

## FINITE ELEMENT POINTWISE RESULTS ON CONVEX POLYHEDRAL DOMAINS\*

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**Abstract.** The main goal of the paper is to establish that the  $L^1$  norm of jumps of the normal derivative across element boundaries and the  $L^1$  norm of the Laplacian of a piecewise polynomial finite element function can be controlled by corresponding weighted discrete  $H^2$  norm on convex polyhedral domains. In the finite element literature such results are only available for piecewise linear elements in two dimensions and the extension to convex polyhedral domains is rather technical. As a consequence of this result, we establish almost pointwise stability of the Ritz projection and the discrete resolvent estimate in  $L^\infty$  norm.

**Key words.** elliptic problems, finite elements, maximum norm, error estimates, pointwise error estimates, resolvent

**AMS subject classifications.** 65N30, 65N15

**DOI.** 10.1137/15M1013912

**1. Introduction.** As a simple model of a second order elliptic partial differential equation we consider,

$$(1) \quad \begin{aligned} -\Delta u(x) &= f(x), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned}$$

with a right-hand side  $f \in L^r(\Omega)$ ,  $r > \frac{3}{2}$  on a convex polyhedral domain  $\Omega \subset \mathbb{R}^3$ . Let  $u_h$  be the Ritz projection of  $u$  onto the space of continuous piecewise polynomial finite element functions  $V_h$  on the mesh  $\mathcal{T}$  consisting of elements  $\tau$ . The aim of this paper is to establish some important weighted and pointwise results for  $u_h$  in three space dimensions that are not available in the literature but required in a number of applications (cf. [22, 23, 24]).

Our main technical result is Lemma 6 which shows

$$(2) \quad \begin{aligned} &\sum_{\tau \in \mathcal{T}} (\|\Delta v_h\|_{L^1(\tau)} + \|[\partial_n v_h]\|_{L^1(\partial\tau)}) \\ &\leq C |\ln h|^{\frac{1}{2}} \left( \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{L^2(\Omega)} + \|\sigma^{\frac{1}{2}} \nabla v_h\|_{L^2(\Omega)} \right) \quad \forall v_h \in V_h, \end{aligned}$$

where the weight  $\sigma$  describing an  $h$ -dependent regularized distance is introduced in (13) and  $\Delta_h$  is the discrete Laplace operator defined in (11). This estimate says that the  $L^1$  norm of jumps of the normal derivative across element boundary as well as the  $L^1$  norm of the Laplacian of any piecewise polynomial function can be controlled by the properly weighted discrete  $H^2$ -norm. A corresponding result for piecewise linear functions was proved by Rannacher [31] in two dimensions and was used to establish pointwise stability of the semidiscrete and fully discrete backward Euler solution of

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\*Received by the editors March 25, 2015; accepted for publication (in revised form) January 7, 2016; published electronically March 8, 2016.

<http://www.siam.org/journals/sinum/54-2/M101391.html>

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parabolic problems on convex polygonal domains. However, in order to extend this result to three dimensions one has to overcome some serious technical obstacles. To accomplish this, we require several additional technical lemmas. Some results are standard; however, Lemma 5 is rather peculiar and can be thought of as a weighted Gagliardo–Nirenberg interpolation inequality. It shows that for any  $w \in H_0^1(\Omega)$ , any  $\alpha, \beta \in \mathbb{R}$  with  $\alpha \geq -\frac{1}{2}$ , and any  $1 \leq p \leq \infty$ , there holds

$$(3) \quad \|\sigma^\alpha w\|_{L^2(\Omega)}^2 \leq C \|\sigma^{\alpha-\beta} w\|_{L^p(\Omega)} \|\sigma^{\alpha+1+\beta} \nabla w\|_{L^q(\Omega)}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This result is an extension and a generalization of Lemma 3.4 from [1] and provides one with great flexibility in manipulating weighted spaces, especially for Galerkin finite element solutions. Thus, by choosing  $\beta = 0$  and  $p = q = 2$ , we obtain

$$\|\sigma^\alpha w\|_{L^2(\Omega)} \leq C \|\sigma^{\alpha+1} \nabla w\|_{L^2(\Omega)} \quad \forall w \in H_0^1(\Omega), \quad \forall \alpha \geq -\frac{1}{2};$$

i.e., the estimate allows, for example, to “trade” derivatives for weights. An estimate similar to (3) with a weight function  $|x|$  instead of  $\sigma(x)$  can be found in [6].

These two technical lemmas, having an independent interest, are powerful results and have a variety of applications. One application provided in this paper is almost the stability of the Ritz projection in  $L^\infty$  norm (cf. Theorem 12),

$$(4) \quad \|u_h\|_{L^\infty(\Omega)} \leq C |\ln h| \|u\|_{L^\infty(\Omega)}.$$

Due to the fact that the Ritz projection is an identity on  $V_h$ , the above stability result is equivalent to the almost best approximation property of the error

$$(5) \quad \|u - u_h\|_{L^\infty(\Omega)} \leq C |\ln h| \inf_{\chi \in V_h} \|u - \chi\|_{L^\infty(\Omega)}.$$

We would like to mention that estimates (4) and (5) for piecewise linear elements are asymptotically sharp and  $|\ln h|$  is necessary as was shown in the example of Haverkamp [20] in two dimensional setting; cf. also [16]. It is known that the factor  $|\ln h|$  can be removed for higher order finite elements on smooth domains; see, e.g., [35]. However, whether  $|\ln h|$  is necessary for higher order elements on convex polyhedral domains is an open question.

The second application provided in this paper is the following discrete resolvent estimate (cf. Theorem 15):

$$(6) \quad \|(z + \Delta_h)^{-1} \chi\|_{L^\infty(\Omega)} \leq \frac{C |\ln h|}{|z - \lambda|} \|\chi\|_{L^\infty(\Omega)} \quad \text{for } z \in \mathbb{C} \setminus \Sigma_{\lambda, \gamma}, \quad \forall \chi \in V_h,$$

where

$$(7) \quad \Sigma_{\lambda, \gamma} = \{z \in \mathbb{C} : |\arg(z - \lambda)| \leq \gamma\}$$

for any  $\gamma \in (0, \frac{\pi}{2})$  and  $\lambda \in [0, \lambda_0]$ , where  $\lambda_0 > 0$  is the smallest eigenvalue of  $-\Delta$  with homogeneous Dirichlet boundary conditions. Such resolvent estimates are useful in treatment of certain fully discrete schemes in  $L^\infty$  norm (cf. [34], [43, chap. 9]) and we require them to establish fully discrete maximal parabolic regularity in [22]. In two space dimensions, such resolvent estimates were established in [3, Lemma 6.1]. For

smooth bounded domains in  $\mathbb{R}^N$  with  $N \geq 2$  the logarithmic term can be removed [4], but the analysis there requires the following continuous resolvent estimate:

$$\|(z + \Delta)^{-1}v\|_{W^{1,\infty}(\Omega)} \leq \frac{C}{(1 + |z|)^{\frac{1}{2}}} \|v\|_{L^\infty(\Omega)},$$

which for convex polyhedral domains we were not able to locate in the literature. If such a resolvent estimate is valid on convex polyhedral domains, then following the analysis similar to [4], the logarithmic term can be removed as well. However, for our applications in [22, 24] the above estimate with  $|\ln h|$  is sufficient. Results (2)–(6) constitute the main results of this paper.

Pointwise error estimates in the finite element literature for the second order elliptic problems started in the works of Nitsche [28, 27], Natterer [26], Scott [41], and Frehse and Rannacher [15]. Since then a lot of work was done in various settings [11, 12, 30, 32, 33, 35, 36, 37, 38, 39]. Nevertheless, results in three dimensions are rather scarce, especially on nonsmooth domains. Rannacher [30] and Schatz and Wahlbin [35] were the first who showed best approximation property in  $L^\infty$  and  $W^{1,\infty}$  norms on smooth bounded domains in  $\mathbb{R}^N$  with  $N \geq 2$ ; however, results in [35] targeted interior error estimates and the global results were just byproducts. Later in [38], the effect of a “skin” layer was analyzed. The first stability result for the Ritz projection in  $W^{1,\infty}$  norm without a logarithmic term on nonsmooth (convex polygonal) domains were obtained by Rannacher and Scott in [32]. Using a similar technique, such a result for three dimensional polyhedral domains was first provided in the book [5] with some additional geometrical restrictions beyond convexity. This restriction was removed in Guzmán et al. [19] and later extended to more general meshes (cf. Demlow et al. [8]). However, the estimate (4) is known only on smooth domains (cf. [38]) or two dimensional polygonal domains [33]. To cover the case of a three dimensional convex polyhedral domain, we among other things exploited the idea from [19] and used the  $C^{1,\lambda}$  regularity (cf. Lemma 9 and the proof of Lemma 10) instead of the  $W^{2,p}(\Omega)$  regularity with  $p > 3$ , which would require additional geometrical restrictions; cf. [5].

Currently, in the finite element literature on pointwise error estimates on unstructured meshes, there exist two popular techniques, global weighted technique, due to Nitsche, and the technique based on local energy estimates due to Schatz and Wahlbin. Both techniques are natural from an analytical point of view and more or less equivalent, meaning that the same results can be established by either technique. Thus, adapting a certain technique for a proof is just a matter of taste. In this paper both techniques are used. Although, the proof of Lemma 6 is based on a weighted technique, the proof of Lemma 10 is based on the local energy technique, which appeared more convenient for us in that particular proof. There is also a technique based on Campanato spaces due to Dolzmann [9], but it is technically more involved and has never gained much support.

The rest of this paper is organized as follows. In section 2 we introduce the notation, and define the weight function and weighted norms. Then we continue with a series of lemmas including the key results, Lemmas 5 and 6. In section 3, we establish (4) in Theorem 12, and in section 4, we establish the resolvent estimate (6) in Theorem 15.

**2. Notation and weighted norm estimates.** Throughout this paper we use the usual notation for Lebesgue and Sobolev spaces. For any set  $D$ , we will denote the  $L^2(D)$  norm by  $\|\cdot\|_D$ . Other norms we will write explicitly. We denote by  $(\cdot, \cdot)$  the  $L^2(\Omega)$  inner product and we will specify a subdomain by a subscript in the case

it is not the whole  $\Omega$ .

