$A_0^2 := \|u_0\|_{H^1(\Omega_s)}^2 + \|\Delta u_0\|_{L^2(\Omega_s)}^2 + \|u_1\|_{H^1(\Omega_s)}^2 + \|v_0\|_{H^1(\Omega_f)}^2 + \|\Delta v_0\|_{L^2(\Omega_f)}^2 + \|p_0\|_{H^1(\Omega_f)}^2. \tag{3}$$

Remark 2.1 The assumptions on the initial data made here are not necessary to guarantee the existence of solutions to (2). Existence and regularity results for less regular initial data can be found, e.g., in [26–29]. However, we assume more regular initial data and the thereby implied higher regularity of solutions to derive in Sect. 3.3 an optimality system for the considered optimal control problem.

2.1 Notation

To keep the notation as compact as possible, we introduce the vector-valued spaces

$$V := \{ \varphi \in H^1(\Omega)^d : \varphi = 0 \text{ on } \Gamma \},$$

$$V_f := \{ \varphi \in H^1(\Omega_f)^d : \varphi = 0 \text{ on } \Gamma_f \}, \quad \text{and} \quad V_s := \{ \varphi \in H^1(\Omega_s)^d : \varphi = 0 \text{ on } \Gamma_s \}$$

with trace zero on parts of the boundary. In addition, we will need the spaces of divergence-free functions

$$V_{\text{div}} := \{ v \in V : \text{div } v = 0 \text{ on } \Omega_f \} \quad \text{and} \quad V_{f,\text{div}} := \{ v \in V_f : \text{div } v = 0 \text{ on } \Omega_f \}.$$

Therefore, the divergence condition is only demanded on the fluid domain Ω_f . Furthermore, we introduce the vector-valued L^2 spaces $H := L^2(\Omega)^d$, $H_f := L^2(\Omega_f)^d$, and $H_s := L^2(\Omega_s)^d$. Finally, for the pressure variable, we introduce the space $L_f := \{ p \in L^2(\Omega_f) : (p, 1)_f = 0 \}$. Here and in what follows, let

$$(u, v) := (u, v)_\Omega, \quad (u, v)_f := (u, v)_{\Omega_f}, \quad (u, v)_s := (u, v)_{\Omega_s}, \quad \langle u, v \rangle_i := \langle u, v \rangle_{\Gamma_i}$$

be the L^2 inner products on Ω , on its sub-domains Ω_f and Ω_s , and on Γ_i . Furthermore, we use the notation

$$\langle\langle u, v \rangle\rangle = \int_I (u, v) \, dt$$

for inner products on the space–time cylinder $I \times \Omega$ and the analogously defined expressions $\langle\langle u, v \rangle\rangle_f$, $\langle\langle u, v \rangle\rangle_s$, and $\langle\langle u, v \rangle\rangle_i$.

2.2 Monolithic Formulation

Monolithic formulations are well known for nonlinear FSI problems, and they are used, for example, in [39,40] to obtain robust numerical algorithms, keeping errors occurring from the coupling conditions small. Such formulations are usually obtained by transforming a weak formulation of the FSI equations in a system of first order in time by introducing a structure velocity variable. Then, the kinematic coupling condition is enforced by choosing a smooth trial space for the common velocity variable defined on the whole domain. Furthermore, due to a test function defined in the same space, the dynamic coupling condition is automatically fulfilled. In the case of the here considered linear FSI problem (2), this leads to a velocity $v \in L^2(I; V)$, a pressure $p \in L^2(I; L_f)$, and a structure displacement $u \in L^2(I; V_s)$ fulfilling the weak formulation

$$\rho_f((\partial_t v, \varphi))_f + v((\nabla v, \nabla \varphi))_f - ((p, \operatorname{div} \varphi))_f + \rho_s((\partial_t v, \varphi))_s + \mu((\nabla u, \nabla \varphi))_s = ((B_f q, \varphi))_f + ((B_s q, \varphi))_s \quad \forall \varphi \in L^2(I; V), \tag{4}$$

$$((v, \psi))_s - ((\partial_t u, \psi))_s = 0 \quad \forall \psi \in L^2(I; H_s), \tag{5}$$

$$((\xi, \operatorname{div} v))_f = 0 \quad \forall \xi \in L^2(I; L_f), \tag{6}$$

together with the initial conditions

$$u(0) = u_0, \quad v(0)|_{\Omega_s} = u_1, \quad \text{and} \quad v(0)|_{\Omega_f} = v_0. \tag{7}$$

The velocity v now describes the fluid velocity on Ω_f and the velocity of the structure on Ω_s .

For optimal control, this formulation has some drawbacks due to its asymmetry, see Sect. 3.2.1 for details. Because of this, we favor a slightly different formulation, which is motivated by an approach used by Johnson in [41] in the context of the wave equation. He suggests to introduce a velocity variable v , which fulfills

$$\mu((\nabla v, \nabla \psi))_s - \mu((\nabla \partial_t u, \nabla \psi))_s = 0 \quad \forall \psi \in L^2(I; V_s) \tag{8}$$

instead of (5). As the resulting weak equations (4), (6), and (8) are symmetric and test-and-trial spaces coincide, the resulting linear FSI system is self-adjoint.

2.3 Known Results from the Literature

The stated linear fluid–structure interaction problem (2) was intensively studied in [24] and [25] by Du, Gunzburger, and coworkers. The following proposition taken from there ensures existence and uniqueness of a solution to (2) admitting optimal regularity with respect to the assumptions on the right-hand sides:

Proposition 2.1 (Theorems 3.2 and 3.4 in [24] and Theorem 2.3 in [25]) *Let u_0, u_1 , and v_0 satisfy Assumption 1, and let $f_f \in H^1(I; V_f^*)$ and $f_s \in H^1(I; H_s)$. Then, there exists a unique triplet (v, p, u) with*

$$v \in H^1(I; V_f) \cap W^{1,\infty}(I; H_f), \quad p \in L^2(I; L_f), \quad u \in W^{1,\infty}(I; V_s) \cap W^{2,\infty}(I; H_s)$$

satisfying the initial conditions $v(0) = v_0$ in H_f , $u(0) = u_0$ in V_s , and $\partial_t u(0) = u_1$ in H_s , as well the coupling condition

$$v|_{\Gamma_i} = \partial_t u|_{\Gamma_i} \text{ in } L^2(I; H^{\frac{1}{2}}(\Gamma_i)^d),$$

and almost everywhere in I

$$\begin{aligned} \rho_f(\partial_t v, \varphi)_f + v(\nabla v, \nabla \varphi)_f - (p, \operatorname{div} \varphi)_f + \rho_s(\partial_{tt} u, \varphi)_s \\ + \mu(\nabla u, \nabla \varphi)_s = (f_f, \varphi)_f + (f_s, \varphi)_s \quad \forall \varphi \in V, \\ (\xi, \operatorname{div} v)_f = 0 \quad \forall \xi \in L_f. \end{aligned}$$

Furthermore, the solution fulfills the a priori estimates

- a) $\|v\|_{L^2(I; H^1(\Omega_f))}^2 + \|v\|_{L^\infty(I; L^2(\Omega_f))}^2 + \|u\|_{L^\infty(I; H^1(\Omega_s))}^2 + \|\partial_t u\|_{L^\infty(I; L^2(\Omega_s))}^2 \leq C[\|f_f\|_{L^2(I; V_f^*)}^2 + \|f_s\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2],$
- b) $\|\partial_t v\|_{L^2(I; H^1(\Omega_f))}^2 + \|\partial_t v\|_{L^\infty(I; L^2(\Omega_f))}^2 + \|\partial_t u\|_{L^\infty(I; H^1(\Omega_s))}^2 + \|\partial_{tt} u\|_{L^\infty(I; L^2(\Omega_s))}^2 \leq C[\|f_f\|_{H^1(I; V_f^*)}^2 + \|f_s\|_{H^1(I; L^2(\Omega_s))}^2 + A_0^2],$
- c) $\|p\|_{L^2(I; L^2(\Omega_f))}^2 \leq C[\|f_f\|_{H^1(I; V_f^*)}^2 + \|f_s\|_{H^1(I; L^2(\Omega_s))}^2 + A_0^2],$

where A_0^2 is defined in (3).

Remark 2.2 Under similar assumptions on the initial data, the stated result on existence, uniqueness, and regularity of solutions has also been proven in [26–29] by Avalos and Triggiani for an elastic solid completely surrounded by a fluid.

Remark 2.3 Clearly, the solution given by Proposition 2.1 also fulfills the weak space-time formulation

$$\begin{aligned} \rho_f((\partial_t v, \varphi))_f + v((\nabla v, \nabla \varphi))_f - ((p, \operatorname{div} \varphi))_f + \rho_s((\partial_{tt} u, \varphi))_s \\ + \mu((\nabla u, \nabla \varphi))_s = ((f_f, \varphi))_f + ((f_s, \varphi))_s \quad \forall \varphi \in L^2(I; V), \quad (9) \\ ((\xi, \operatorname{div} v))_f = 0 \quad \forall \xi \in L^2(I; L_f). \end{aligned}$$

Later, we are going to control the system by right-hand sides, which are just in L^2 with respect to time. To do so, we will need the existence of solutions even if the right-hand side has less regularity than assumed in Proposition 2.1. The existence of such a solution operator is guaranteed by the results presented in [24–29]. Here, we will make use of the following result given in [24, 25]:

Proposition 2.2 (Theorem 2.5 in [24] and Theorem 2.2 in [25]) *Let u_0, u_1 , and v_0 satisfy Assumption 1 and let $f_f \in L^2(I; V_f^*)$ and $f_s \in L^2(I; H_s)$. Then, there exists a unique $\tilde{v} \in H^1(I; V_{\operatorname{div}}^*)$ with*

$$v = \tilde{v}|_{\Omega_f} \in L^2(I; V_{f,\text{div}}) \cap L^\infty(I; H_f)$$

and $u = \int_0^t \tilde{v}(s)|_{\Omega_s} ds + u_0 \in L^\infty(I; V_s) \cap W^{1,\infty}(I; H_s)$

satisfying the initial conditions $v(0) = v_0$ in H_f , $u(0) = u_0$ in V_s , and $\partial_t u(0) = u_1$ in H_s , as well as the coupling condition

$$\int_0^t v(s)|_{\Gamma_i} ds = u(t)|_{\Gamma_i} - u_0|_{\Gamma_i} \text{ in } L^2(I; H^{\frac{1}{2}}(\Gamma_i)^d)$$

and almost everywhere in I

$$\rho_f(\partial_t \tilde{v}, \varphi)_f + \nu(\nabla v, \nabla \varphi)_f + \rho_s(\partial_t \tilde{v}, \varphi)_s + \mu(\nabla u, \nabla \varphi)_s = (f_f, \varphi)_f + (f_s, \varphi)_s \quad \forall \varphi \in V_{\text{div}}.$$

Furthermore, the solution (u, v) fulfills the a priori estimate a) in Proposition 2.1.

Remark 2.4 Clearly, the solution given by Proposition 2.2 fulfills also the space–time weak formulation

$$\begin{aligned} &\rho_f((\partial_t \tilde{v}, \varphi))_f + \nu((\nabla v, \nabla \varphi))_f + \rho_s((\partial_t \tilde{v}, \varphi))_s + \mu((\nabla u, \nabla \varphi))_s \\ &= ((f_f, \varphi))_f + ((f_s, \varphi))_s \quad \forall \varphi \in L^2(I; V_{\text{div}}), \end{aligned} \tag{10}$$

and for right-hand sides fulfilling the assumptions of Proposition 2.1, it coincides with the solution given there.

Remark 2.5 In [24, 25], the authors proved Proposition 2.2 under weaker assumptions on the initial data than in Assumption 1 here. For even weaker assumptions on the initial data, the results in [26–29] show the existence of a unique solution for linear FSI problems. There, the authors use semigroup theory and a technique to derive a pressure-free formulation. Therefore, they were able to prove the existence of a mild solution and optimal regularity results. As the goal of this paper was to present an optimality system which can be discretized using a Galerkin finite element discretization in space and time, we require the solution to fulfill a weak formulation with explicit pressure and including the coupling conditions. Hence, we will throughout make use of initial data fulfilling Assumption 1, which ensures a regular solution as far as the right-hand sides fulfill the assumptions in Proposition 2.1.

Remark 2.6 In [24, 25], the authors demand $f_f \in L^2(I; H_f)$ in Proposition 2.2 and $f_f \in H^1(I; H_f)$, $u_0 \in H^2(\Omega_s)^d$, $v_0 \in H^2(\Omega_f)^d$ in Proposition 2.1. However, the proofs can directly be extended to $f_f \in L^2(I; V_f^*)$ and $f_f \in H^1(I; V_f^*)$ with initial data fulfilling Assumption 1 as stated above.

Remark 2.7 The results in [24, 25] and [28] are more general and also apply for Stokes flow coupled with linear elasticity equations. Therefore, all the results presented in the following are also extendable to formulations with stress tensors.

2.4 Novel Symmetric Weak Formulation

Now, we closer regard the symmetric weak formulation given by (4), (6), and (8), which was already motivated in Sect. 2.2. In doing so, we consider the following system of equations, where—for the remaining part of this section—the control terms are replaced by the right-hand sides f_f and f_s :

$$\begin{aligned}
 \rho_f((\partial_t v, \varphi))_f + \nu((\nabla v, \nabla \varphi))_f - ((p, \operatorname{div} \varphi))_f + \rho_s((\partial_t v, \varphi))_s \\
 + \mu((\nabla u, \nabla \varphi))_s = ((f_f, \varphi))_f + ((f_s, \varphi))_s \quad \forall \varphi \in L^2(I; V), \\
 \mu((\nabla v, \nabla \psi))_s - \mu((\nabla \partial_t u, \nabla \psi))_s = ((g, \psi))_s \quad \forall \psi \in L^2(I; V_s), \\
 ((\xi, \operatorname{div} v))_f = 0 \quad \forall \xi \in L^2(I; L_f).
 \end{aligned}
 \tag{11}$$

Additionally, as before, the initial conditions given by (7) have to be fulfilled. Note that the volume force g , appearing on the right-hand side of the equation introducing the structure velocity, has no physical interpretation but will occur later in the adjoint equation, see Sect. 3.

The results in Sect. 2.3 enable us to prove the existence and uniqueness of a solution for this weak formulation.

Theorem 2.1 *Let u_0, u_1 , and v_0 satisfy Assumption 1, and let the right-hand sides fulfill $f_f \in H^1(I; V_f^*)$, $f_s \in H^1(I; H_s)$, and $g \in L^2(I; H_s)$. Then, there exists a unique triplet (v, p, u) with*

$$\begin{aligned}
 v \in L^2(I; V) \cap W^{1;\infty}(I, H), \quad v|_{\Omega_f} \in H^1(I; V_f), \quad v|_{\Omega_s} \in L^\infty(I; V_s), \\
 u \in L^\infty(I; V_s) \cap H^1(I; V_s), \quad p \in L^2(I; L_f),
 \end{aligned}$$

which satisfies the initial conditions (7) and solves (11). Furthermore, the solution fulfills the a priori estimates

$$\begin{aligned}
 a) \quad & \|v\|_{L^\infty(I; L^2(\Omega))}^2 + \|v\|_{L^2(I; H^1(\Omega_f))}^2 + \|u\|_{L^\infty(I; H^1(\Omega_s))}^2 \\
 & \leq C[\|f_f\|_{L^2(I; V_f^*)}^2 + \|f_s\|_{L^2(I; L^2(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2], \\
 b) \quad & \|\partial_t v\|_{L^\infty(I; L^2(\Omega))}^2 + \|\partial_t v\|_{L^2(I; H^1(\Omega_f))}^2 + \|v\|_{L^\infty(I; H^1(\Omega_s))}^2 + \|\partial_t u\|_{L^2(I; H^1(\Omega_s))}^2 \\
 & \leq C[\|f_f\|_{H^1(I; V_f^*)}^2 + \|f_s\|_{H^1(I; L^2(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2], \\
 c) \quad & \|p\|_{L^2(I; L^2(\Omega_f))}^2 \leq C[\|f_f\|_{H^1(I; V_f^*)}^2 + \|f_s\|_{H^1(I; L^2(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2],
 \end{aligned}$$

where A_0^2 is defined in (3).

Proof Let $\hat{f}_f := f_f$ and $\hat{f}_s := f_s + \int_0^t g(s) \, ds$. Due to the assumptions on the data, we have $\hat{f}_f \in H^1(I; V_f^*)$ and $\hat{f}_s \in H^1(I; H_s)$. Therefore, Proposition 2.1 ensures for these right-hand sides and the given initial data existence of a unique triplet $(\hat{v}_f, \hat{p}, \hat{u})$ solving (9). Next, we introduce a structure velocity \hat{v}_s by the setting $\hat{v}_s = \partial_t \hat{u} \in L^\infty(I; V_s)$. Thus, \hat{v}_s fulfills

$$\mu(\nabla \hat{v}_s, \nabla \psi)_s = \mu(\nabla \partial_t \hat{u}, \nabla \psi)_s \quad \forall \psi \in L^2(I; V_s). \tag{12}$$

Now, we are prepared to introduce the global velocity \hat{v} by setting $\hat{v}|_{\Omega_f} := \hat{v}_f$ and $\hat{v}|_{\Omega_s} := \hat{v}_s$. As $\hat{v}_f \in L^2(I; V_f)$ and $\hat{v}_s \in L^2(I; V_s)$, we get immediately $\hat{v} \in L^2(I; H)$. To obtain $\hat{v} \in L^2(I; V)$, we have to check that the weak partial derivatives \hat{w}_i with $\hat{w}_i|_{\Omega_f} := \partial_{x_i} \hat{v}_f$ and $\hat{w}_i|_{\Omega_s} := \partial_{x_i} \hat{v}_s$ constitute the weak partial derivatives $\partial_{x_i} \hat{v}$ of \hat{v} for $i = 1, 2, \dots, d$. To this end, let $\varphi \in L^2(I; C_0^\infty(\Omega)^d)$. We obtain by the definition of the weak derivatives

$$\begin{aligned} (\hat{v}, \partial_{x_i} \varphi) &= (\hat{v}_f, \partial_{x_i} \varphi)_f + (\hat{v}_s, \partial_{x_i} \varphi)_s \\ &= -(\partial_{x_i} \hat{v}_f, \varphi)_f - (\partial_{x_i} \hat{v}_s, \varphi)_s + \langle \hat{v}_f, \varphi n_f^T e_i \rangle_i + \langle \hat{v}_s, \varphi n_s^T e_i \rangle_i \\ &= -(\hat{w}_i, \varphi) + \langle \partial_t \hat{u} - \hat{v}_f, \varphi n_s^T e_i \rangle_i = -(\hat{w}_i, \varphi), \end{aligned}$$

where the last step holds, since the kinematic coupling condition is valid due to Proposition 2.1. Therefore, it holds $\hat{v} \in L^2(I; V)$.