Let  $\mathcal{T}$  denote a quasi-uniform triangulation of  $\Omega$  with a mesh size  $h$ , i.e.,  $\mathcal{T} = \{\tau\}$  is a partition of  $\Omega$  into tetrahedrons  $\tau$  of diameter  $h_\tau$  such that for  $h = \max_\tau h_\tau$ ,

$$\text{diam}(\tau) \leq h \leq C|\tau|^{\frac{1}{3}} \quad \forall \tau \in \mathcal{T}$$

hold. Let  $V_h$  be the set of all functions in  $H_0^1(\Omega)$  that are polynomial of degree  $k$ ,  $k \geq 1$  on each  $\tau$ ; i.e.,  $V_h$  is the usual space of Lagrangian finite elements of degree  $k$ . For the space  $V_h$  we will utilize the  $L^2$  projection  $P_h: L^2(\Omega) \rightarrow V_h$  defined by

$$(8) \quad (P_h v, \chi)_\Omega = (v, \chi)_\Omega \quad \forall \chi \in V_h,$$

the Ritz projection  $R_h: H_0^1(\Omega) \rightarrow V_h$  defined by

$$(9) \quad (\nabla R_h v, \nabla \chi)_\Omega = (\nabla v, \nabla \chi)_\Omega \quad \forall \chi \in V_h,$$

and the usual nodewise interpolant  $I_h: C_0(\Omega) \rightarrow V_h$  with usual approximation properties (cf., e.g., [7, Theorem 3.1.5])

$$(10) \quad \|u - I_h u\|_{L^q(\Omega)} \leq Ch^{2+3(\frac{1}{q}-\frac{1}{p})} \|u\|_{W^{2,p}(\Omega)} \quad \text{for } q \geq p > \frac{3}{2}.$$

Moreover, we introduce the discrete Laplace operator  $\Delta_h: V_h \rightarrow V_h$  by

$$(11) \quad (-\Delta_h v_h, \chi)_\Omega = (\nabla v_h, \nabla \chi)_\Omega \quad \forall \chi \in V_h.$$

Let  $x_0 \in \Omega$  be a fixed (but arbitrary) point. Associated to this point we introduce a smooth Delta function [44, Lemma 2.2], which we will denote by  $\tilde{\delta} = \tilde{\delta}_{x_0}$ ; cf. also [40]. This function is supported in one cell, denoted by  $\tau_0$ , and satisfies

$$(\chi, \tilde{\delta})_{\tau_0} = \chi(x_0) \quad \forall \chi \in \mathbb{P}^k(\tau_0).$$

In addition from [44, Lemma 2.2] we also have

$$(12) \quad \|\tilde{\delta}\|_{W^{s,p}(\Omega)} \leq Ch^{-s-3(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty, \quad s = 0, 1, 2.$$

Thus, in particular,  $\|\tilde{\delta}\|_{L^1(\Omega)} \leq C$ ,  $\|\tilde{\delta}\|_\Omega \leq Ch^{-\frac{3}{2}}$ , and  $\|\tilde{\delta}\|_{L^\infty(\Omega)} \leq Ch^{-3}$ . Next, we introduce a weight function

$$(13) \quad \sigma(x) = \sqrt{|x - x_0|^2 + K^2 h^2},$$

where  $K > 0$  is a sufficiently large constant to be chosen later. One can easily check that  $\sigma$  satisfies the following properties (cf., e.g., [7, sect. 3.3]):

$$(14a) \quad \|\sigma^{-\frac{3}{2}}\|_\Omega \leq C |\ln h|^{\frac{1}{2}},$$

$$(14b) \quad |\nabla^l \sigma| \leq C \sigma^{1-l}, \quad l = 1, 2, \dots,$$

$$(14c) \quad \max_\tau \sigma \leq C \min_\tau \sigma \quad \forall \tau \in \mathcal{T}.$$

Next, we require an estimate in weighted norms for the  $L^2$  projection  $P_h$ . For piecewise linear case in two dimensions this result is established in Lemma 7.1 in [13]. In the appendix we provide a proof for arbitrary polynomial order.

LEMMA 1. *There exist  $\varepsilon > 0$  and a constant  $C > 0$ , such that for any positive function  $\varphi : \Omega \rightarrow \mathbb{R}_+$  satisfying  $h^l |\nabla^l \varphi| \leq \varepsilon \varphi$  for all  $l = 1, 2, \dots, k$  and any  $v \in L^2(\Omega)$ , we have*

$$\|\varphi P_h v\|_{\Omega} \leq C \|\varphi v\|_{\Omega}.$$

The next lemma provides error estimates for the projection and the interpolation errors in weighted norms. In what follows we will use the discrete  $L^2$  norm  $\|\cdot\|_{h,\Omega}$  for functions defined cellwise:

$$\|z\|_{h,\Omega}^2 = \sum_{\tau \in \mathcal{T}} \|z\|_{L^2(\tau)}^2.$$

LEMMA 2. *Let  $v \in C_0(\Omega) \cap H_0^1(\Omega)$  be cellwise in  $H^l$ , i.e.,  $v|_{\tau} \in H^l(\tau)$  for all  $\tau \in \mathcal{T}$  and some  $2 \leq l \leq k + 1$ . Then the following estimates hold for any  $\alpha \in \mathbb{R}$  and  $K$  large enough:*

$$(15) \quad \|\sigma^\alpha(v - I_h v)\|_{\Omega} + h \|\sigma^\alpha \nabla(v - I_h v)\|_{\Omega} + h^2 \|\sigma^\alpha \nabla^2(v - I_h v)\|_{h,\Omega} \leq ch^l \|\sigma^\alpha \nabla^l v\|_{h,\Omega},$$

$$(16) \quad \|\sigma^\alpha(v - P_h v)\|_{\Omega} + h \|\sigma^\alpha \nabla(v - P_h v)\|_{\Omega} + h^2 \|\sigma^\alpha \nabla^2(v - P_h v)\|_{h,\Omega} \leq ch^l \|\sigma^\alpha \nabla^l v\|_{h,\Omega}.$$

*Proof.* The estimates (15) are straightforward due to the local nature of the node-wise interpolant  $I_h$  and due to estimate (14c). The estimate for the first term in (16) is standard for  $\alpha = 0$ . For  $\alpha \neq 0$  we consider  $\varphi = \sigma^\alpha$ . There holds for any  $l = 1, 2, \dots, k$

$$|\nabla^l \varphi| = |\nabla^l(\sigma^\alpha)| \leq C_\alpha \sigma^{\alpha-l} \leq C_\alpha \sigma^{-l} \varphi \leq C_\alpha K^{-l} h^{-l} \varphi,$$

where we used (14b) and the fact that  $\sigma \geq Kh$ . For  $K$  large enough this  $\varphi$  fulfills the condition of Lemma 1 and we get

$$\|\sigma^\alpha(P_h v - I_h v)\|_{\Omega} = \|\sigma^\alpha P_h(v - I_h v)\|_{\Omega} \leq C \|\sigma^\alpha(v - I_h v)\|_{\Omega}.$$

Therefore,

$$\|\sigma^\alpha(v - P_h v)\|_{\Omega} \leq C \|\sigma^\alpha(v - I_h v)\|_{\Omega},$$

and we get the estimate of the first term (16) using (15). The estimate for the second term in (16) is obtained by the inverse inequality, which holds in weighted norms again due to (14c). □

The next lemma is a superconvergence result in weighted norms.

LEMMA 3. *Let  $v_h \in V_h$ . Then the following estimates hold for any  $\alpha, \beta \in \mathbb{R}$  and  $K$  large enough:*

$$(17) \quad \|\sigma^\alpha(\text{Id} - I_h)(\sigma^\beta v_h)\|_{\Omega} + h \|\sigma^\alpha \nabla(\text{Id} - I_h)(\sigma^\beta v_h)\|_{\Omega} \leq ch \|\sigma^{\alpha+\beta-1} v_h\|_{\Omega},$$

$$(18) \quad \|\sigma^\alpha(\text{Id} - P_h)(\sigma^\beta v_h)\|_{\Omega} + h \|\sigma^\alpha \nabla(\text{Id} - P_h)(\sigma^\beta v_h)\|_{\Omega} \leq ch \|\sigma^{\alpha+\beta-1} v_h\|_{\Omega}.$$

*Proof.* We prove the estimate for the first term in (17). By the estimate (15) from the previous lemma we obtain

$$\|\sigma^\alpha(\text{Id} - I_h)(\sigma^\beta v_h)\|_{\Omega} \leq ch^{k+1} \|\sigma^\alpha \nabla^{k+1}(\sigma^\beta v_h)\|_{h,\Omega}.$$

Using the multi-index notation with  $\gamma \in \mathbb{N}^3$ , we have

$$\|\sigma^\alpha \nabla^{k+1}(\sigma^\beta v_h)\|_{h,\Omega}^2 = \sum_{|\gamma|=k+1} \|\sigma^\alpha \partial^\gamma(\sigma^\beta v_h)\|_{h,\Omega}^2.$$

Using the Leibniz's formula for  $\partial^\gamma(\sigma^\beta v_h)$  and noticing that  $\partial^\gamma v_h = 0$  on a cell  $\tau$  for  $|\gamma| = k + 1$  (since  $v_h$  is a polynomial of degree at most  $k$  on  $\tau$ ), for each  $\tau \in \mathcal{T}$  we obtain

$$\partial^\gamma(\sigma^\beta v_h) = \sum_{|\delta| > 0, \delta \leq \gamma} \binom{\gamma}{\delta} \partial^\delta(\sigma^\beta) \partial^{\gamma-\delta} v_h,$$

where the inequality  $\delta \leq \gamma$  for multi-indices is understood as  $\delta_i \leq \gamma_i$  for  $i = 1, 2, 3$  and the binomial coefficient is defined as

$$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1} \binom{\gamma_2}{\delta_2} \binom{\gamma_3}{\delta_3}.$$

By the property of  $\sigma$ , namely (14b), we have

$$|\partial^\delta(\sigma^\beta)| \leq c\sigma^{\beta-|\delta|}.$$

Therefore, we get for any  $\tau \in \mathcal{T}$  and any  $\gamma$  with  $|\gamma| = k + 1$ ,

$$\begin{aligned} h^{k+1} \|\sigma^\alpha \partial^\gamma(\sigma^\beta v_h)\|_\tau &\leq ch^{k+1} \sum_{|\delta| > 0, \delta \leq \gamma} \|\sigma^\alpha \partial^\delta(\sigma^\beta) \partial^{\gamma-\delta} v_h\|_\tau \\ &\leq ch^{k+1} \sum_{|\delta| > 0, \delta \leq \gamma} \|\sigma^{\alpha+\beta-|\delta|} \partial^{\gamma-\delta} v_h\|_\tau. \end{aligned}$$

Using property (14c) of  $\sigma$  and the inverse inequality  $\|\partial^{\gamma-\delta} v_h\|_\tau \leq ch^{|\delta|-k-1} \|v_h\|_\tau$ , we obtain

$$h^{k+1} \sum_{|\delta| > 0, \delta \leq \gamma} \|\sigma^{\alpha+\beta-|\delta|} \partial^{\gamma-\delta} v_h\|_\tau \leq Ch \sum_{|\delta| > 0, \delta \leq \gamma} h^{|\delta|-1} \|\sigma^{\alpha+\beta-|\delta|} v_h\|_\tau.$$

Finally, using  $h \leq \sigma$  we obtain the desired estimate.

The estimates for the second term in (17) and estimates (18) follow by similar arguments as in the proof of Lemma 2.  $\square$

The next lemma shows how the weight  $\sigma$  compensates the singularity of the regularized Delta function  $\tilde{\delta}$ .

LEMMA 4. *It holds that*

$$(19) \quad \|\sigma^{\frac{3}{2}} \tilde{\delta}\|_\Omega + h \|\sigma^{\frac{3}{2}} \nabla \tilde{\delta}\|_\Omega + h^2 \|\sigma^{\frac{3}{2}} \nabla^2 \tilde{\delta}\|_\Omega + \|\sigma^{\frac{3}{2}} P_h \tilde{\delta}\|_\Omega \leq C.$$

*Proof.* Using that the support of  $\tilde{\delta}_{x_0}$  is in a single element  $\tau_0$  and using (12), we have

$$\|\sigma^{\frac{3}{2}} \tilde{\delta}\|_\Omega^2 = \int_{\tau_0} |\sigma^{\frac{3}{2}} \tilde{\delta}|^2 dx \leq \|\tilde{\delta}\|_{L^\infty(\Omega)}^2 \int_{\tau_0} (|x-x_0|^2 + K^2 h^2)^{\frac{3}{2}} dx \leq Ch^{-6} h^3 |\tau_0| \leq C.$$

Similarly, we get

$$\|\sigma^{\frac{3}{2}} \nabla \tilde{\delta}\|_\Omega^2 = \int_{\tau_0} |\sigma^{\frac{3}{2}} \nabla \tilde{\delta}|^2 dx \leq \|\nabla \tilde{\delta}\|_{L^\infty(\Omega)}^2 \int_{\tau_0} (|x-x_0|^2 + K^2 h^2)^{\frac{3}{2}} dx \leq Ch^{-8} h^3 |\tau_0| \leq Ch^{-2}$$

and thus  $\|\sigma^{\frac{3}{2}} \nabla \tilde{\delta}\|_\Omega \leq ch^{-1}$ . Analogously, we obtain the estimate  $\|\sigma^{\frac{3}{2}} \nabla^2 \tilde{\delta}\|_\Omega \leq ch^{-2}$ . For the last term in (19) we obtain by (16)

$$\|\sigma^{\frac{3}{2}} P_h \tilde{\delta}\|_\Omega \leq \|\sigma^{\frac{3}{2}} \tilde{\delta}\|_\Omega + \|\sigma^{\frac{3}{2}} (P_h - I) \tilde{\delta}\|_\Omega \leq \|\sigma^{\frac{3}{2}} \tilde{\delta}\|_\Omega + ch^2 \|\sigma^{\frac{3}{2}} \nabla^2 \tilde{\delta}\|_\Omega \leq C. \quad \square$$

The idea of the next lemma comes from the two dimensional argument used within the proof of Lemma 3.4 from [1]. However, the proof of the three dimensional result is different. A similar estimate with the weight being the distance function  $|x|$  instead of  $\sigma(x)$  can be found in [6].