It remains to prove that $(\hat{v}, \hat{p}, \hat{u})$ solves the weak formulation (11). Due to the construction of \hat{v}_s by (12), we directly get

$$\rho_s(\partial_t \hat{u}, \varphi)_s = \rho_s(\partial_t \hat{v}, \varphi)_s \quad \forall \varphi \in L^2(I; V_s). \tag{13}$$

If we enter (13) in the weak formulation (9), we immediately obtain with (12) that the triplet $(\hat{v}, \hat{p}, \hat{u})$ solves the weak formulation (11) with the right-hand sides \hat{f}_f, \hat{f}_s , and $g = 0$.

In what follows, we construct a solution to (11) with the original right-hand sides f_f, f_s , and g . We define $\tilde{u}: I \rightarrow V_s$ for almost all $t \in I$ by

$$\mu(\nabla \tilde{u}(t), \nabla \psi)_s := (-g(t), \psi)_s \quad \forall \psi \in V_s. \tag{14}$$

Standard elliptic theory guarantees the existence and uniqueness of $\tilde{u}(t)$ together with the estimate

$$\|\tilde{u}(t)\|_{H^1(\Omega_s)} \leq C \|g(t)\|_{L^2(\Omega_s)} \quad \text{for almost all } t \in I. \tag{15}$$

As $g \in L^2(I; H_s)$, integration in time of the above inequality immediately leads to $\tilde{u} \in L^2(I; V_s)$. Furthermore, integrating (14) in time twice implies

$$\mu \left(\left(\nabla \int_0^t \tilde{u}(s) \, ds, \nabla \psi \right) \right)_s = - \left(\left(\int_0^t g(s) \, ds, \psi \right) \right)_s \quad \forall \psi \in L^2(I; V_s).$$

Defining $u := \hat{u} + \int_0^t \tilde{u} \, ds$, we directly obtain $u \in L^2(I; V_s)$. Since for $\varphi \in \hat{L}^2(I; V)$ it holds $\psi = \varphi|_{\Omega_s} \in L^2(I; V_s)$, we get for all $\varphi \in L^2(I; V)$ the identity

$$\begin{aligned} \mu(\nabla \hat{u}, \nabla \varphi)_s &= \mu(\nabla \hat{u}, \nabla \varphi)_s + \mu \left(\left(\nabla \int_0^t \tilde{u}(s) \, ds, \nabla \varphi \right) \right)_s + \left(\left(\int_0^t g(s) \, ds, \varphi \right) \right)_s \\ &= \mu(\nabla u, \nabla \varphi)_s + \left(\left(\int_0^t g(s) \, ds, \varphi \right) \right)_s. \end{aligned}$$

Together with the definition of \hat{f}_s , this implies that $u, v := \hat{v}$, and $p := \hat{p}$ solves the first equation of (11). Furthermore, since \hat{u} and \hat{v} solve the second equation of (11) with $g = 0$, we obtain for all $\psi \in L^2(I; V_s)$

$$\begin{aligned} \mu(\nabla v, \nabla \psi)_s - \mu(\nabla \partial_t u, \nabla \psi)_s &= \mu(\nabla \hat{v}, \nabla \psi)_s - \mu(\nabla \partial_t \hat{u}, \nabla \psi)_s - \mu(\nabla \tilde{u}, \nabla \psi)_s \\ &= -\mu(\nabla \tilde{u}, \nabla \psi)_s = (g, \psi)_s. \end{aligned}$$

Therefore (v, p, u) solves the weak formulation (11) for the right-hand sides f_f, f_s , and g .

It remains to prove the uniqueness. Let (v_1, p_1, u_1) and (v_2, p_2, u_2) be two solutions fulfilling the weak formulation (11) and the regularities assumed in Theorem 2.1. Define $\bar{v} := v_1 - v_2, \bar{p} := p_1 - p_2, \bar{u} := u_1 - u_2$. It holds

$$\bar{v}|_{\Omega_f}(0) = 0, \quad \bar{v}|_{\Omega_s}(0) = 0, \quad \bar{u}(0) = 0,$$

and for almost all $t \in I$

$$\begin{aligned} \rho_f(\partial_t \bar{v}(t), \varphi)_f - (\bar{p}(t), \operatorname{div} \varphi)_f + \nu(\nabla \bar{v}(t), \nabla \varphi)_f \\ + \rho_s(\partial_t \bar{v}(t), \varphi)_s + \mu(\nabla \bar{u}(t), \nabla \varphi)_s &= 0 \quad \forall \varphi \in V, \\ \mu(\nabla \bar{v}(t), \nabla \psi)_s - \mu(\nabla \partial_t \bar{u}(t), \nabla \psi)_s &= 0 \quad \forall \psi \in V_s, \\ (\xi, \operatorname{div} \bar{v}(t))_f &= 0 \quad \forall \xi \in L_f. \end{aligned}$$

Choosing the test functions $\varphi = \bar{v}(t), \psi = \bar{u}(t)$, and $\xi = \bar{p}(t)$, we get

$$\begin{aligned} \rho_f(\partial_t \bar{v}(t), \bar{v}(t))_f - (\bar{p}(t), \operatorname{div} \bar{v}(t))_f + \nu(\nabla \bar{v}(t), \nabla \bar{v}(t))_f \\ + \rho_s(\partial_t \bar{v}(t), \bar{v}(t))_s + \mu(\nabla \bar{u}(t), \nabla \bar{v}(t))_s &= 0, \\ \mu(\nabla \bar{v}(t), \nabla \bar{u}(t))_s - \mu(\nabla \partial_t \bar{u}(t), \nabla \bar{u}(t))_s &= 0, \\ (\bar{p}(t), \operatorname{div} \bar{v}(t))_f &= 0. \end{aligned}$$

Because of the symmetry of the bilinear forms we obtain, for almost all $t \in I$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \rho_f \|\bar{v}(t)\|_{L^2(\Omega_f)}^2 + \nu \|\nabla \bar{v}(t)\|_{L^2(\Omega_f)}^2 + \frac{1}{2} \frac{d}{dt} \rho_s \|\bar{v}(t)\|_{L^2(\Omega_s)}^2 \\ + \frac{1}{2} \frac{d}{dt} \mu \|\nabla \bar{u}(t)\|_{L^2(\Omega_s)}^2 = 0. \end{aligned}$$

Integrating this identity in time and noting the initial conditions, we are led to

$$\frac{1}{2} \rho_f \|\bar{v}(t)\|_{L^2(\Omega_f)}^2 + \nu \int_0^t \|\nabla \bar{v}(s)\|_{L^2(\Omega_f)}^2 ds + \frac{1}{2} \rho_s \|\bar{v}(t)\|_{L^2(\Omega_s)}^2 + \frac{1}{2} \mu \|\nabla \bar{u}(t)\|_{L^2(\Omega_s)}^2 = 0$$

for almost all $t \in I$. This implies $\bar{v} = 0$ and, as \bar{u} vanishes on $\Gamma_s \subset \Gamma$ with $|\Gamma_s| > 0$, also $\bar{u} = 0$. Thus, we get, in particular for almost all $t \in I$

$$(\bar{p}, \operatorname{div} \varphi)_f = 0 \quad \forall \varphi \in V,$$

and thus $\bar{p} = 0$ since $\bar{p} \in L_f$. Therefore, the solution is unique.

According to Proposition 2.1, the solution $(\hat{v}_f, \hat{p}, \hat{u})$ of (9) fulfills the estimates given in Proposition 2.1 with right-hand side $\hat{f}_f := f_f$ and $\hat{f}_s := f_s + \int_0^t g(s) ds$. As \hat{v}_s of the formulation (11) coincides to $\partial_t \hat{u}$, the estimates for $\partial_t \hat{u}$ from the Propositions 2.2 and 2.1 are valid for $\hat{v}|_{\Omega_s} = \hat{v}_s$, too. Hence, we have

$$\begin{aligned} & \|\hat{v}\|_{L^\infty(I; L^2(\Omega))}^2 + \|\hat{v}\|_{L^2(I; H^1(\Omega_f))}^2 + \|\hat{u}\|_{L^\infty(I; H^1(\Omega_s))}^2 \\ & \leq C[\|f_f\|_{L^2(I; V_f^*)}^2 + \|f_s\|_{L^2(I; L^2(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2], \\ \|\partial_t \hat{v}\|_{L^\infty(I; L^2(\Omega))}^2 + \|\partial_t \hat{v}\|_{L^2(I; H^1(\Omega_f))}^2 + \|\hat{v}\|_{L^\infty(I; H^1(\Omega_s))}^2 \\ & \leq C[\|f_f\|_{H^1(I; V_f^*)}^2 + \|f_s\|_{H^1(I; L^2(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2], \\ \|\hat{p}\|_{L^2(I; L^2(\Omega_f))}^2 & \leq C[\|f_f\|_{H^1(I; V_f^*)}^2 + \|f_s\|_{H^1(I; L^2(\Omega_s))}^2] \end{aligned}$$

Due to the setting $v = \hat{v}$ and $p = \hat{p}$, these estimates directly transfer to v and p . To estimate $u := \hat{u} + \int_0^t \tilde{u}(s) ds$, $\tilde{u} \in V_s$ given by (14) has to be bounded. By (15), we get for almost all $t \in I$ that

$$\begin{aligned} \|u(t)\|_{H^1(\Omega_s)}^2 & = \left\| \hat{u}(t) + \int_0^t \tilde{u}(s) ds \right\|_{H^1(\Omega_s)}^2 \\ & \leq C \left[\|\hat{u}(t)\|_{H^1(\Omega_s)}^2 + T \int_I \|\tilde{u}(s)\|_{H^1(\Omega_s)}^2 ds \right] \\ & \leq C[\|\hat{u}(t)\|_{H^1(\Omega_s)}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2], \end{aligned}$$

which implies

$$\|u\|_{L^\infty(I; H^1(\Omega_s))}^2 \leq C[\|\hat{u}\|_{L^\infty(I; H^1(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2].$$

Furthermore, we get with $\partial_t \hat{u} = \hat{v}|_{\Omega_s}$ for almost all $t \in I$ that

$$\|\partial_t u(t)\|_{H^1(\Omega_s)}^2 = \|\partial_t \hat{u}(t) + \tilde{u}(t)\|_{H^1(\Omega_s)}^2 \leq C[\|\hat{v}(t)\|_{H^1(\Omega_s)}^2 + \|g(t)\|_{L^2(\Omega_s)}^2],$$

and consequently (limited through $g \in L^2(I; H_s)$)

$$\|\partial_t u\|_{L^2(I; H^1(\Omega_s))}^2 \leq C[\|\hat{v}\|_{L^2(I; H^1(\Omega_s))}^2 + \|g\|_{L^2(I; L^2(\Omega_s))}^2].$$

Together with the above estimates to for $(\hat{v}, \hat{p}, \hat{u})$, we obtain the stated estimates for (v, p, u) . □

Remark 2.8 If the right-hand side g lies in $L^\infty(I; H_s)$, then we also get an estimate for $\|\partial_t u\|_{L^\infty(I; H^1(\Omega_s))}$, as in Proposition 2.1.

In what follows, we analyze in which sense the weak solution of (11) fulfills the original fluid–structure interaction problem, and especially in which sense the coupling

conditions are fulfilled. To this end, we introduce the space $\tilde{H}^{\frac{1}{2}}(\Gamma_i)$ in the spirit of [42, Definition 1.3.2.5] by

$$\tilde{H}^{\frac{1}{2}}(\Gamma_i) = \{v \in H^{\frac{1}{2}}(\Gamma_i) : \tilde{v} \in H^{\frac{1}{2}}(\Gamma)\},$$

where \tilde{v} denotes the continuation of v on Γ by zero.

Theorem 2.2 *Let the assumptions of Theorem 2.1 be fulfilled, and let in addition $f_f \in L^2(I; H_f)$ and (v, p, u) be the solution of (11). Then, the kinematic coupling condition*

$$v|_{\Omega_f} = v|_{\Omega_s}$$

is valid in the sense of $L^2(I; H^{\frac{1}{2}}(\Gamma_i)^d) \cap H^{\frac{1}{2}}(I; L^2(\Gamma_i)^d)$. Furthermore, the dynamic coupling condition

$$v \partial_{n_f} v - p n_f + \mu \partial_{n_s} u = 0$$

holds in $L^2(I; (\tilde{H}^{\frac{1}{2}}(\Gamma_i)^d)^)$.*

Proof By Theorem 2.1, we have that $v \in L^2(I, V)$ and $v \in W^{1,\infty}(I; H) \subset H^1(I; H)$. Hence, the trace results in [43, Theorem 2.1] imply that the kinematic coupling condition $v|_{\Omega_f} = v|_{\Omega_s}$ holds on Γ_i in the space $L^2(I; H^{\frac{1}{2}}(\Gamma_i)^d) \cap H^{\frac{1}{2}}(I; L^2(\Gamma_i)^d)$.

In the remaining part of the proof, we derive the validity of the stated dynamic coupling condition. In (11), we choose test functions φ with $\varphi|_{\Omega_f} \in L^2(I; C_0^\infty(\Omega_f)^d)$ and $\varphi|_{\Omega_s} = 0$ to get

$$((\rho_f \partial_t v, \varphi))_f - ((\text{div}(v \nabla v + p \text{Id}), \varphi))_f = ((f_f, \varphi))_f \quad \forall \varphi \in L^2(I; C_0^\infty(\Omega_f)^d),$$

where $\text{div}(v \nabla v + p \text{Id})$ is defined in the distributional sense. This is equivalent to

$$\text{div}(v \nabla v + p \text{Id}) = \rho_f \partial_t v - f_f \quad \text{in } L^2(I; C_0^\infty(\Omega_f)^d)^*.$$

As Theorem 2.1 yields $v|_{\Omega_f} \in W^{1,\infty}(I; H_f)$, we get by the assumption on f_f that $\rho_f \partial_t v + f_f \in L^2(I; H_f)$. We immediately obtain that

$$\text{div}(v \nabla v + p \text{Id}) = \rho_f \partial_t v - f_f \quad \text{in } L^2(I; H_f) \tag{16}$$

and

$$\|\text{div}(v \nabla v + p \text{Id})\|_{L^2(I; L^2(\Omega_f))} \leq \|f_f\|_{L^2(I; L^2(\Omega_f))} + C \|\partial_t v\|_{L^2(I; L^2(\Omega_f))},$$

where the right-hand side is bounded according to Theorem 2.1. The same approach, but choosing $\varphi|_{\Omega_f} = 0$ and $\varphi|_{\Omega_s} \in L^2(I; C_0^\infty(\Omega_s)^d)$ leads to

$$\text{div}(\mu \nabla u) = \rho_s \partial_t v - f_s \quad \text{in } L^2(I; H_s) \tag{17}$$

and

$$\|\operatorname{div}(\mu \nabla u)\|_{L^2(I; L^2(\Omega_s))} \leq \|f_s\|_{L^2(I; L^2(\Omega_s))} + C \|\partial_t v\|_{L^2(I; L^2(\Omega_s))},$$

where the right-hand side is again bounded according to Theorem 2.1. Thus, we obtain that

$$\begin{aligned} v \nabla v + p \operatorname{Id} \in E(\Omega_f) &:= \{ \varphi \in L^2(I; H_f) : \|\operatorname{div} \varphi\|_{L^2(I; L^2(\Omega_f))} < \infty \}, \\ \mu \nabla u \in E(\Omega_s) &:= \{ \varphi \in L^2(I; H_s) : \|\operatorname{div} \varphi\|_{L^2(I; L^2(\Omega_s))} < \infty \}. \end{aligned}$$

According to [44, Ch. I §1 Theorem 1.1], the space $L^2(I; C_0^\infty(\Omega_f)^d)$ is dense in $E(\Omega_f)$ and $L^2(I; C_0^\infty(\Omega_s)^d)$ is dense in $E(\Omega_s)$. Therefore, following [45, p. 114] or [42, Theorems 1.5.3.10 and 1.5.3.11], we get

$$\begin{aligned} \|(v \nabla v + p \operatorname{Id})^T n_f\|_{L^2(I; (\tilde{H}^{\frac{1}{2}}(\Gamma_i)^d)^*)} &\leq \|v \nabla v + p \operatorname{Id}\|_{L^2(I; L^2(\Omega_f))} \\ &\quad + \|f_f\|_{L^2(I; L^2(\Omega_f))} + C \|\partial_t v\|_{L^2(I; L^2(\Omega_f))} \\ \|(\mu \nabla u)^T n_s\|_{L^2(I; (\tilde{H}^{\frac{1}{2}}(\Gamma_i)^d)^*)} &\leq \|\mu \nabla u\|_{L^2(I; L^2(\Omega_s))} \\ &\quad + \|f_s\|_{L^2(I; L^2(\Omega_s))} + C \|\partial_t v\|_{L^2(I; L^2(\Omega_s))}. \end{aligned}$$

According to [42, 45], this enables us to apply Gauß’ theorem in (11) to obtain

$$\begin{aligned} \langle v \partial_{n_f} v - p n_f, \varphi \rangle_i + \langle \mu \partial_{n_s} u, \varphi \rangle_i &= \langle (\operatorname{div}(v \nabla v + p \operatorname{Id}) - \rho_f \partial_t v + f_f, \varphi) \rangle_f \\ &\quad + \langle (\operatorname{div}(\mu \nabla u) - \rho_s \partial_t v + f_s, \varphi) \rangle_s \quad \forall \varphi \in L^2(I; V). \end{aligned}$$

This immediately implies by (17) and (16) that

$$\langle v \partial_{n_f} v - p n_f + \mu \partial_{n_s} u, \varphi \rangle_i = 0 \quad \forall \varphi \in L^2(I; V),$$

and thus, the dynamic coupling condition is fulfilled in $L^2(I; (\tilde{H}^{\frac{1}{2}}(\Gamma_i)^d)^*)$. □

3 Optimal Control Problem

In the following, we regard the optimal control problem of a linearized FSI configuration given by (1) subject to the state equation (2) and the control constraints $q_a \leq q \leq q_b$. Therefore, the initial data are assumed to satisfy Assumption 1, and for the desired states, we require $v_d \in H^1(I; V_f^*) \cap L^2(I; H_f)$ and $u_d \in L^2(I; H_s)$.