LEMMA 5. *There exists a constant  $C$  independent of  $K$  and  $h$  such that for any  $f \in H_0^1(\Omega)$ , any  $\alpha, \beta \in \mathbb{R}$  with  $\alpha \geq -\frac{1}{2}$ , and any  $1 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$  holds:*

$$\|\sigma^\alpha f\|_\Omega^2 \leq C \|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)} \|\sigma^{\alpha+1+\beta} \nabla f\|_{L^q(\Omega)},$$

provided  $\|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)}$  and  $\|\sigma^{\alpha+1+\beta} \nabla f\|_{L^q(\Omega)}$  are bounded.

*Proof.* We assume that the domain  $\Omega$  is contained in a ball  $B_R(x_0)$  centered in  $x_0$  with the radius  $R$  sufficiently large and denote by  $r = r(x) = |x - x_0|$  the distance to  $x_0$ . We define  $f$  on the whole  $B_R(x_0)$  by extending it by 0 outside of  $\Omega$ . In the following we will use spherical coordinates  $(r, \theta, \varphi)$  and the notation

$$\omega = (\theta, \varphi) \in \mathcal{S} := (0, \pi) \times (0, 2\pi).$$

Moreover, we will use the convention  $f = f(x) = f(r, \omega)$  as well as  $\sigma = \sigma(x) = \sigma(r)$ . Notice that

$$(20) \quad \frac{d}{dr}(\sigma(r)) = \frac{d}{dr}(\sigma(r) - \sigma(0)) = \frac{r}{\sigma}.$$

Then transforming the integral, we get

$$(21) \quad \|\sigma^\alpha f\|_\Omega^2 = \int_\Omega \sigma^{2\alpha} f^2 dx = \int_{\mathcal{S}} \int_0^R \sigma^{2\alpha}(r) f^2 r^2 dr d\omega.$$

Using (20) and integrating by parts, we obtain

$$\begin{aligned} \int_0^R \sigma^{2\alpha}(r) f^2 r^2 dr &= \int_0^R \frac{d}{dr}(\sigma(r) - \sigma(0)) \sigma^{2\alpha+1} f^2 r dr \\ &= - \int_0^R (\sigma(r) - \sigma(0)) \frac{d}{dr}(\sigma^{2\alpha+1} f^2 r) dr \\ &= - \int_0^R (\sigma(r) - \sigma(0)) \\ &\quad \times \left( (2\alpha + 1) \sigma^{2\alpha} \frac{r}{\sigma} f^2 r + \sigma^{2\alpha+1} f^2 + 2\sigma^{2\alpha+1} f f_r r \right) dr. \end{aligned}$$

Using the assumption  $\alpha \geq -\frac{1}{2}$ , we have

$$- \int_0^R (\sigma(r) - \sigma(0)) \left( (2\alpha + 1) \sigma^{2\alpha} \frac{r}{\sigma} f^2 r + \sigma^{2\alpha+1} f^2 \right) dr \leq 0$$

and as a result

$$\int_0^R \sigma^{2\alpha}(r) f^2 r^2 dr \leq - \int_0^R (\sigma(r) - \sigma(0)) 2\sigma^{2\alpha+1} f f_r r dr.$$

Using that

$$\frac{\sigma(r) - \sigma(0)}{r^2} = \frac{\sqrt{r^2 + K^2 h^2} - Kh}{r^2} = \frac{1}{\sqrt{r^2 + K^2 h^2} + Kh} \leq \sigma^{-1},$$

and applying the Cauchy–Schwarz inequality together with the fact that  $r \leq \sigma$ , for  $1 < p < \infty$ , we have

$$\begin{aligned} \|\sigma^\alpha f\|_\Omega^2 &\leq C \int_S \int_0^R \left| \frac{\sigma(r) - \sigma(0)}{r^2} \right| \sigma^{2\alpha+1} |f| |f_r| r^3 dr d\omega \leq C \int_S \int_0^R \sigma^{2\alpha} |f| |f_r| r^3 dr d\omega \\ &\leq C \int_S \int_0^R \sigma^{2\alpha+1} |f| |f_r| r^2 dr d\omega \\ &\leq C \int_S \int_0^R \left( \sigma^{\alpha-\beta} |f| r^{\frac{2}{p}} \right) \left( \sigma^{\alpha+1+\beta} |f_r| r^{\frac{2}{q}} \right) dr d\omega \\ &\leq C \left( \int_S \int_0^R \sigma^{p(\alpha-\beta)} |f|^p r^2 dr d\omega \right)^{1/p} \left( \int_S \int_0^R \sigma^{q(\alpha+1+\beta)} |f_r|^q r^2 dr d\omega \right)^{1/q} \\ &\leq C \|\sigma^{\alpha-\beta} f\|_{L^p(\Omega)} \|\sigma^{\alpha+1+\beta} \nabla f\|_{L^q(\Omega)}. \end{aligned}$$

For  $p = 1$  or  $p = \infty$  the estimate is similar.  $\square$

The next lemma is a three dimensional version of Lemma 2.4 in [31]. The analysis in three dimensions is more involved than the corresponding analysis in two dimensions. The main difficulty lies in the fact that we need to deal with odd powers of the weight function  $\sigma$  which causes serious technical difficulties. In the proof we mix two popular techniques from the finite element literature on pointwise estimates.

LEMMA 6. *There exists a constant  $C > 0$  independent on  $h$ , such that for any  $v_h \in V_h$ ,*

$$\sum_{\tau \in \mathcal{T}} (\|\Delta v_h\|_{L^1(\tau)} + \|[\partial_n v_h]\|_{L^1(\partial\tau)}) \leq C |\ln h|^{\frac{1}{2}} \left( \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_\Omega + \|\sigma^{\frac{1}{2}} \nabla v_h\|_\Omega \right).$$

*Proof.* We define  $v$  as the solution of

$$(22) \quad \begin{aligned} -\Delta v &= -\Delta_h v_h && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Thus, by construction  $v_h$  is the Ritz projection of  $v$ ; i.e.,  $R_h v = v_h$  and  $v \in H^2(\Omega) \cap H_0^1(\Omega)$  with

$$\|v\|_{H^2(\Omega)} \leq C \|\Delta v\|_\Omega = C \|\Delta_h v_h\|_\Omega.$$

Using the fact that jumps of  $v$  are zero, the trace inequality, and the inverse inequality



we have

$$\begin{aligned}
 \sum_{\tau} \|[\partial_n v_h]\|_{L^1(\partial\tau)} &= \sum_{\tau} \|[\partial_n[v_h - v]]\|_{L^1(\partial\tau)} \\
 &\leq C \sum_{\tau} (h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + \|\nabla^2(v - v_h)\|_{L^1(\tau)}) \\
 &\leq C \sum_{\tau} (h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)} \\
 (23) \quad &\quad + \|\nabla^2(v_h - I_h v)\|_{L^1(\tau)}) \\
 &\leq C \sum_{\tau} (h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)} \\
 &\quad + h^{-1}\|\nabla(v_h - I_h v)\|_{L^1(\tau)}) \\
 &\leq C \sum_{\tau} (h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} \\
 &\quad + \|\nabla^2(v - I_h v)\|_{L^1(\tau)}).
 \end{aligned}$$

Similarly, we obtain for the Laplacian of  $v_h$

$$\begin{aligned}
 (24) \quad \sum_{\tau} \|\Delta v_h\|_{L^1(\tau)} &\leq \sum_{\tau} (\|\Delta v\|_{L^1(\tau)} + \|\Delta(v_h - v)\|_{L^1(\tau)}) \\
 &\leq C \sum_{\tau} (\|\nabla^2 v\|_{L^1(\tau)} + h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} \\
 &\quad + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} + \|\nabla^2(v - I_h v)\|_{L^1(\tau)}).
 \end{aligned}$$

Combining (23) and (24), using the properties of  $\sigma$  and Lemma 2, we obtain

$$\begin{aligned}
 \sum_{\tau} (\|\Delta v_h\|_{L^1(\tau)} + \|[\partial_n[v_h - v]]\|_{L^1(\partial\tau)}) \\
 \leq C \sum_{\tau} (\|\nabla^2 v\|_{L^1(\tau)} + h^{-1}\|\nabla(v - v_h)\|_{L^1(\tau)} + h^{-1}\|\nabla(v - I_h v)\|_{L^1(\tau)} \\
 \quad + \|\nabla^2(v - I_h v)\|_{L^1(\tau)}) \\
 \leq C \|\sigma^{-\frac{3}{2}}\|_{\Omega} \left( \|\sigma^{\frac{3}{2}}\nabla^2 v\|_{\Omega} + h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega} + h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - I_h v)\|_{\Omega} \right. \\
 \quad \left. + \|\sigma^{\frac{3}{2}}\nabla^2(v - I_h v)\|_{h,\Omega} \right) \\
 \leq C |\ln h|^{\frac{1}{2}} \left( h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega} + \|\sigma^{\frac{3}{2}}\nabla^2 v\|_{\Omega} \right).
 \end{aligned}$$

To conclude the proof, we need to establish that

$$h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega} + \|\sigma^{\frac{3}{2}}\nabla^2 v\|_{\Omega} \leq C \left( \|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega} + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega} \right),$$

which we will show in the next two lemmas separately. □

In the next lemma we will treat the  $\|\sigma^{\frac{3}{2}}\nabla^2 v\|_{\Omega}$  term.

LEMMA 7. *There exists a constant  $C > 0$  independent on  $h$ , such that for any  $v_h \in V_h$  and the corresponding  $v \in H^2(\Omega) \cap H_0^1(\Omega)$  defined by (22) the following estimate holds:*

$$\|\sigma^{\frac{3}{2}}\nabla^2 v\|_{\Omega} \leq C \left( \|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega} + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega} \right).$$

*Proof.* Since

$$\partial_{ij}^2(\sigma^{\frac{3}{2}}v) = \sigma^{\frac{3}{2}}\partial_{ij}^2v + (\partial_i(\sigma^{\frac{3}{2}})\partial_jv + \partial_j(\sigma^{\frac{3}{2}})\partial_iv) + \partial_{ij}^2(\sigma^{\frac{3}{2}})v,$$

and by properties of  $\sigma$ ,  $|\nabla(\sigma^{\frac{3}{2}})| \leq C|\sigma^{\frac{1}{2}}|$  and  $|\nabla^2(\sigma^{\frac{3}{2}})| \leq C\sigma^{-\frac{1}{2}}$ , we obtain

$$(25) \quad \|\sigma^{\frac{3}{2}}\nabla^2v\|_{\Omega} \leq \|\nabla^2(\sigma^{\frac{3}{2}}v)\|_{\Omega} + c\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + c\|\sigma^{-\frac{1}{2}}v\|_{\Omega}.$$

Using Lemma 5 with  $\alpha = -\frac{1}{2}$ ,  $\beta = 0$ , and  $p = 2$  we have

$$(26) \quad \|\sigma^{-\frac{1}{2}}v\|_{\Omega} \leq C\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}.$$

By the global  $H^2(\Omega)$  regularity, we have

$$\|\nabla^2(\sigma^{\frac{3}{2}}v)\|_{\Omega} \leq C\|\Delta(\sigma^{\frac{3}{2}}v)\|_{\Omega}.$$