3.1 Existence and Uniqueness of Solutions

We analyze two concrete configurations for the considered control problem:

Configuration C1 Let the control space given by $Q := (L^2(\Omega)^d)^N$ with $N \in \mathbb{N}$, and let $B_f : Q \rightarrow H^1(I; H_f)$, as well as $B_s : Q \rightarrow H^1(I; H_s)$ be linear and continuous

operators, given for $q = (q^1, q^2, \dots, q^N) \in Q$ by

$$B_f q = \sum_{i=1}^N g_f^i q^i \Big|_{\Omega_f} \quad \text{and} \quad B_s q = \sum_{i=1}^N g_s^i q^i \Big|_{\Omega_s}.$$

Therefore, $g_s^i, g_f^i \in H^1(I), i = 1, 2, \dots, N$ are given functions. The admissible set Q_{ad} is defined as

$$Q_{ad} := \{ q \in Q : q_a \leq q^i(x) \leq q_b, \text{ for almost all } x \in \Omega \text{ and } i = 1, 2, \dots, N \}.$$

Note that the inequality in the definition of Q_{ad} has to be understood componentwise for $q^i \in L^2(\Omega)^d$.

Configuration C2 Let the control space given by $Q := L^2(I)^N$ with $N \in \mathbb{N}$, and let $B_f : Q \rightarrow L^2(I; H_f)$, as well as $B_s : Q \rightarrow L^2(I; H_s)$ be linear continuous operators given for $q = (q^1, q^2, \dots, q^N) \in Q$ by

$$B_f q = \sum_{i=1}^N q^i h^i \Big|_{\Omega_f} \quad \text{and} \quad B_s q = \sum_{i=1}^N q^i h^i \Big|_{\Omega_s}.$$

Therefore, $h^i \in V_{div}, i = 1, 2, \dots, N$ are given functions. The admissible set Q_{ad} is defined as

$$Q_{ad} := \{ q \in Q : q_a \leq q^i(t) \leq q_b, \text{ for almost all } t \in I \text{ and } i = 1, 2, \dots, N \}.$$

The assumption that h^i has to be divergence-free in Ω_f is taken for simplicity of the presentation. All results can be extended to $h^i \in V$ using a Helmholtz decomposition.

Since for both configurations, $B_f q \in L^2(I; H_f)$ and $B_s q \in L^2(I; H_s)$, Proposition 2.2 ensures the well-posedness of the so-defined control to state mapping $G : q \mapsto (v(q), u(q))$, with $(v(q), u(q))$ the solution of (10) for $f_f := B_s q$ and $f_s := B_s q$. The linearity of (10) and the estimate given in Proposition 2.2 imply the continuity of G :

Lemma 3.1 *The control to state mapping $G : Q \rightarrow L^2(I; H_f) \times L^2(I; H_s)$ is an affine-linear and continuous operator for both configurations C1 and C2.*

Proof Let (\hat{v}, \hat{u}) be the solution of (10) for $f_f = f_s = 0$. Furthermore, we denote by $G_0 : Q \rightarrow L^2(I; H_f) \times L^2(I; H_s)$ the linear part of G , defined by (10) with zero initial data for $f_f := B_s q$ and $f_s := B_s q$. Hence, the mapping $G : Q \rightarrow L^2(I; H_f) \times L^2(I; H_s)$ can be expressed as

$$(v(q), u(q)) = Gq = (\hat{v}, \hat{u}) + G_0 q.$$

Proposition 2.2 yields that (\hat{v}, \hat{u}) is bounded in $L^\infty(I; H_f) \times L^\infty(I; V_s)$ and G_0 is a bounded linear operator. Thus, the control to state mapping G is continuous in both considered configurations. □

By means of the control to state mapping G , the reduced functional $j: Q \rightarrow \mathbb{R}$ can be defined as

$$j(q) := J(q, u(q), v(q)), \tag{18}$$

and the optimal control problem under consideration can for both configurations be written in the compact form

$$\min_{q \in Q_{\text{ad}}} j(q). \tag{19}$$

Theorem 3.1 *For both configurations C1 and C2, the considered optimal control problem (19) admits a unique solution.*

Proof Standard arguments, see, e.g., [46, Theorem 2.14], guarantee the existence of a unique optimal control $\bar{q} \in Q_{\text{ad}}$. \square

We emphasize that this existence result is also valid if the control is acting only on the domain Ω_s or Ω_f and if reference solutions are only given on sub-domains.

3.2 The Adjoint Equation

The necessary optimality conditions to be derived in Sect. 3.3 will make use of an adjoint equation for the linear FSI problem (2).

3.2.1 Discussion of the Adjoint Equations for a Non-symmetric Formulation

Before analyzing the adjoint equations of the weak formulation (11), we investigate for a moment the optimal control problem with the state equation formulated by (4), (5), and (6). For this formulation, the formal Lagrange approach leads to the following adjoint equation:

$$\begin{aligned} -((\varphi, z^v)_f + v((\nabla\varphi, \nabla z^v)_f + (z^p, \text{div } \varphi)_f) \\ -((\varphi, z^v)_s - ((\varphi, z^u)_s = \gamma_f(v - v_d, \varphi)_f \quad \forall \varphi \in L^2(I; V), \\ -((\psi, z^u)_s + \mu((\nabla\psi, \nabla z^v)_s = \gamma_s(u - u_d, \psi)_s \quad \forall \psi \in L^2(I; V_s), \\ -((\xi, \text{div } z^v)_f = 0 \quad \forall \xi \in L^2(I; L_f). \end{aligned}$$

Here, $z^v|_{\Omega_f}$ describes the solution of an adjoint Stokes equation, and $z^v|_{\Omega_s}$ the solution of an adjoint linear wave equation. However, as the system (4), (5), (6) is not symmetric, the adjoint equation is a Stokes–wave system with new coupling conditions on Γ_i :

$$z^v|_{\Omega_f} = z^v|_{\Omega_s}, \quad v\partial_{n_f} z^v - z^p n_f = 0, \quad \text{and} \quad \mu\partial_{n_s} z^v = 0.$$

In contrast to this, the advantage of the following optimality system lies in the fact that the adjoint equation is again a linear FSI problem and all numerical methods developed to solve the primal FSI problem can be utilized. Therefore, no additional difficulty occurs in the implementation and analysis.

3.2.2 Existence and Regularity for the Adjoint Equation

For the symmetric weak formulation (11), we can derive the adjoint equation using the formal Lagrange technique. As the coupling conditions do not occur explicitly, but are embedded in the weak formulation, we obtain immediately the following adjoint equation:

$$\begin{aligned}
 & -\rho_f((\varphi, \partial_t z^v))_f + \nu((\nabla\varphi, \nabla z^v))_f + ((z^p, \operatorname{div} \varphi))_f \\
 & -\rho_s((\varphi, \partial_t z^v))_s + \mu((\nabla\varphi, \nabla z^u))_s = \gamma_f(v - v_d, \varphi)_f \quad \forall \varphi \in L^2(I; V), \\
 & \mu((\nabla\psi, \nabla z^v))_s + \mu((\nabla\psi, \nabla \partial_t z^u))_s = \gamma_s(u - u_d, \psi)_s \quad \forall \psi \in L^2(I; V_s), \\
 & -((\xi, \operatorname{div} z^v))_f = 0 \quad \forall \xi \in L^2(I; L_f).
 \end{aligned} \tag{20}$$

Later, we will prove that together with zero terminal conditions, this is indeed the correct adjoint equation appearing in the optimality system.

Due to the symmetry in (11), the adjoint equation is again a linear FSI problem. Therefore, we can use the already proved results, and we get the following result on the existence of a unique adjoint solution:

Theorem 3.2 *Let $v_d \in H^1(I; V_f^*) \cap L^2(I; H_f)$, $u_d \in L^2(I; H_s)$, and let the initial data u_0, u_1 , and v_0 satisfy Assumption 1. Furthermore, let $q \in Q$ be given as in configuration C1 or $q \in Q \cap H^1(I)^N$ be given for configuration C2, and let the triple (v, p, u) be the corresponding solution of (11) with $f_f = B_f q$, $f_s = B_s q$, and $g = 0$. Then, there exists a unique triple (z^v, z^p, z^u) with*

$$\begin{aligned}
 z^v & \in L^2(I; V) \cap W^{1,\infty}(I; H), \quad z^v|_{\Omega_f} \in H^1(I; V_f), \quad z^v|_{\Omega_s} \in L^\infty(I; V_s), \\
 z^u & \in L^\infty(I; V_s) \cap H^1(I; V_s), \quad z^p \in L^2(I; L_f)
 \end{aligned}$$

satisfying the terminal condition $z^v(T) = 0, z^u(T) = 0$ and the adjoint equation (20). Furthermore, the adjoint solution triple (z^v, z^p, z^u) fulfills the a priori estimates

$$\begin{aligned}
 \text{a) } & \|z^v\|_{L^\infty(I; L^2(\Omega))}^2 + \|z^v\|_{L^2(I; H^1(\Omega_f))}^2 + \|z^u\|_{L^\infty(I; H^1(\Omega_s))}^2 \\
 & \leq C[\|v_d\|_{L^2(I; V_f^*)}^2 + \|u_d\|_{L^2(I; L^2(\Omega_s))}^2] \\
 & \quad + C[\|B_f q\|_{L^2(I; V_f^*)}^2 + \|B_s q\|_{L^2(I; L^2(\Omega_s))}^2 + A_0^2], \\
 \text{b) } & \|\partial_t z^v\|_{L^\infty(I; L^2(\Omega))}^2 + \|\partial_t z^v\|_{L^2(I; H^1(\Omega_f))}^2 + \|z^v\|_{L^\infty(I; H^1(\Omega_s))}^2 + \|\partial_t z^u\|_{L^2(I; H^1(\Omega_s))}^2 \\
 & \leq C[\|v_d\|_{H^1(I; V_f^*)}^2 + \|u_d\|_{L^2(I; L^2(\Omega_s))}^2] \\
 & \quad + C[\|B_f q\|_{H^1(I; V_f^*)}^2 + \|B_s q\|_{H^1(I; L^2(\Omega_s))}^2 + A_0^2], \\
 \text{c) } & \|z^p\|_{L^2(I; L^2(\Omega_f))}^2 \leq C[\|v_d\|_{H^1(I; V_f^*)}^2 + \|u_d\|_{L^2(I; L^2(\Omega_s))}^2] \\
 & \quad + C[\|B_f q\|_{H^1(I; V_f^*)}^2 + \|B_s q\|_{H^1(I; L^2(\Omega_s))}^2 + A_0^2],
 \end{aligned}$$

where A_0^2 is defined in (3).

Proof Based on the assumptions on the control q , Theorem 2.1 ensures a solution $v \in H^1(I; H)$, $u \in L^2(I; V_s)$ of (11) with $f_f = B_f q$, $f_s = B_s q$, and $g = 0$. Hence, the right-hand sides of the adjoint equation $f_f := \gamma_f(v - v_d)$ and $g := \gamma_s(u - u_d)$ fulfill the required regularities $f_f \in H^1(I; V_f^*)$ and $g \in L^2(I; H_s)$ of Theorem 2.1. Furthermore, the initial conditions for the adjoint equation backwards in time $z_T^v = 0$ and $z_T^u = 0$ fulfill with $z_T^p = 0$ the assumptions on u_0, v_0, p_0 , and u_1 of Theorem 2.1. As after the transformation $t \mapsto -t$, the considered adjoint equation (20) coincides with the state equation (11), and there exists a unique adjoint solution (z^v, z^p, z^u) due to Theorem 2.1. The estimates follow immediately from Theorem 2.1, too. \square

3.3 Necessary Optimality Conditions

Since the reduced functional j is convex due to the (affine) linear-quadratic structure of the considered control problem, the necessary and sufficient optimality condition for the optimal solution $\bar{q} \in Q_{ad}$ of (19) reads as

$$j'(\bar{q})(\delta q - \bar{q}) \geq 0 \quad \forall \delta q \in Q_{ad}. \tag{21}$$

Based on this, we derive in the sequel an optimality system separately for the configurations C1 and C2. Therefore, we make use of the self-adjoint formulation (11). Due to this symmetry, the derivation of an optimality system for configuration C1 is straightforward, see Sect. 3.3.1. For configuration C2 however, this is not directly possible since for $q \in Q = L^2(I)^N$, the right-hand sides $B_f q$ and $B_s q$ do not fulfill the prerequisites of Theorem 2.1. Therefore, an additional approximation step will be necessary, see Sect. 3.3.2.

3.3.1 Control Distributed in Space (Configuration C1)

Here, the control $q \in Q = (L^2(\Omega)^d)^N$ acts as volume force through the linear operators B_f and B_s , as described in configuration C1. Since in this case it holds $B_f q \in H^1(I; H_f)$ and $B_s q \in H^1(I; H_s)$, the weak formulation (11) is applicable for $f_f = B_f q$ and $f_s = B_s q$ by Theorem 2.1. For the derivative of the reduced functional given by (18), we directly obtain the following representation:

Lemma 3.2 *Let the initial data u_0, u_1 , and v_0 satisfy Assumption 1, and let the desired states fulfill $v_d \in H^1(I; V_f^*) \cap L^2(I; H_f)$ and $u_d \in L^2(I; H_s)$. Let for given $q \in Q$, the triple (v, p, u) be the solution of (11) with $f_f = B_f q$, $f_s = B_s q$, and $g = 0$ guaranteed by Theorem 2.1. Furthermore, let (z^v, z^p, z^u) be the solution of the adjoint equation (20) guaranteed by Theorem 3.2. Then, the directional derivative of the reduced cost functional at q in direction $\delta q \in Q$ is given by*

$$j'(q)(\delta q) = \sum_{i=1}^N [(\langle g_f^i \delta q^i, z^v \rangle_f + \langle g_s^i \delta q^i, z^v \rangle_s + \alpha(q^i, \delta q^i)].$$

Proof By Theorem 2.1, the control to state map G can be understood as mapping from Q to $L^2(I; V) \times L^2(I; L_f) \times L^2(I; V_s)$. Similar to the proof of Lemma 3.1, let $(\hat{v}, \hat{p}, \hat{u})$ be the solution of (11) for $f_f = f_s = g = 0$. Furthermore, we denote by $G_0: Q \rightarrow L^2(I, H_f) \times L^2(I; L_f) \times L^2(I; H_s)$ the linear part G given by (11) for zero initial data and $f_f = B_f q, f_s = B_s q, g = 0$. Then, G can be written for $q \in Q$ as

$$(v(q), p(q), u(q)) = Gq = (\hat{v}, \hat{p}, \hat{u}) + G_0q.$$

Hence, we get directly

$$j'(q)(\delta q) = \gamma_f(v - v_d, \delta v)_f + \gamma_s(u - u_d, \delta u)_s + \alpha \sum_{i=1}^N (q^i, \delta q^i) \tag{22}$$

for all $\delta q \in Q$ where $(\delta v, \delta p, \delta u) = G_0 \delta q$.

By construction, $(\delta v, \delta p, \delta u)$ solves (11) for the right-hand sides $f_f = B_f \delta q, f_s = B_s \delta q, g = 0$, and zero initial data. Hence, by testing this equation with $(\varphi, \xi, \psi) = (z^v, z^p, z^u) \in L^2(I; V) \times L^2(I; L_f) \times L^2(I; V_s)$, we obtain the identity

$$\begin{aligned} &\rho_f((\delta v_t, z^v)_f - (\delta p, \operatorname{div} z^v)_f + v((\nabla \delta v, \nabla z^v)_f \\ &\quad + \rho_s((\delta v_t, z^v)_s + \mu((\nabla \delta u, \nabla z^v)_s) = \sum_{i=1}^N [(g_f^i \delta q^i, z^v)_f + (g_s^i \delta q^i, z^v)_s], \\ &\mu((\nabla \delta v, \nabla z^u)_s - \mu((\nabla \delta u_t, \nabla z^u)_s) = 0, \\ &((z^p, \operatorname{div} \delta v)_f = 0. \end{aligned}$$

Testing with $(\varphi, \xi, \psi) = (\delta v, \delta p, \delta u) \in L^2(I; V) \times L^2(I; L_f) \times L^2(I; V_s)$ in the adjoint equation (20) yields

$$\begin{aligned} &-\rho_f((\delta v, \partial_t z^v)_f + v((\nabla \delta v, \nabla z^v)_f + (z^p, \operatorname{div} \delta v)_f \\ &\quad - \rho_s((\delta v, \partial_t z^v)_s + \mu((\nabla \delta v, \nabla z^u)_s) = \gamma_f(v - v_d, \delta v)_f, \\ &\quad -(\delta p, \operatorname{div} z^v)_f = 0, \\ &\mu((\nabla \delta u, \nabla z^v)_s + \mu((\nabla \delta u, \nabla \partial_t z^u)_s) = \gamma_s(u - u_d, \delta u)_s. \end{aligned}$$

As the adjoint solution (z^v, z^p, z^u) has zero initial conditions at $t = T$, and as $(\delta v, \delta p, \delta u)$ has zero initial conditions at $t = 0$, the boundary terms vanish when using integration by parts in time. If we insert the equations into each other, then we obtain for any $\delta q \in Q$

$$\gamma_f(v - v_d, \delta v)_f + \gamma_s(u - u_d, \delta u)_s = \sum_{i=1}^N [(g_f^i \delta q^i, z^v)_f + (g_s^i \delta q^i, z^v)_s].$$

Together with (22), this implies the assertion. □

Combining the condition (21) and Lemma 3.2 implies the following representation and regularity for the optimal control \bar{q} in terms of the pointwise projection $P_{Q_{\text{ad}}}$ on

the admissible set Q_{ad} , given by

$$P_{Q_{ad}} : L^2(\Omega)^d \rightarrow L^2(\Omega)^d, \quad P_{Q_{ad}}(r)(x) := \max(q_a, \min(q_b, r(x)))$$

for almost all $x \in \Omega$, where the projection has to be applied componentwise for $r \in L^2(\Omega)^d$.

Lemma 3.3 *Let the assumptions of Lemma 3.2 be fulfilled. Then, the optimal solution $\bar{q} \in Q_{ad}$ of the considered optimal control problem (19) for configuration C1 fulfills for $i = 1, 2, \dots, N$:*

$$\bar{q}^i|_{\Omega_f} = P_{Q_{ad}} \left(-\frac{1}{\alpha} \int_I g_f^i(t) z^v(t, \cdot) dt \right), \quad \bar{q}^i|_{\Omega_s} = P_{Q_{ad}} \left(-\frac{1}{\alpha} \int_I g_s^i(t) z^v(t, \cdot) dt \right).$$

Thus, for the optimal control holds $\bar{q}|_{\Omega_f} \in (H^1(\Omega_f)^d)^N$ and $\bar{q}|_{\Omega_s} \in (H^1(\Omega_s)^d)^N$.