For  $\Delta(\sigma^{\frac{3}{2}}v)$  we obtain

$$\Delta(\sigma^{\frac{3}{2}}v) = \Delta(\sigma^{\frac{3}{2}})v + 3\sigma^{\frac{1}{2}}\nabla\sigma \cdot \nabla v + \sigma^{\frac{3}{2}}\Delta v,$$

and thus by the properties of  $\sigma$  we get

$$\|\nabla^2(\sigma^{\frac{3}{2}}v)\|_{\Omega} \leq C\|\sigma^{-\frac{1}{2}}v\|_{\Omega} + C\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + C\|\sigma^{\frac{3}{2}}\Delta v\|_{\Omega},$$

and by using (26) we obtain

$$(27) \quad \|\nabla^2(\sigma^{\frac{3}{2}}v)\|_{\Omega} \leq C\left(\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + \|\sigma^{\frac{3}{2}}\Delta v\|_{\Omega}\right) = C\left(\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega} + \|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}\right),$$

where we applied the definition of  $v$  (22) in the last step. It remains to estimate  $\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}$ . It holds that

$$\begin{aligned} \|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 &= (\sigma\nabla v, \nabla v) = (\sigma\nabla v, \nabla(v - v_h)) + (\sigma^{\frac{1}{2}}\nabla v, \sigma^{\frac{1}{2}}\nabla v_h) \\ &\leq (\sigma\nabla v, \nabla(v - v_h)) + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega}^2, \end{aligned}$$

and therefore

$$\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 \leq 2(\sigma\nabla v, \nabla(v - v_h)) + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega}^2.$$

For the first term we get

$$\begin{aligned} (\sigma\nabla v, \nabla(v - v_h)) &= -(\nabla \cdot (\sigma\nabla v), v - v_h) = -(\nabla\sigma \cdot \nabla v + \sigma\Delta v, v - v_h) \\ &= -(\sigma^{\frac{1}{2}}\nabla\sigma \cdot \nabla v + \sigma^{\frac{3}{2}}\Delta v, \sigma^{-\frac{1}{2}}(v - v_h)) \\ &\leq \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta v\|_{\Omega}^2 + Ch^{-1}\|v - v_h\|_4^2 \\ &\leq \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2 + Ch^{-1}h^4\|v\|_{H^2(\Omega)}^2 \\ &\leq \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2 + Ch^3\|\Delta v\|_{\Omega}^2 \\ &\leq \frac{1}{4}\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 + C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2, \end{aligned}$$

where we used again (22) and that  $h \leq \sigma$ . This results in

$$\|\sigma^{\frac{1}{2}}\nabla v\|_{\Omega}^2 \leq C\|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega}^2,$$

which together with (27) completes the proof. □

As the next step we have to estimate  $h^{-1}\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega}$ . In the proof of this estimate we will make a heavy use of pointwise estimates for the Green's function and its derivatives. The proof of the next lemma for a general second order elliptic equation can be found in [18].

LEMMA 8. *Let  $\Omega \subset \mathbb{R}^3$  be a convex domain of polyhedral type. Let  $G(x, y)$  denotes the elliptic Green's function of the Laplace operator on the domain  $\Omega$ . Then the following estimates hold:*

$$(28a) \quad |G(x, y)| \leq C|x - y|^{-1},$$

$$(28b) \quad |\nabla_x G(x, y)| \leq C|x - y|^{-2}.$$

Sharper Hölder type estimates for the Green's function are derived for three dimensional polyhedral domains in [19, Theorem 1]. We summarize them in the following lemma.

LEMMA 9. *Let  $\Omega \subset \mathbb{R}^3$  be a convex domain of polyhedral type. There exist  $0 < \lambda < 1$  that depends on geometry of  $\Omega$  and a constant  $C$  such that the estimates*

$$\frac{|\partial_k G(x, \xi) - \partial_k G(y, \xi)|}{|x - y|^\lambda} \leq C (|x - \xi|^{-2-\lambda} + |y - \xi|^{-2-\lambda}) \quad \text{for } k = 1, 2, 3,$$

are satisfied for all  $x, y, \xi \in \Omega, x \neq y$ .

Now we are ready to establish an estimate for  $\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega}$ .

LEMMA 10. *There exists a constant  $C > 0$  independent on  $h$ , such that for any  $v_h \in V_h$  and the corresponding  $v \in H^2(\Omega) \cap H_0^1(\Omega)$  defined by (22) and for  $K$  sufficiently large, the following estimate holds:*

$$\|\sigma^{\frac{3}{2}}\nabla(v - v_h)\|_{\Omega} \leq Ch \left( \|\sigma^{\frac{3}{2}}\Delta_h v_h\|_{\Omega} + \|\sigma^{\frac{1}{2}}\nabla v_h\|_{\Omega} \right).$$

*Proof.* To obtain the estimate we use a dyadic decomposition of  $\Omega$ . Denote  $d_j = 2^{-j} \text{diam}(\Omega)$  for  $j = 0, 1, \dots$  and define the innermost set

$$\Omega_* = \{x \in \Omega \mid |x - x_0| \leq C_*h\},$$

where the constant  $C_*$  to be determined later and

$$\Omega_j = \{x \in \Omega \mid d_{j+1} < |x - x_0| \leq d_j\}.$$

Let  $J$  be chosen such that  $d_{J+1} \leq C_*h_* \leq d_J$ . Note that by construction  $\text{diam}(\Omega_j) \leq d_j, J \leq C|\ln h|$ , and  $\sigma \leq \sqrt{d_j^2 + (Kh)^2} \leq d_j + Kh$  on  $\Omega_j$ . Furthermore, for  $k > j + 1$  so that  $d_j > d_k$  there holds

$$(29) \quad \frac{1}{2}d_j \leq \text{dist}(\Omega_k, \Omega_j) \leq d_j.$$

We have the decomposition

$$\Omega = \Omega_* \cup \bigcup_{j=0}^J \Omega_j.$$

Moreover, in the following analysis we will need the following sets:

$$\Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1} \quad \text{and} \quad \Omega''_j = \Omega_{j-2} \cup \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1} \cup \Omega_{j+2}.$$

Denote by  $e := v - v_h$ . We have

$$\begin{aligned} \|\sigma^{\frac{3}{2}} \nabla(v - v_h)\|_{\Omega}^2 &= \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega}^2 \leq (C_* + K)^3 h^3 \|\nabla e\|_{\Omega_*}^2 + C \sum_{j=0}^J (d_j + Kh)^3 \|\nabla e\|_{\Omega_j}^2 \\ &\leq C(C_* + K)^3 h^3 \|\nabla e\|_{\Omega}^2 + C \sum_{j=0}^J d_j^3 \|\nabla e\|_{\Omega_j}^2. \end{aligned}$$

By global best approximation result, elliptic  $H^2$ -regularity, and  $h \leq \frac{\sigma}{K}$ , we have

$$\begin{aligned} h^{\frac{3}{2}} \|\nabla e\|_{\Omega} &\leq Ch^{\frac{3}{2}} \|\nabla(v - I_h v)\|_{\Omega} \leq Ch^{\frac{3}{2}+1} \|\nabla^2 v\|_{\Omega} \\ &\leq Ch^{\frac{3}{2}+1} \|\Delta v\|_{\Omega} \leq CK^{-\frac{3}{2}} h \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{\Omega}. \end{aligned}$$

Using local energy estimates [29], we have for any  $\chi \in V_h$ ,

$$(30) \quad \|\nabla e\|_{\Omega_j}^2 \leq C \left( \|\nabla(v - \chi)\|_{\Omega'_j}^2 + d_j^{-2} \|v - \chi\|_{\Omega'_j}^2 + d_j^{-2} \|e\|_{\Omega'_j}^2 \right).$$

Taking  $\chi = I_h v$  and using the approximation theory,  $h \leq d_j$ , and that  $d_j \leq c\sigma$  on  $\Omega'_j$ , we have

$$(31) \quad \sum_{j=0}^J d_j^3 \left( \|\nabla(v - \chi)\|_{\Omega'_j}^2 + d_j^{-2} \|v - \chi\|_{\Omega'_j}^2 \right) \leq Ch^2 \sum_{j=0}^J d_j^3 \|\nabla^2 v\|_{\Omega'_j}^2 \leq Ch^2 \|\sigma^{\frac{3}{2}} \nabla^2 v\|_{\Omega}^2.$$

Later on, we will control the term on the right-hand side of (31) using Lemma 7. Thus, we need to estimate  $\sum_{j=0}^J d_j \|e\|_{\Omega'_j}^2$ . First, we notice that  $\sum_{j=0}^J d_j \|e\|_{\Omega'_j}^2 \leq \frac{7}{2} \sum_{j=0}^{J,*} d_j \|e\|_{\Omega_j}^2$ , where the notation  $\sum_{i=0}^{J,*}$  means that the sum includes the innermost set  $\Omega_*$ . For  $j = *, J, J-1$  using that  $d_{J-1} = 2d_J \leq 4C_*h$  and the global approximation result, we obtain

$$(32) \quad \sum_{j=J-1}^{J,*} d_j \|e\|_{\Omega_j}^2 \leq CC_* h^5 \|\nabla^2 v\|_{\Omega}^2 \leq CC_* h^5 \|\Delta v\|_{\Omega}^2 = CC_* K^{-3} h^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{\Omega}^2.$$

To estimate  $\sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2$ , we use a duality argument. Let  $w$  be the solution of the following problem:

$$(33) \quad \begin{aligned} -\Delta w &= e \cdot \mathbb{1}_{\Omega_j} && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where  $\mathbb{1}_{\Omega_j}$  is the characteristic function of  $\Omega_j$ . Then

$$\|e\|_{\Omega_j}^2 = (\nabla w, \nabla e)_{\Omega} = (\nabla(w - I_h w), \nabla e)_{\Omega} = \sum_{i=0}^{J,*} (\nabla(w - I_h w), \nabla e)_{\Omega_i}.$$

We now consider three cases.

*Case 1.*  $|i - j| \leq 2$ . In this case  $\frac{1}{2}d_i \leq d_j \leq 2d_i$  and  $\Omega_i \subset \Omega'_j$ . Using the Cauchy-Schwarz inequality and the  $H^2$ -regularity, we obtain

$$\sum_{|i-j| \leq 1} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \leq Ch \|\nabla^2 w\|_{\Omega} \|\nabla e\|_{\Omega_j} \leq Ch \|e\|_{\Omega_j} \|\nabla e\|_{\Omega_j},$$

and as a result, we obtain

$$\sum_{j=0}^{J-2} d_j \sum_{|i-j|\leq 1} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \leq Ch \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j} \|\nabla e\|_{\Omega'_j} := S_1.$$

Hence,

$$\begin{aligned} S_1 &\leq C \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{J-1} d_j h^2 \|\nabla e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \\ &\leq C \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} K^{-1} \left( \sum_{j=0}^{J-1} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega'_j}^2 \right)^{\frac{1}{2}} \\ &\leq CK^{-1} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega}, \end{aligned}$$

where we used that  $\frac{h}{\sigma} \leq K^{-1}$  and  $d_j \leq 2\sigma$  on  $\Omega_j$ .