Proof The necessary optimality condition (21) can be written as

$$\left(\int_I g_f^i z^v dt, \delta q^i - \bar{q}^i \right)_f + \left(\int_I g_s^i z^v dt, \delta q^i - \bar{q}^i \right)_s + \alpha (\bar{q}^i, \delta q^i - \bar{q}^i) \geq 0 \quad \forall \delta q^i \in Q_{ad}.$$

Using the projection $P_{Q_{ad}}$, this can be expressed for $i = 1, 2, \dots, N$ as

$$\bar{q}^i|_{\Omega_f} = P_{Q_{ad}} \left(-\frac{1}{\alpha} \int_I g_f^i(t) z^v(t, \cdot) dt \right), \quad \bar{q}^i|_{\Omega_s} = P_{Q_{ad}} \left(-\frac{1}{\alpha} \int_I g_s^i(t) z^v(t, \cdot) dt \right).$$

Theorem 3.2 ensures $z^v \in L^2(I; V)$, and together with

$$\|P_{Q_{ad}}(r)\|_{H^1(\Omega)} \leq \|r\|_{H^1(\Omega)},$$

we conclude that $\bar{q}^i|_{\Omega_f} \in H^1(\Omega_f)^d$ and $\bar{q}^i|_{\Omega_s} \in H^1(\Omega_s)^d$. □

The optimal solution $\bar{q} \in Q_{ad}$ solves the optimality system presented in the following theorem:

Theorem 3.3 *Let the initial data u_0, u_1 , and v_0 satisfy Assumption 1. Furthermore, let $v_d \in H^1(I; V_f^*) \cap L^2(I; H_f)$ and $u_d \in L^2(I; H_s)$. Then, the optimal solution $\bar{q} \in Q_{ad}$ of optimal control problem (19) for configuration C1 fulfills $\bar{q}|_{\Omega_f} \in (H^1(\Omega_f)^d)^N$ and $\bar{q}|_{\Omega_s} \in (H^1(\Omega_s)^d)^N$ as well as the following necessary optimality condition:*

1. *The optimal state $(\bar{v}, \bar{p}, \bar{u}) = (v(\bar{q}), p(\bar{q}), u(\bar{q}))$ solves*

$$\begin{aligned} \rho_f (\partial_t \bar{v}, \varphi)_f - ((\bar{p}, \operatorname{div} \varphi))_f + v(\nabla \bar{v}, \nabla \varphi)_f \\ + \rho_s (\partial_t \bar{v}, \varphi)_s + \mu (\nabla \bar{u}, \nabla \varphi)_f = ((B_f \bar{q}, \varphi))_f + ((B_s \bar{q}, \varphi))_s \quad \forall \varphi \in L^2(I; V), \\ \mu (\nabla \bar{v}, \nabla \psi)_s - \mu (\nabla \partial_t \bar{u}, \nabla \psi)_s = 0 \quad \forall \psi \in L^2(I; V_s), \\ ((\xi, \operatorname{div} \bar{v}))_f = 0 \quad \forall \xi \in L^2(I; L_f). \end{aligned}$$

2. The optimal adjoint $(\bar{z}^v, \bar{z}^p, \bar{z}^u) = (z^v(\bar{q}), z^p(\bar{q}), z^u(\bar{q}))$ solves

$$\begin{aligned}
 -\rho_f \langle \varphi, \partial_t \bar{z}^v \rangle_f + \nu \langle \nabla \varphi, \nabla \bar{z}^v \rangle_f + \langle \bar{z}^p, \operatorname{div} \varphi \rangle_f \\
 -\rho_s \langle \varphi, \partial_t \bar{z}^v \rangle_s + \mu \langle \nabla \varphi, \nabla \bar{z}^u \rangle_s &= \gamma_f \langle \bar{v} - v_d, \varphi \rangle_f \quad \forall \varphi \in L^2(I; V), \\
 \mu \langle \nabla \psi, \nabla \bar{z}^v \rangle_s + \mu \langle \nabla \psi, \nabla \partial_t \bar{z}^u \rangle_s &= \gamma_s \langle \bar{u} - u_d, \psi \rangle_s \quad \forall \psi \in L^2(I; V_s), \\
 -\langle \xi, \operatorname{div} \bar{z}^v \rangle_f &= 0 \quad \forall \xi \in L^2(I; L_f).
 \end{aligned}$$

3. It holds for $i = 1, 2, \dots, N$ that

$$\bar{q}^i \Big|_{\Omega_f} = P_{Q_{\text{ad}}} \left(-\frac{1}{\alpha} \int_I g_f^i(t) \bar{z}^v(t, \cdot) dt \right), \quad \bar{q}^i \Big|_{\Omega_s} = P_{Q_{\text{ad}}} \left(-\frac{1}{\alpha} \int_I g_s^i(t) \bar{z}^v(t, \cdot) dt \right).$$

3.3.2 Time-Dependent Control (Configuration C2)

In the following, the control $q \in Q = (L^2(I))^N$ is controlling the volume force through the linear operators B_f and B_s described in configuration C2. As Theorem 2.1 does not guarantee the existence of a unique solution of (11) for a right-hand side $f_f = B_f q \in L^2(I; H_f)$ and $f_s = B_s q \in L^2(I; H_s)$, we cannot directly proceed as in Sect. 3.3.1. Therefore, we will make use of a smooth sequence in $Q \cap (H^1(I))^N$ converging against the optimal solution. For smooth controls, the symmetric formulation (11) can be utilized, and a priori estimates for the adjoint then lead to higher regularity also for the limit. Then, we are able to derive the optimality system similar as for the configuration C1.

Lemma 3.4 *Let the initial data u_0, u_1 , and v_0 satisfy Assumption 1. Furthermore, let $v_d \in H^1(I; V_f^*) \cap L^2(I; H_f)$ and $u_d \in L^2(I; H_s)$. Additionally, for a given control $q \in Q \cap (H^1(I))^N$, let the triple (v, p, u) be the solution of (11) with $f_f = B_f q$, $f_s = B_s q$, and $g = 0$ guaranteed by Theorem 2.1. Furthermore, let (z^v, z^p, z^u) be the solution of the adjoint equation (20) guaranteed by Theorem 3.2. Then, the directional derivative of the reduced cost functional j at q in direction $\delta q \in Q$ is given by*

$$j'(q)(\delta q) = \sum_{i=1}^N \left[\langle h^i \delta q^i, z^v \rangle + \alpha \int_I q^i \delta q^i dt \right].$$

Proof Since for $q \in Q \cap (H^1(I))^N$ it holds $B_f q \in H^1(I, H_f)$ and $B_s q \in H^1(I; H_s)$, we proceed as in Lemma 3.2 to obtain

$$j'(q)(\delta q) = \sum_{i=1}^N \left[\langle h^i \delta q^i, z^v \rangle + \alpha \int_I q^i \delta q^i dt \right]$$

for all $\delta q \in Q \cap (H^1(I))^N$. By the density of $(H^1(I))^N$ in $(L^2(I))^N$ with respect to the $(L^2(I))^N$ topology, we obtain the assertion. \square

In the next lemma, we prove that the optimal control \bar{q} lies in $Q \cap (H^1(I))^N$ such that the representation derived in Lemma 3.4 is also valid for \bar{q} . Therefore, we will

introduce also for configuration C2 the pointwise projection $P_{Q_{ad}}$ on the admissible set Q_{ad} given here by

$$P_{Q_{ad}} : L^2(I) \rightarrow L^2(I), \quad P_{Q_{ad}}(r)(t) := \max(q_a, \min(q_b, r(t))) \quad \text{for almost all } t \in I.$$

Lemma 3.5 *Let the assumptions of Lemma 3.4 be fulfilled. Then, the optimal solution $\bar{q} \in Q_{ad}$ of the considered optimal control problem (19) for configuration C2 lies in $(H^1(I))^N$.*

Proof Let $\bar{q} \in Q_{ad}$ be the optimal solution. We consider a smooth sequence (q_n) with $q_n \in Q \cap (H^1(I))^N$ and $q_n \rightarrow \bar{q}$ in Q . As in the proof of Lemma 3.2, according to Theorem 2.1, we have that $(v_n, p_n, u_n) = Gq_n$ solves (11) with right-hand sides $f_f = B_f q_n$, $f_s = B_s q_n$, and $g = 0$. The velocity and displacement have at least the regularities $v_n \in H^1(I; H_f)$ and $u_n \in L^2(I; V_s)$. Therefore, Theorem 3.2 guarantees the existence of a unique adjoint solution (z_n^v, z_n^p, z_n^u) of (20) with $v_n - v_d$ and $u_n - u_d$ in the right-hand side. By Lemma 3.4, we have

$$j'(q_n)(\delta q) = \sum_{i=1}^N \left[\langle h_f^i \delta q^i, z_n^v \rangle + \alpha \int_I q_n^i \delta q^i \, dt \right]$$

for all $\delta q \in Q$. Due to estimate a) in Theorem 2.1, the linearity of (11), and the boundedness of h^i in V_{div} , we get for $m, n \in \mathbb{N}$ the estimate

$$\begin{aligned} & \|v_n - v_m\|_{L^2(I; H^1(\Omega_f))}^2 + \|u_n - u_m\|_{L^2(I; H^1(\Omega_s))}^2 \\ & \leq C \left[\sum_{i=1}^N \|h_f^i(q_n^i - q_m^i)\|_{L^2(I; V_f^*)}^2 + \sum_{i=1}^N \|h_s^i(q_n^i - q_m^i)\|_{L^2(I; L^2(\Omega_s))}^2 \right] \\ & \leq C \sum_{i=1}^N \|q_n^i - q_m^i\|_{L^2(I)}^2. \end{aligned}$$