Case 2.  $i < j - 2$ . In this case  $d_i > d_j$  and  $\frac{1}{2}d_i \leq \text{dist}(\Omega_j, \Omega_i) \leq d_i$ . We then have

$$\sum_{i < j-2} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \leq \sum_{i < j-2} \|\nabla(w - I_h w)\|_{\Omega_i} \|\nabla e\|_{\Omega_i} \leq Ch \sum_{i < j-2} \|\nabla^2 w\|_{\Omega_i} \|\nabla e\|_{\Omega_i}.$$

Using that  $w$  is harmonic on  $\Omega_i$  and the Hölder inequality, we have

$$\|\nabla^2 w\|_{\Omega_i} \leq Cd_i^{-1} \|\nabla w\|_{\Omega'_i} \leq Cd_i^{\frac{1}{2}} \|\nabla w\|_{L^\infty(\Omega'_i)}.$$

Using Green's function representation, Green's function estimate (28b), and that  $\frac{1}{2}d_i \leq \text{dist}(\Omega_j, \Omega'_i) \leq d_i$ , for  $x \in \Omega'_i$  we have

$$\begin{aligned} |\nabla w(x)| &= \left| \int_{\Omega_j} \nabla_x G(x, y) e(y) dy \right| \leq C \int_{\Omega_j} \frac{|e(y)|}{|x - y|^2} dy \\ &\leq Cd_i^{-2} \|e\|_{L^1(\Omega_j)} \leq Cd_i^{-2} d_j^{\frac{3}{2}} \|e\|_{\Omega_j}. \end{aligned}$$

As a result

$$\|\nabla^2 w\|_{\Omega_i} \leq Cd_i^{-\frac{3}{2}} d_j^{\frac{3}{2}} \|e\|_{\Omega_j},$$

and hence the total contribution of this term to the sum is

$$\begin{aligned} \sum_{j=0}^{J-2} d_j \sum_{i < j-2} (\nabla(w - I_h w), \nabla e)_{\Omega_i} &\leq \sum_{j=0}^{J-2} d_j \sum_{i < j-2} h \|\nabla^2 w\|_{\Omega_i} \|\nabla e\|_{\Omega_i} \\ &\leq Ch \sum_{j=0}^{J-2} d_j^{\frac{5}{2}} \|e\|_{\Omega_j} \sum_{i < j-2} d_i^{-\frac{3}{2}} \|\nabla e\|_{\Omega_i} := S_2. \end{aligned}$$

Changing the order of summation, we obtain

$$S_2 \leq Ch \sum_{i=0}^{J-2} d_i^{-\frac{3}{2}} \|\nabla e\|_{\Omega_i} \sum_{j=i+3}^{J-2} d_j^{\frac{5}{2}} \|e\|_{\Omega_j}.$$

Now using the properties of geometric series, we have

$$\sum_{j=i+3}^{J-2} d_j^{\frac{5}{2}} \|e\|_{\Omega_j} \leq \left( \sum_{j=i+2}^J d_j^4 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \leq C d_i^2 \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}.$$

As a result, using that  $d_i \leq 2\sigma$  on  $\Omega_i$

$$\begin{aligned} S_2 &\leq Ch \sum_{i=0}^{J-2} d_i^{\frac{1}{2}} \|\nabla e\|_{\Omega_i} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \\ (34) \quad &\leq C \left( \sum_{i=0}^J \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega_i}^2 \right)^{\frac{1}{2}} \left( \sum_{i=0}^J \left( \frac{h}{d_i} \right)^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \\ &\leq CC_*^{-1} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where we used that  $\sum_{i=0}^J \left( \frac{h}{d_i} \right)^2 \leq CC_*^{-2}$ .

*Case 3.  $i > j + 2$ .* In this case,  $d_i < d_j$  and  $\frac{1}{2}d_j \leq \text{dist}(\Omega_j, \Omega_i) \leq d_j$ . We then have, using the Hölder inequality and the approximation theory,

$$\begin{aligned} \sum_{i=j+3}^{J,*} (\nabla(w - I_h w), \nabla e)_{\Omega_i} &\leq \sum_{i=j+3}^{J,*} \|\nabla(w - I_h w)\|_{L^\infty(\Omega_i)} \|\nabla e\|_{L^1(\Omega_i)} \\ &\leq Ch^\lambda \sum_{i=j+3}^J \|w\|_{C^{1,\lambda}(\Omega_i)} d_i^{\frac{3}{2}} \|\nabla e\|_{\Omega_i} \\ &\quad + Ch^\lambda \|w\|_{C^{1,\lambda}(\Omega_*)} (C_* h)^{\frac{3}{2}} \|\nabla e\|_{\Omega_*}, \end{aligned}$$

where  $\lambda$  depends on the domain  $\Omega$  and such that  $w \in C^{1,\lambda}(\Omega)$ ; cf. [25]. Following [19], for  $x, y \in \Omega_i$  by Lemma 9 for  $k = 1, 2, 3$ , we have

$$\begin{aligned} \frac{|\partial_k w(x) - \partial_k w(y)|}{|x - y|^\lambda} &\leq \int_{\Omega_j} \frac{|\partial_{x_i} G(x, \xi) - \partial_{y_i} G(y, \xi)|}{|x - y|^\lambda} |e(\xi)| d\xi \\ &\leq C \max_{\xi \in \Omega_j} (|x - \xi|^{-2-\lambda} + |y - \xi|^{-2-\lambda}) \int_{\Omega_j} |e(\xi)| d\xi \\ &\leq C d_j^{-2-\lambda} d_j^{\frac{3}{2}} \|e\|_{\Omega_j} \leq C d_j^{-\frac{1}{2}-\lambda} \|e\|_{\Omega_j}. \end{aligned}$$

Thus, we obtain

$$\|w\|_{C^{1,\lambda}(\Omega_i)} \leq C d_j^{-\frac{1}{2}-\lambda} \|e\|_{\Omega_j}, \quad i = j + 2, \dots, J, *.$$

As a result, the total contribution to the sum is

$$\begin{aligned} &\sum_{j=0}^{J-2} d_j \sum_{i=j+3}^{J,*} (\nabla(w - I_h w), \nabla e)_{\Omega_i} \\ &\leq Ch^\lambda \sum_{j=0}^{J-2} d_j^{\frac{1}{2}-\lambda} \|e\|_{\Omega_j} \left( \sum_{i=j+3}^J d_i^{\frac{3}{2}} \|\nabla e\|_{\Omega_i} + (C_* h)^{\frac{3}{2}} \|\nabla e\|_{\Omega_*} \right) := S_3. \end{aligned}$$

Changing the order of summation, we obtain

$$S_3 \leq Ch^\lambda \left( \sum_{i=0}^J d_i^{\frac{3}{2}} \|\nabla e\|_{\Omega_i} + (C_*h)^{\frac{3}{2}} \|\nabla e\|_{\Omega_*} \right) \sum_{j=0}^{i-3} d_j^{\frac{1}{2}-\lambda} \|e\|_{\Omega_j}.$$

Using the properties of the geometric series, we get

$$\sum_{j=0}^{i-3} d_j^{\frac{1}{2}-\lambda} \|e\|_{\Omega_j} \leq \left( \sum_{j=0}^{i-3} d_j^{-2\lambda} \right)^{\frac{1}{2}} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \leq Cd_i^{-\lambda} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned} (35) \quad S_3 &\leq C \left( \sum_{i=0}^J \left( \frac{h}{d_i} \right)^{2\lambda} + \left( \frac{C_*}{K} \right)^3 \right)^{\frac{1}{2}} \left( \sum_{i=0}^{J,*} \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega_i}^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}} \\ &\leq C \left( C_*^{-\lambda} + \left( \frac{C_*}{K} \right)^{\frac{3}{2}} \right) \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega} \left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where we used that  $\sum_{i=0}^J \left( \frac{h}{d_i} \right)^{2\lambda} \leq CC_*^{-2\lambda}$  and  $h \leq \frac{\sigma}{K}$ . Using (32) and combining cases 1, 2, and 3 and canceling by  $\left( \sum_{j=0}^{J-2} d_j \|e\|_{\Omega_j}^2 \right)^{\frac{1}{2}}$ , we obtain

$$(36) \quad \sum_{j=0}^{J,*} d_j \|e\|_{\Omega_j}^2 \leq C \left( K^{-2} + C_*^{-2} + C_*^{-2\lambda} + \left( \frac{C_*}{K} \right)^3 \right) \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega}^2 + Ch^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{\Omega}^2.$$

Combining the above estimate with (30) and (31), we obtain

$$\begin{aligned} &\|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega}^2 \leq Ch^2 \|\sigma^{\frac{3}{2}} \nabla^2 v\|_{\Omega}^2 \\ &+ C \left( K^{-2} + C_*^{-2} + C_*^{-2\lambda} + \left( \frac{C_*}{K} \right)^3 \right) \|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega}^2 + Ch^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{\Omega}^2. \end{aligned}$$

Taking  $K = C_*^2$  and selecting  $C_*$  sufficiently large, we conclude that

$$\|\sigma^{\frac{3}{2}} \nabla e\|_{\Omega}^2 \leq C_{C_*,K} h^2 \|\sigma^{\frac{3}{2}} \nabla^2 v\|_{\Omega}^2 + Ch^2 \|\sigma^{\frac{3}{2}} \Delta_h v_h\|_{\Omega}^2.$$

Applying Lemma 7 to the first term on the right-hand side and taking the square root concludes the proof.  $\square$

The following lemma provides a discrete analog of the embedding  $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$  and will be used in section 4.

LEMMA 11. *There exists a constant  $C$  independent of  $h$  such that*

$$\|v_h\|_{L^\infty(\Omega)} \leq C \|\Delta_h v_h\|_{\Omega} \quad \forall v_h \in V_h.$$

*Proof.* To establish this lemma, we, similarly to the proof of Lemma 6, define  $v$  as the solution of

$$(37) \quad \begin{aligned} -\Delta v &= -\Delta_h v_h && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Thus, by construction  $v_h$  is the Ritz projection of  $v$ , i.e.,  $R_h v = v_h$  and  $v \in H^2(\Omega) \cap H_0^1(\Omega)$  with

$$\|v\|_{H^2(\Omega)} \leq C\|\Delta v\|_{\Omega} = C\|\Delta_h v_h\|_{\Omega}.$$

By the triangle inequality,

$$\|v_h\|_{L^\infty(\Omega)} \leq \|P_h v\|_{L^\infty(\Omega)} + \|v_h - P_h v\|_{L^\infty(\Omega)}.$$

Using the stability of the  $L^2$ -projection in  $L^\infty$  norm (see, e.g., [10] and the Sobolev embedding theorem), we have

$$\|P_h v\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)} \leq C\|v\|_{H^2(\Omega)} \leq C\|\Delta_h v_h\|_{\Omega}.$$

On the other hand, using the inverse inequality first, then the triangle inequality and the standard error estimates, we obtain

$$\begin{aligned} \|v_h - P_h v\|_{L^\infty(\Omega)} &\leq Ch^{-\frac{3}{2}}\|v_h - P_h v\|_{\Omega} \\ &\leq Ch^{-\frac{3}{2}}(\|v - P_h v\|_{\Omega} + \|v - v_h\|_{\Omega}) \leq Ch^{\frac{1}{2}}\|\Delta_h v_h\|_{\Omega}. \end{aligned}$$

This establishes the lemma.  $\square$

**3. Stability of the Ritz projection in  $L^\infty$ -norm.** Using the technical lemmas, Lemmas 5 and 6, we can establish several important results. The first result shows stability (modulo logarithm) of the Ritz projection in  $L^\infty$ -norm. This result is known for smooth domains  $\Omega \subset \mathbb{R}^N$  and for polygonal domains in  $\mathbb{R}^2$ . To the best of our knowledge this result is new for convex polyhedral domains  $\Omega \subset \mathbb{R}^3$ .

**THEOREM 12.** *There exists a constant  $C$  independent of  $h$  such that for the solution  $u$  of (1) and its Ritz projection  $u_h = R_h u \in V_h$ , it holds that*

$$\|u_h\|_{L^\infty(\Omega)} \leq C|\ln h|\|u\|_{L^\infty(\Omega)}.$$

*Proof.* Let  $x_0 \in \bar{\tau}_0$  such that  $\|u_h\|_{L^\infty(\Omega)} = |u_h(x_0)|$ . For such  $x_0$  define a regularized Green's function  $g$  that satisfies

$$(38) \quad \begin{aligned} -\Delta g(x) &= \tilde{\delta}(x), & x \in \Omega, \\ g(x) &= 0, & x \in \partial\Omega, \end{aligned}$$

where  $\tilde{\delta}(x) := \tilde{\delta}_{x_0}(x)$  is the regularized delta function from (12). We define  $g_h = R_h g \in V_h$ ; i.e.,

$$(39) \quad (\nabla g_h, \nabla \chi) = (\tilde{\delta}, \chi) \quad \forall \chi \in V_h.$$

Then using the Galerkin orthogonality, integration by parts, and Lemma 6, we obtain

$$\begin{aligned} (40) \quad u_h(x_0) &= (\nabla u_h, \nabla g_h) = (\nabla u, \nabla g_h) = \sum_{\tau \in \mathcal{T}} ((u, [\partial_n g_h])_{\partial\tau} + (u, -\Delta g_h)_\tau) \\ &\leq \|u\|_{L^\infty(\Omega)} \sum_{\tau \in \mathcal{T}} (\|[\partial_n g_h]\|_{L^1(\partial\tau)} + \|\Delta g_h\|_{L^1(\tau)}) \\ &\leq C|\ln h|^{\frac{1}{2}}\|u\|_{L^\infty(\Omega)} \left( \|\sigma^{\frac{3}{2}} \Delta_h g_h\|_{\Omega} + \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega} \right). \end{aligned}$$

Equation (39) for  $g_h$  is equivalent to  $-\Delta_h g_h = P_h \tilde{\delta}$ , and then using Lemma 4 we easily obtain

$$(41) \quad \|\sigma^{\frac{3}{2}} \Delta_h g_h\|_{\Omega} = \|\sigma^{\frac{3}{2}} P_h \tilde{\delta}\|_{\Omega} \leq C.$$



To estimate  $\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}$ , we set  $\chi = P_h(\sigma g_h)$  in (39) and obtain

$$(\nabla g_h, \nabla P_h(\sigma g_h)) = (\tilde{\delta}, P_h(\sigma g_h)).$$

Using the fact that

$$\begin{aligned} (\nabla g_h, \nabla P_h(\sigma g_h)) &= (\nabla g_h, \nabla(\sigma g_h)) + (\nabla g_h, \nabla(P_h(\sigma g_h) - \sigma g_h)) \\ &= \|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}^2 + (\nabla g_h, \nabla\sigma g_h) + (\nabla g_h, \nabla(P_h(\sigma g_h) - \sigma g_h)) \end{aligned}$$

and

$$(\tilde{\delta}, P_h(\sigma g_h)) = (P_h\tilde{\delta}, \sigma g_h) = (\sigma^{\frac{3}{2}}P_h\tilde{\delta}, \sigma^{-\frac{1}{2}}g_h),$$

we have

$$\begin{aligned} \|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}^2 &\leq \|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega} \left( \|\sigma^{-\frac{1}{2}}\nabla\sigma g_h\|_{\Omega} + \|\sigma^{-\frac{1}{2}}\nabla(P_h(\sigma g_h) - \sigma g_h)\|_{\Omega} \right) \\ &\quad + \|\sigma^{-\frac{1}{2}}g_h\|_{\Omega} \|\sigma^{\frac{3}{2}}P_h\tilde{\delta}\|_{\Omega}. \end{aligned}$$

Using that  $|\nabla\sigma| \leq C$ , Lemmas 3 and 4, and kicking back  $\|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}^2$ , we obtain

$$(42) \quad \|\sigma^{\frac{1}{2}}\nabla g_h\|_{\Omega}^2 \leq C \left( \|\sigma^{-\frac{1}{2}}g_h\|_{\Omega}^2 + 1 \right).$$

To estimate  $\|\sigma^{-\frac{1}{2}}g_h\|_{\Omega}$  we use Lemma 5, with  $\alpha = \beta = -\frac{1}{2}$  and  $p = 3$ , to obtain

$$(43) \quad \|\sigma^{-\frac{1}{2}}g_h\|_{\Omega} \leq C \|g_h\|_{L^3(\Omega)}^{\frac{1}{2}} \|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}.$$

Using the inverse and the triangle inequalities,

$$\begin{aligned} \|g_h\|_{L^3(\Omega)} &\leq \|g\|_{L^3(\Omega)} + \|g - g_h\|_{L^3(\Omega)} \leq \|g\|_{L^3(\Omega)} + \|I_h g - g_h\|_{L^3(\Omega)} + \|g - I_h g\|_{L^3(\Omega)} \\ &\leq \|g\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}}\|I_h g - g_h\|_{\Omega} + \|g - I_h g\|_{L^3(\Omega)} \\ &\leq \|g\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}}\|g - g_h\|_{\Omega} + Ch^{-\frac{1}{2}}\|g - I_h g\|_{\Omega} + \|g - I_h g\|_{L^3(\Omega)}. \end{aligned}$$

Using the approximation theory (10), the standard  $L^2$  estimate, and the properties of  $\tilde{\delta}$  function, we have

$$(44) \quad h^{-\frac{1}{2}}\|g - g_h\|_{\Omega} + h^{-\frac{1}{2}}\|g - I_h g\|_{\Omega} + \|g - I_h g\|_{L^3(\Omega)} \leq Ch^{\frac{3}{2}}\|g\|_{H^2(\Omega)} \leq Ch^{\frac{3}{2}}\|\tilde{\delta}\|_{\Omega} \leq C.$$

Next, we will show

$$(45) \quad \|g\|_{L^3(\Omega)} \leq C|\ln h|^{\frac{1}{3}}.$$

To establish that we use a Green's function representation

$$g(x) = \int_{\tau_0} G(x, y)\tilde{\delta}(y)dy.$$

Define  $B_h = B_{3h}(x_0) \cap \Omega$  and  $B_h^c = \Omega \setminus B_h$  and consider two cases:  $x \in B_h$  and  $x \in B_h^c$ . In the case  $x \in B_h$ , we obtain using spherical coordinates centered at  $x$  as well as (12) and (28a)

$$|g(x)| \leq \|\tilde{\delta}\|_{L^\infty(\tau_0)} \int_{\tau_0} |G(x, y)|dy \leq Ch^{-3} \int_0^{ch} \frac{1}{\rho} \rho^2 d\rho \leq Ch^{-1}.$$

Hence,

$$(46) \quad \|g\|_{L^3(B_h)}^3 \leq Ch^{-3} \int_{B_h} dx \leq C.$$

In the case  $x \in B_h^c$ , we have for any  $y \in \tau_0$  by the triangle inequality

$$|x - y| \geq |x - x_0| - |y - x_0| \geq |x - x_0| - h,$$

and therefore, again by (12) and (28a)

$$|g(x)| \leq \|\tilde{\delta}\|_{L^1(\tau_0)} \frac{C}{|x - x_0| - h} \leq \frac{C}{|x - x_0| - h}.$$

Hence,

$$\|g\|_{L^3(B_h^c)}^3 \leq C \int_{B_h^c} \frac{dx}{(|x - x_0| - h)^3} \leq C|\ln h|.$$

Together with (46), the above equation shows (45). Combining (44) and (45) we have established

$$(47) \quad \|g_h\|_{L^3(\Omega)} \leq C|\ln h|^{\frac{1}{3}}.$$

To treat  $\|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}$  we use the Hölder's inequality and (14a). We have

$$\begin{aligned} \|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{3}{2}} &\leq \|\sigma^{-\frac{3}{4}}\|_{L^4(\Omega)} \|\sigma^{\frac{3}{4}} |\nabla g_h|^{\frac{3}{2}}\|_{L^{\frac{4}{3}}(\Omega)} \leq \|\sigma^{-\frac{3}{2}}\|_{\Omega}^{\frac{1}{2}} \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega}^{\frac{3}{2}} \\ &\leq C|\ln h|^{\frac{1}{4}} \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega}^{\frac{3}{2}}. \end{aligned}$$

Hence,

$$(48) \quad \|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)} \leq C|\ln h|^{\frac{1}{6}} \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega}.$$

Thus, combining (42), (43), (47), and the above estimate, we have

$$\begin{aligned} \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega}^2 &\leq C \left( \|\sigma^{-\frac{1}{2}} g_h\|_{\Omega}^2 + 1 \right) \leq C \left( \|g_h\|_{L^3(\Omega)} \|\nabla g_h\|_{L^{\frac{3}{2}}(\Omega)} + 1 \right) \\ (49) \quad &\leq C \left( |\ln h|^{\frac{1}{3} + \frac{1}{6}} \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega} + 1 \right) \\ &\leq C|\ln h|^{\frac{1}{2}} \|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega}. \end{aligned}$$

Dividing both sides by  $\|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega}$ , we finally obtain

$$\|\sigma^{\frac{1}{2}} \nabla g_h\|_{\Omega} \leq C|\ln h|^{\frac{1}{2}},$$

which together with (40) and (41) establishes the theorem.  $\square$

**4. Resolvent estimates.** In this section we establish some resolvent estimates. Since we will be dealing with complex valued function spaces, we need to modify the definition of the  $L^2$ -inner product as

$$(u, v)_{\Omega} = \int_{\Omega} u(x) \bar{v}(x) dx,$$

where  $\bar{v}$  is the complex conjugate of  $v$  and the finite element space as  $\mathbb{V}_h = V_h + iV_h$ .

In the continuous case for Lipschitz domains the following result was shown in [42]: There exists a constant  $C$  such that

$$(50) \quad \|(z + \Delta)^{-1}v\|_{L^p(\Omega)} \leq \frac{C}{1 + |z|} \|v\|_{L^p(\Omega)}, \quad z \in \mathbb{C} \setminus \Sigma_{0,\gamma}, \quad 1 \leq p \leq \infty,$$

where  $\Sigma_{0,\gamma}$  is defined in (7). Using the identity  $\Delta(z + \Delta)^{-1} = \text{Id} - z(z + \Delta)^{-1}$ , one immediately obtains

$$(51) \quad \|\Delta(z + \Delta)^{-1}v\|_{L^p(\Omega)} \leq C \|v\|_{L^p(\Omega)}, \quad z \in \mathbb{C} \setminus \Sigma_{0,\gamma}, \quad 1 \leq p \leq \infty, \quad v \in L^p(\Omega).$$

In the following analysis we will also require a Green's function estimate for the resolvent equation with a real parameter  $s > 0$ ; i.e., for  $(s - \Delta)^{-1}$ .

LEMMA 13. *Let  $s > 0$  and let  $\Gamma_s(x, y)$  be the Green's function for the operator  $s - \Delta$  with zero Dirichlet boundary conditions. Then there exists a constant  $C$  independent of  $s$  such that*

$$(52) \quad \Gamma_s(x, y) \leq \frac{C}{|x - y|}.$$

*Proof.* The Green's function  $\Gamma_s(x, y)$  for  $s - \Delta$  has a representation

$$(53) \quad \Gamma_s(x, y) = \int_0^\infty e^{-st} H(t, x, y) dt,$$

where  $H(t, x, y)$  is the Green's function for the heat equation. Because of zero Dirichlet boundary data,  $H(t, x, y)$  is bounded by the fundamental solution of the heat equation and satisfies (cf. [14, Chapter 2.3])

$$H(t, x, y) \leq \frac{C}{t^{\frac{3}{2}}} e^{-c\frac{|x-y|^2}{t}}.$$

The integral estimate (cf. [21, Appendix])

$$(54) \quad \int_0^\infty \frac{e^{-\frac{d^2}{t} - st}}{t^{\frac{3}{2}}} dt \leq \frac{C}{d} \quad \forall d > 0$$

gives us the lemma. □

First, we prove a discrete resolvent estimate with respect to  $L^2(\Omega)$  norm.

LEMMA 14. *For any  $\gamma \in (0, \frac{\pi}{2})$  there exists a constant  $C = C_\gamma$  independent of  $h$  and  $z$  such that for any  $\lambda \in [0, \lambda_0]$  with  $\lambda_0 > 0$  being the smallest eigenvalue of  $-\Delta$  with homogeneous Dirichlet boundary conditions holds*

$$\|(z + \Delta_h)^{-1}\chi\|_\Omega \leq \frac{C_\gamma}{|z - \lambda|} \|\chi\|_\Omega \quad \forall z \in \mathbb{C} \setminus \Sigma_{\lambda,\gamma}, \quad \forall \chi \in V_h,$$

where  $\Sigma_{\lambda,\gamma}$  is defined in (7). The constant  $C_\gamma$  behaves like  $\gamma^{-1}$  for small values of  $\gamma$ .