Furthermore, due to estimate a) in Theorem 3.2, the adjoint variables fulfill the estimate

$$\begin{aligned} & \|z_n^v - z_m^v\|_{L^\infty(I; L^2(\Omega))}^2 + \|z_n^v - z_m^v\|_{L^2(I; H^1(\Omega_f))}^2 + \|z_n^u - z_m^u\|_{L^2(I; H^1(\Omega_s))}^2 \\ & \leq C \left[\sum_{i=1}^N \|h_f^i(q_n^i - q_m^i)\|_{L^2(I; V_f^*)}^2 + \sum_{i=1}^N \|h_s^i(q_n^i - q_m^i)\|_{L^2(I; L^2(\Omega_s))}^2 \right] \\ & \leq C \sum_{i=1}^N \|q_n^i - q_m^i\|_{L^2(I)}^2. \end{aligned}$$

If we regard in (20) test functions $\varphi \in L^2(I; V_{div})$ that are divergence-free in the fluid domain Ω_f , we get the estimate

$$\|\partial_t z_n^v - \partial_t z_m^v\|_{L^2(I; V_{\text{div}}^*)}^2 \leq C \left[\|z_n^v - z_m^v\|_{L^2(I; H^1(\Omega_f))}^2 + \|z_n^u - z_m^u\|_{L^2(I; H^1(\Omega_s))}^2 + \|v_n - v_m\|_{L^2(I; L^2(\Omega_f))} \right].$$

By combining the above estimates, we derive for the adjoint $z_n^v - z_m^v$ the bound

$$\|\partial_t z_n^v - \partial_t z_m^v\|_{L^2(I; V_{\text{div}}^*)}^2 + \|z_n^v - z_m^v\|_{L^\infty(I; L^2(\Omega))}^2 \leq C \sum_{i=1}^N \|q_n^i - q_m^i\|_{L^2(I)}^2.$$

As $q_n \rightarrow \bar{q}$ in $(L^2(I))^N$, it holds $\|q_n^i - q_m^i\|_{L^2(I)}^2 \rightarrow 0$ for $m, n \rightarrow \infty$. Thus, z_n^v is a Cauchy sequence in $H^1(I; V_{\text{div}}^*) \cap L^\infty(I; H)$, and therefore, there exists the limit $\tilde{z}^v \in H^1(I; V_{\text{div}}^*) \cap L^\infty(I; H)$ such that

$$z_n^v \rightarrow \tilde{z}^v \text{ in } H^1(I; V_{\text{div}}^*) \cap L^\infty(I; H).$$

Since we assumed $h^i \in V_{\text{div}}$ and $\delta q^i \in L^2(I)$, the product fulfills $h^i \delta q^i \in L^2(I; V_{\text{div}})$. This implies, due to $L^2(I; V_{\text{div}}) \hookrightarrow L^2(I; H)$ and due to the convergence of z_n^v in $L^\infty(I; H)$, that

$$\begin{aligned} j'(q_n)(\delta q) &= \sum_{i=1}^N \left[\langle h^i \delta q^i, z_n^v \rangle + \alpha \int_I q_n^i \delta q^i \, dt \right] \\ &\rightarrow \sum_{i=1}^N \left[\langle h^i \delta q^i, \tilde{z}^v \rangle + \alpha \int_I \bar{q}^i \delta q^i \, dt \right]. \end{aligned}$$

In addition, the directional derivative of the reduced cost functional $j'(\cdot)(\delta q)$ is continuous as the control to state mapping $G: Q \rightarrow L^2(I; H_f) \times L^2(I; H_s)$ is affine-linear and continuous accordingly to Lemma 3.1. Therefore, the convergence $q_n \rightarrow \bar{q}$ in Q implies in addition that $j'(q_n)(\delta q) \rightarrow j'(\bar{q})(\delta q)$ and we obtain the identity

$$j'(\bar{q})(\delta q) = \sum_{i=1}^N \left[\langle h^i \delta q^i, \tilde{z}^v \rangle + \alpha \int_I \bar{q}^i \delta q^i \, dt \right].$$

As the optimal solution \bar{q} fulfills the necessary optimality condition (21), we get the optimality condition

$$\sum_{i=1}^N \left[\langle h^i (\delta q^i - \bar{q}^i), \tilde{z}^v \rangle + \alpha \int_I \bar{q}^i (\delta q^i - \bar{q}^i) \, dt \right] \geq 0 \quad \forall \delta q \in Q_{\text{ad}}.$$

Using the projection $P_{Q_{\text{ad}}}$ on the admissible set Q_{ad} , this can be expressed as

$$\bar{q}^i = P_{Q_{\text{ad}}}(r^i) \quad \text{with} \quad r^i = -\frac{1}{\alpha} \int_{\Omega} h^i(x) \tilde{z}^v(\cdot, x) \, dx, \quad i = 1, 2, \dots, N.$$

The time regularity of the limit $\tilde{z}^v \in H^1(I; V_{\text{div}}^*)$ and the assumed regularity of $h \in V_{\text{div}}$ imply

$$\partial_t r^i(t) = -\frac{1}{\alpha} \langle h^i, \partial_t \bar{z}^v(t, \cdot) \rangle_{V_{\text{div}} \times V_{\text{div}}^*}, \quad i = 1, 2, \dots, N$$

for almost all $t \in I$, and consequently that $r^i \in H^1(I)$. Using the stability of the projection

$$\|P_{Q_{\text{ad}}}(r)\|_{H^1(I)} \leq \|r\|_{H^1(I)},$$

we obtain the asserted regularity $\bar{q} \in (H^1(I))^N$. □

Then, the optimal solution $\bar{q} \in Q_{\text{ad}}$ of the considered optimal control problem in configuration C2 fulfills the following theorem:

Theorem 3.4 *Let the initial data u_0, u_1 , and v_0 satisfy Assumption 1. Furthermore, let the desired states fulfill $v_d \in H^1(I; V_f^*) \cap L^2(I; H_f)$ and $u_d \in L^2(I; H_s)$. Then, the optimal solution $\bar{q} \in Q_{\text{ad}}$ of the considered optimal control problem (19) for configuration C2 fulfills $\bar{q} \in (H^1(I))^N$ and the following necessary optimality condition:*

1. *The optimal state $(\bar{v}, \bar{p}, \bar{u}) = (v(\bar{q}), p(\bar{q}), u(\bar{q}))$ solves*

$$\begin{aligned} \rho_f \langle \partial_t \bar{v}, \varphi \rangle_f - \langle \bar{p}, \text{div } \varphi \rangle_f + \nu \langle \nabla \bar{v}, \nabla \varphi \rangle_f \\ + \rho_s \langle \partial_t \bar{v}, \varphi \rangle_s + \mu \langle \nabla \bar{u}, \nabla \varphi \rangle_f = \langle B_f \bar{q}, \varphi \rangle_f + \langle B_s \bar{q}, \varphi \rangle_s \quad \forall \varphi \in L^2(I; V), \\ \mu \langle \nabla \bar{v}, \nabla \psi \rangle_s - \mu \langle \nabla \partial_t \bar{u}, \nabla \psi \rangle_s = 0 \quad \forall \psi \in L^2(I; V_s), \\ \langle \xi, \text{div } \bar{v} \rangle_f = 0 \quad \forall \xi \in L^2(I; L_f). \end{aligned}$$

2. *The optimal adjoint $(\bar{z}^v, \bar{z}^p, \bar{z}^u) = (z^v(\bar{q}), z^p(\bar{q}), z^u(\bar{q}))$ solves*

$$\begin{aligned} -\rho_f \langle \varphi, \partial_t \bar{z}^v \rangle_f + \nu \langle \nabla \varphi, \nabla \bar{z}^v \rangle_f + \langle \bar{z}^p, \text{div } \varphi \rangle_f \\ - \rho_s \langle \varphi, \partial_t \bar{z}^v \rangle_s + \mu \langle \nabla \varphi, \nabla \bar{z}^u \rangle_s = \gamma_f \langle \bar{v} - v_d, \varphi \rangle_f \quad \forall \varphi \in L^2(I; V), \\ \mu \langle \nabla \psi, \nabla \bar{z}^v \rangle_s + \mu \langle \nabla \psi, \nabla \partial_t \bar{z}^u \rangle_s = \gamma_s \langle \bar{u} - u_d, \psi \rangle_s \quad \forall \psi \in L^2(I; V_s), \\ -\langle \xi, \text{div } \bar{z}^v \rangle_f = 0 \quad \forall \xi \in L^2(I; L_f). \end{aligned}$$

3. *It holds for $i = 1, 2, \dots, N$ that*

$$\bar{q}^i = P_{Q_{\text{ad}}} \left(-\frac{1}{\alpha} \int_{\Omega} h^i(x) \bar{z}^v(\cdot, x) dx \right).$$

Proof As $\bar{q} \in Q \cap (H^1(I))^N$, one can choose in the proof of Lemma 3.5 the sequence $q_n = \bar{q}$. This immediately implies that $\bar{z}^v = \bar{z}^v$. □

4 Conclusions

Using a novel symmetric formulation, we derived an optimality system for an optimal control problem constrained by a linear fluid–structure interaction problem. Therefore,

two different types of controls were analyzed: $q \in L^2(\Omega)$ (see Configuration C1) and $q \in L^2(I)$ (see Configuration C2). In both cases, we showed H^1 regularity of the optimal control. This justifies the usage of the introduced symmetric formulation for state and adjoint which also is advantageous from a computational point of view.

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