*Proof.* For given  $\chi \in V_h$  let  $u_h \in \mathbb{V}_h$  be the solution of

$$-zu_h - \Delta_h u_h = \chi.$$

The existence and uniqueness of  $u_h$  (cf., e.g., [17]), followed by the fact that all eigenvalues of the discrete Laplacian  $\Delta_h$  are real and positive and for the smallest discrete eigenvalue  $\lambda_{0,h}$ , there holds  $\lambda_{0,h} \geq \lambda_0$ . Testing this equation with  $\bar{u}_h$ , we obtain

$$-z\|u_h\|_{\Omega}^2 + \|\nabla u_h\|_{\Omega}^2 = (\chi, u_h)_{\Omega},$$

and therefore,

$$(55) \quad (\lambda - z)\|u_h\|_{\Omega}^2 + \|\nabla u_h\|_{\Omega}^2 - \lambda\|u_h\|_{\Omega}^2 = (\chi, u_h)_{\Omega}.$$

By definition of  $\lambda_0$  we have

$$\|\nabla u_h\|_{\Omega}^2 \geq \lambda_0\|u_h\|_{\Omega}^2$$

and hence

$$\delta := \|\nabla u_h\|_{\Omega}^2 - \lambda\|u_h\|_{\Omega}^2 \geq (\lambda_0 - \lambda)\|u_h\|_{\Omega}^2 \geq 0.$$

Thus (55) can be rewritten as

$$|\lambda - z|\|u_h\|_{\Omega}^2 e^{i\phi} + \delta = (\chi, u_h)_{\Omega},$$

with  $|\phi| \leq \pi - \gamma$ . Multiplying this equation with  $e^{-\frac{i\phi}{2}}$ , taking real part and exploiting  $\delta \geq 0$  and  $\cos(\frac{\phi}{2}) > 0$ , we obtain

$$|\lambda - z|\|u_h\|_{\Omega}^2 \leq \frac{1}{\cos(\frac{\phi}{2})}|(\chi, u_h)_{\Omega}| \leq \frac{1}{\sin(\frac{\gamma}{2})}|(\chi, u_h)_{\Omega}| = C_{\gamma}|(\chi, u_h)_{\Omega}|.$$

This results in

$$\|u_h\|_{\Omega} \leq \frac{C_{\gamma}}{|\lambda - z|}\|\chi\|_{\Omega},$$

which completes the proof.  $\square$

Using the continuous resolvent results, (52), Lemma 14, and results from section 2, we establish the discrete resolvent estimate for the  $L^{\infty}$  norm.

**THEOREM 15.** *For any  $\gamma \in (0, \frac{\pi}{2})$ , there exists a constant  $C = C_{\gamma}$  independent of  $h$  and  $z$  such that*

$$\|(z + \Delta_h)^{-1}\chi\|_{L^{\infty}(\Omega)} \leq \frac{C|\ln h|}{|z - \lambda|}\|\chi\|_{L^{\infty}(\Omega)} \quad \forall z \in \mathbb{C} \setminus \Sigma_{\lambda, \gamma}, \quad \forall \chi \in V_h, \quad \forall \lambda \in [0, \lambda_0],$$

where  $\Sigma_{\lambda, \gamma}$  is defined in (6) and  $\lambda_0$  is the smallest eigenvalue of  $-\Delta$ .

*Proof.* First, we establish the theorem with  $\lambda = 0$ ; i.e.,

$$(56) \quad \|(z + \Delta_h)^{-1}\chi\|_{L^{\infty}(\Omega)} \leq \frac{C|\ln h|}{|z|}\|\chi\|_{L^{\infty}(\Omega)} \quad \forall z \in \mathbb{C} \setminus \Sigma_{0, \gamma}, \quad \forall \chi \in V_h,$$

and then following the argument of [3] at the end of section 6, we establish the theorem with  $\lambda \in [0, \lambda_0]$ . To show (56), we follow ideas of [43, Thm. 6.5], and [2]. The argument in [43] is purely two dimensional and we have to adapt it to our three dimensional setting.

Let  $x_0 \in \Omega$  be a fixed point and let  $\tilde{\delta} = \tilde{\delta}_{x_0}^h$  be the smooth delta function introduced in section 2. Then,

$$|(z + \Delta_h)^{-1}\chi(x_0)| = |((z + \Delta_h)^{-1}\chi, P_h\tilde{\delta})| = |(\chi, (\bar{z} + \Delta_h)^{-1}P_h\tilde{\delta})|.$$

We define an adjoint regularized Green's function  $G = G^{x_0}(x, \bar{z})$  by

$$G = G^{x_0}(x, \bar{z}) = (\bar{z} + \Delta)^{-1}\tilde{\delta}$$

and its discrete analog  $G_h = G_h^{x_0}(x, \bar{z}) \in \mathbb{V}_h$  by

$$G_h = G_h^{x_0}(x, \bar{z}) = (\bar{z} + \Delta_h)^{-1}P_h\tilde{\delta},$$

which we can write in the weak form as

$$(57) \quad z(\varphi, G_h) - (\nabla\varphi, \nabla G_h) = (\varphi, \tilde{\delta}) \quad \forall \varphi \in \mathbb{V}_h.$$

Using (14a) we get

$$|(z + \Delta_h)^{-1}\chi(x_0)| = |(\chi, G_h)| \leq \|\sigma^{-\frac{3}{2}}\|_{\Omega}\|\chi\|_{L^\infty}\|\sigma^{\frac{3}{2}}G_h\|_{\Omega} \leq C|\ln h|^{\frac{1}{2}}\|\chi\|_{L^\infty}\|\sigma^{\frac{3}{2}}G_h\|_{\Omega}.$$

Thus we need only establish

$$(58) \quad \|\sigma^{\frac{3}{2}}G_h\|_{\Omega} \leq C|\ln h|^{\frac{1}{2}}|z|^{-1}.$$

Consider the expression

$$(59) \quad -z\|\sigma^{\frac{3}{2}}G_h\|_{\Omega}^2 + \|\sigma^{\frac{3}{2}}\nabla G_h\|_{\Omega}^2 = -z(\sigma^3G_h, G_h) + (\nabla(\sigma^3G_h), \nabla G_h) - 3(\sigma^2\nabla\sigma G_h, \nabla G_h).$$

By taking  $\chi = -P_h(\sigma^3G_h)$  in (57) and subtracting it from (59), we obtain

$$(60) \quad -z\|\sigma^{\frac{3}{2}}G_h\|_{\Omega}^2 + \|\sigma^{\frac{3}{2}}\nabla G_h\|_{\Omega}^2 = F,$$

where

$$F = F_1 + F_2 + F_3 := -(P_h(\sigma^3G_h), \tilde{\delta}) + (\nabla(\sigma^3G_h - P_h(\sigma^3G_h)), \nabla G_h) - 3(\sigma^2\nabla\sigma G_h, \nabla G_h).$$

Since  $\gamma \leq |\arg z| \leq \pi$ , this equation is of the form

$$e^{i\alpha}a + b = f \quad \text{with} \quad a, b > 0, \quad 0 \leq |\alpha| \leq \pi - \gamma,$$

by multiplying it by  $e^{-\frac{i\alpha}{2}}$  and taking real parts, we have

$$a + b \leq \left(\cos\left(\frac{\alpha}{2}\right)\right)^{-1}|f| \leq \left(\sin\left(\frac{\gamma}{2}\right)\right)^{-1}|f| = C_\gamma|f|.$$

From (60) we therefore conclude that

$$|z|\|\sigma^{\frac{3}{2}}G_h\|_{\Omega}^2 + \|\sigma^{\frac{3}{2}}\nabla G_h\|_{\Omega}^2 \leq C_\gamma|F| \quad \text{for} \quad z \in \Sigma_{0,\gamma}.$$

Using the Cauchy-Schwarz inequality, Lemma 4, and the arithmetic-geometric mean inequality, we obtain

$$|F_1| = |(\sigma^3G_h, P_h\tilde{\delta})| \leq \|\sigma^{\frac{3}{2}}G_h\|_{\Omega}\|\sigma^{\frac{3}{2}}P_h\tilde{\delta}\|_{\Omega} \leq C\|\sigma^{\frac{3}{2}}G_h\|_{\Omega} \leq CC_\gamma|z|^{-1} + \frac{|z|}{2C_\gamma}\|\sigma^{\frac{3}{2}}G_h\|_{\Omega}^2.$$

To estimate  $F_2$  we use Lemma 3 and the Cauchy–Schwarz and arithmetic-geometric mean inequalities,

$$|F_2| \leq \|\sigma^{-\frac{3}{2}} \nabla(\sigma^3 G_h - P_h(\sigma^3 G_h))\|_{\Omega} \|\sigma^{\frac{3}{2}} \nabla G_h\|_{\Omega} \leq \frac{1}{4C_{\gamma}} \|\sigma^{\frac{3}{2}} \nabla G_h\|_{\Omega}^2 + CC_{\gamma} \|\sigma^{\frac{1}{2}} G_h\|_{\Omega}^2.$$

Finally, using the properties of  $\sigma$ , we obtain

$$|F_3| \leq C \|\sigma^{\frac{1}{2}} G_h\|_{\Omega} \|\sigma^{\frac{3}{2}} \nabla G_h\|_{\Omega} \leq \frac{1}{4C_{\gamma}} \|\sigma^{\frac{3}{2}} \nabla G_h\|_{\Omega}^2 + CC_{\gamma} \|\sigma^{\frac{1}{2}} G_h\|_{\Omega}^2.$$

Combining estimates for  $F'_i$ s and kicking back, we obtain

$$(61) \quad |z| \|\sigma^{\frac{3}{2}} G_h\|_{\Omega}^2 + \|\sigma^{\frac{3}{2}} \nabla G_h\|_{\Omega}^2 \leq C \left( |z|^{-1} + \|\sigma^{\frac{1}{2}} G_h\|^2 \right).$$

Thus, in order to establish (56), we need to show

$$(62) \quad \|\sigma^{\frac{1}{2}} G_h\|_{\Omega}^2 \leq C |\ln h| |z|^{-1}.$$

To accomplish that, we consider the expression

$$-z \|\sigma^{\frac{1}{2}} G_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega}^2 = -z(G_h, \sigma G_h) + (\nabla G_h, \nabla(\sigma G_h)) - (\nabla G_h, \nabla \sigma G_h).$$

Testing (57) with  $\varphi = P_h(\sigma G_h)$ , we obtain similarly as above that

$$|z| \|\sigma^{\frac{1}{2}} G_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega}^2 \leq C_{\gamma} |f| \quad \text{for } z \in \mathbb{C} \setminus \Sigma_{0,\gamma},$$

where

$$f = f_1 + f_2 + f_3 := -(P_h(\sigma G_h), \tilde{\delta}) + (\nabla(\sigma G_h - P_h(\sigma G_h)), \nabla G_h) - (\nabla \sigma G_h, \nabla G_h).$$

Using the Cauchy–Schwarz inequality, Lemma 4, and the arithmetic-geometric mean inequality, we obtain

$$|f_1| = |(\sigma G_h, P_h \tilde{\delta})| \leq \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega} \|\sigma^{\frac{3}{2}} P_h \tilde{\delta}\|_{\Omega} \leq C \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega}.$$

To estimate  $f_2$ , we use Lemma 3 and the Cauchy–Schwarz and the arithmetic-geometric mean inequalities,

$$|f_2| \leq \|\sigma^{-\frac{1}{2}} \nabla(\sigma G_h - (P_h(\sigma G_h)))\|_{\Omega} \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega} \leq \frac{1}{4C_{\gamma}} \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega}^2 + CC_{\gamma} \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega}^2.$$

Finally, using properties of  $\sigma$ , we obtain

$$|f_3| \leq C \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega} \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega} \leq \frac{1}{4C_{\gamma}} \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega}^2 + CC_{\gamma} \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega}^2.$$

Combining estimates for  $f'_i$ s and kicking back, we obtain

$$(63) \quad |z| \|\sigma^{\frac{1}{2}} G_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}} \nabla G_h\|_{\Omega}^2 \leq C \left( \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega} + \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega}^2 \right) \leq C \left( \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega}^2 + 1 \right).$$

Now we proceed as in the proof of Theorem 12. To estimate  $\|\sigma^{-\frac{1}{2}} G_h\|_{\Omega}$  we use Lemma 5, with  $\alpha = \beta = -\frac{1}{2}$  and  $p = 3$ , to obtain

$$(64) \quad \|\sigma^{-\frac{1}{2}} G_h\|_{\Omega} \leq C \|G_h\|_{L^3(\Omega)}^{\frac{1}{2}} \|\nabla G_h\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}.$$

Using the inverse and the triangle inequalities, we obtain

$$\|G_h\|_{L^3(\Omega)} \leq \|G\|_{L^3(\Omega)} + Ch^{-\frac{1}{2}}\|G - G_h\|_{\Omega} + Ch^{-\frac{1}{2}}\|G - I_hG\|_{\Omega} + \|G - I_hG\|_{L^3(\Omega)}.$$

Using the approximation result (10),  $L^2$ -norm error estimates,  $H^2$ -regularity, and the properties of  $\tilde{\delta}$  function, we have

$$(65) \quad h^{-\frac{1}{2}}\|G - G_h\|_{\Omega} + h^{-\frac{1}{2}}\|G - I_hG\|_{\Omega} + \|G - I_hG\|_{L^3(\Omega)} \leq Ch^{\frac{3}{2}}\|G\|_{H^2(\Omega)} \leq Ch^{\frac{3}{2}}\|\tilde{\delta}\|_{\Omega} \leq C.$$

The  $L^2$ -norm error estimates and  $H^2$  regularity is shown, e.g., in [17, Theorem 3.1] for convex polygonal domains, but the proof there works for convex polyhedral domains as well. Next, we will show

$$(66) \quad \|G\|_{L^3(\Omega)} \leq C|\ln h|^{\frac{1}{3}}.$$

To establish that, we use

$$\begin{aligned} \|G\|_{L^3(\Omega)} &= \|(\bar{z} + \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} = \|(|z| - \Delta)(\bar{z} + \Delta)^{-1}(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} \\ &\leq (1 + 2|z|)\|(\bar{z} + \Delta)^{-1}\|_{L^3 \rightarrow L^3}\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} \leq C\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)}, \end{aligned}$$

where we used continuous resolvent estimate (50) with  $p = 3$  and

$$(|z| - \Delta)(\bar{z} + \Delta)^{-1} = (\bar{z} + |z| - \bar{z} - \Delta)(\bar{z} + \Delta)^{-1} = (\bar{z} + |z|)(\bar{z} + \Delta)^{-1} - \text{Id}.$$

Now, to estimate  $\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3}$  we use the Green's function representation

$$\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} = \|(\Gamma_s(\cdot, y), \tilde{\delta}(y))_{\Omega}\|_{L^3(\Omega)},$$

and exactly as in the proof of Theorem 12 using Lemma 13, we obtain

$$\|(|z| - \Delta)^{-1}\tilde{\delta}\|_{L^3(\Omega)} \leq C|\ln h|^{\frac{1}{3}}.$$

Combining the above estimates, we have established

$$(67) \quad \|G_h\|_{L^3(\Omega)} \leq C|\ln h|^{\frac{1}{3}}.$$

Using the same analysis as in Theorem 15, we get

$$(68) \quad \|\nabla G_h\|_{L^{\frac{3}{2}}(\Omega)} \leq C|\ln h|^{\frac{1}{6}}\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}.$$

Thus, using (63) and the above estimates, we have

$$\begin{aligned} |z|\|\sigma^{\frac{1}{2}}G_h\|_{\Omega}^2 + \|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}^2 &\leq C\left(\|G_h\|_{L^3(\Omega)}\|\nabla G_h\|_{L^{\frac{3}{2}}(\Omega)} + 1\right) \\ &\leq C\left(|\ln h|^{\frac{1}{2}}\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega} + 1\right) \\ &\leq C|\ln h| + \frac{1}{2}\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}^2. \end{aligned}$$

Kicking back  $\|\sigma^{\frac{1}{2}}\nabla G_h\|_{\Omega}^2$ , we finally obtain

$$\|\sigma^{\frac{1}{2}}G_h\|_{\Omega}^2 \leq C|\ln h||z|^{-1},$$

which shows (62) and hence the theorem for  $\lambda = 0$ .

To show the result with  $\lambda \in [0, \lambda_0]$ , we use the argument similar to [3, Lemma 6.1]. We decompose  $\mathbb{C} \setminus \Sigma_{\lambda, \gamma}$  as  $\mathbb{C} \setminus \Sigma_{\lambda, \gamma} = D_1 \cup D_2$ , where

$$D_1 = \left\{ z \in \mathbb{C} \setminus \Sigma_{\lambda, \gamma} \mid \frac{\gamma}{2} \leq |\arg z| \leq \pi \text{ and } |z| \geq \frac{\lambda}{2} \right\}$$

and

$$D_2 = \left\{ z \in \mathbb{C} \setminus \Sigma_{\lambda, \gamma} \mid |\arg z| \leq \frac{\gamma}{2} \right\} \cup \left\{ z \in \mathbb{C} \mid |z| \leq \frac{\lambda}{2} \right\}.$$

Since  $|z - \lambda| \leq 3|z|$  for  $z \in D_1$ , the theorem in this case follows from (56). Thus, it remains to establish the bound for  $z \in D_2$ . Using  $\Delta_h(z + \Delta_h)^{-1} = \text{Id} - z(z + \Delta_h)^{-1}$  and Lemma 11, we have

$$(69) \quad \|(z + \Delta_h)^{-1} \chi\|_{L^\infty(\Omega)} \leq C \|\Delta_h(z + \Delta_h)^{-1} \chi\|_\Omega \leq C (1 + |z| \|(z + \Delta_h)^{-1}\|_{L^2 \rightarrow L^2}) \|\chi\|_\Omega.$$

For  $z \in D_2$  using Lemma 14, we obtain

$$\|(z + \Delta_h)^{-1}\|_{L^2 \rightarrow L^2} \leq \frac{C}{|z - \lambda|}.$$

Inserting this into (69) we obtain

$$\|(z + \Delta_h)^{-1} \chi\|_{L^\infty(\Omega)} \leq C \left( 1 + \frac{|z|}{|z - \lambda|} \right) \|\chi\|_\Omega \leq C (|z| + |z - \lambda|) \frac{\|\chi\|_{L^\infty(\Omega)}}{|z - \lambda|}.$$

To complete the proof of Theorem 15, we notice that the term  $|z| + |z - \lambda|$  is uniformly bounded on  $D_2$  by a constant depending only on  $\gamma$  and  $\lambda_0$ .  $\square$

**Appendix.** Proof of Lemma 1. Adding and subtracting  $I_h(\varphi^2 P_h v)$  and using the Cauchy–Schwarz inequality, we have

$$(70) \quad \begin{aligned} \|\varphi P_h v\|_\Omega^2 &= (P_h v, \varphi^2 P_h v)_\Omega = (P_h v, \varphi^2 P_h v - I_h(\varphi^2 P_h v))_\Omega + (v, I_h(\varphi^2 P_h v))_\Omega \\ &\leq \|\varphi P_h v\|_\Omega \|\varphi^{-1} (\varphi^2 P_h v - I_h(\varphi^2 P_h v))\|_\Omega + \|\varphi v\|_\Omega \|\varphi^{-1} I_h(\varphi^2 P_h v)\|_\Omega. \end{aligned}$$

Adapting the notation of [13], define

$$\underline{\varphi}_\tau = \min_\tau \varphi \quad \text{and} \quad \overline{\varphi}_\tau = \max_\tau \varphi$$

for each element  $\tau$ . First, we notice that  $h|\nabla \varphi| \leq \varepsilon \varphi$  implies

$$(71) \quad \overline{\varphi}_\tau \leq C \underline{\varphi}_\tau$$

for  $\varepsilon$  sufficiently small with the constant  $C$  independent of  $\tau$ . Indeed, the inequality

$$\overline{\varphi}_\tau \leq \underline{\varphi}_\tau + h \|\nabla \varphi\|_{L^\infty(\tau)}$$

and the assumption

$$h \|\nabla \varphi\|_{L^\infty(\tau)} \leq \varepsilon \overline{\varphi}_\tau$$

imply (71) for  $\varepsilon$  sufficiently small.

Thus, using (71), the triangle inequality, and the property that for the nodal Lagrange interpolant holds

$$I_h(\varphi^2 P_h v) = I_h(I_h(\varphi^2) P_h v),$$



we have

$$\begin{aligned} \|\varphi^{-1}(\varphi^2 P_h v - I_h(\varphi^2 P_h v))\|_\tau &\leq \underline{\varphi}_\tau^{-1} \|\varphi^2 P_h v - I_h(\varphi^2 P_h v)\|_\tau \\ &\leq \underline{\varphi}_\tau^{-1} \|(\varphi^2 - I_h(\varphi^2)) P_h v\|_\tau \\ &\quad + \underline{\varphi}_\tau^{-1} \|I_h(\varphi^2) P_h v - I_h(I_h(\varphi^2) P_h v)\|_\tau \\ &:= J_1 + J_2. \end{aligned}$$

Using the approximation properties of  $I_h$  and the assumption that  $h|\nabla\varphi| \leq \varepsilon\varphi$ , we obtain

$$(72) \quad J_1 \leq C \underline{\varphi}_\tau^{-1} h \|\nabla(\varphi^2)\|_{L^\infty(\tau)} \|P_h v\|_\tau \leq C\varepsilon \|\varphi P_h v\|_\tau.$$

We first notice that on each element  $\partial^\gamma(P_h v) = 0$  and  $\partial^\gamma(I_h(\varphi^2)) = 0$  for all multi-indices  $\gamma \in \mathbb{N}^3$  with  $|\gamma| = k + 1$ . Thus, using the approximations theory and the Leibniz’s formula, we have

$$J_2 \leq C \underline{\varphi}_\tau^{-1} h^{k+1} |I_h(\varphi^2) P_h v|_{H^{k+1}(\tau)} \leq C \underline{\varphi}_\tau^{-1} h^{k+1} \sum_{l=1}^k |I_h(\varphi^2)|_{W_\infty^l(\tau)} |P_h v|_{H^{k+1-l}(\tau)}.$$

Using the stability of the interpolant in  $W_\infty^l$  and the inverse inequality, we obtain

$$J_2 \leq C \underline{\varphi}_\tau^{-1} \|P_h v\|_\tau \sum_{l=1}^k h^l |\varphi^2|_{W_\infty^l(\tau)}.$$

Finally, using the assumption  $h^l |\nabla^l \varphi| \leq \varepsilon\varphi$ , we obtain

$$J_2 \leq C\varepsilon \|\varphi P_h v\|_\tau.$$

Thus combining the estimates for  $J_1$  and  $J_2$  and summing over the element, we have established that

$$(73) \quad \|\varphi^{-1}(\varphi^2 P_h v - I_h(\varphi^2 P_h v))\|_\Omega \leq C\varepsilon \|\varphi P_h v\|_\Omega.$$

The above estimate by the triangle inequality also implies that

$$(74) \quad \|\varphi^{-1} I_h(\varphi^2 P_h v)\|_\Omega \leq C(\varepsilon + 1) \|\varphi P_h v\|_\Omega.$$

Inserting the estimates (73) and (74) into (70), we obtain

$$\|\varphi P_h v\|_\Omega^2 \leq C\varepsilon \|\varphi P_h v\|_\Omega^2 + C(\varepsilon + 1) \|\varphi v\|_\Omega \|\varphi P_h v\|_\Omega,$$

which for  $\varepsilon$  sufficiently small implies Lemma 1.

**Acknowledgment.** The authors would like to thank Dominik Meidner for the careful reading of the manuscript and providing valuable suggestions that helped improve the presentation of this paper.

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