

DISCRETE MAXIMAL PARABOLIC REGULARITY FOR GALERKIN FINITE ELEMENT METHODS FOR NON-AUTONOMOUS PARABOLIC PROBLEMS

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Abstract. The main goal of the paper is to establish time semidiscrete and space-time fully discrete maximal parabolic regularity for the lowest order time discontinuous Galerkin solution of linear parabolic equations with time-dependent coefficients. Such estimates have many applications. As one of the applications we establish best approximations type results with respect to the $L^p(0, T; L^2(\Omega))$ norm for $1 \leq p \leq \infty$.

Key words. parabolic problems, maximal parabolic regularity, discrete maximal parabolic regularity, finite elements, discontinuous Galerkin methods, optimal error estimates, time-dependent coefficients, non-autonomous problems

AMS subject classifications. 65N30, 65N15

1. Introduction. Let Ω be a Lipschitz domain in \mathbb{R}^d , $d \geq 1$ and $I = (0, T)$. We consider the following second order parabolic partial differential equation with time-dependent coefficients,

$$\begin{aligned} \partial_t u(t, x) + A(t, x)u(t, x) &= f(t, x), & (t, x) \in I \times \Omega, \\ u(t, x) &= 0, & (t, x) \in I \times \partial\Omega, \\ u(0, x) &= u_0(x), & x \in \Omega, \end{aligned} \quad (1.1)$$

with the right-hand side $f \in L^p(I; L^2(\Omega))$ for some $1 \leq p \leq \infty$ and $u_0 \in L^2(\Omega)$, where the time-dependent elliptic operator is given by the formal expression

$$A(t, x)u(t, x) = - \sum_{i, j=1}^d \partial_j (a_{ij}(t, x) \partial_i u(t, x)) \quad (1.2)$$

with $a_{ij}(t, x) \in L^\infty(I \times \Omega)$ for $i, j = 1, \dots, d$ satisfying $a_{ij} = a_{ji}$ and the uniform ellipticity property

$$\sum_{i, j=1}^d a_{ij}(t, x) \xi_i \xi_j \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^d \quad \text{and a.e. } (t, x) \in I \times \Omega, \quad (1.3)$$

for some constant $\alpha > 0$. We also assume that the coefficients $a_{ij}(t, x)$ are continuous in t for almost all $x \in \Omega$ and that the following condition holds:

$$|a_{ij}(t_1, x) - a_{ij}(t_2, x)| \leq \omega(|t_1 - t_2|), \quad 1 \leq i, j \leq d \quad (1.4)$$

for all $t_1, t_2 \in \bar{I}$ and almost all $x \in \Omega$, where $\omega: [0, T] \rightarrow [0, \infty)$ is a nondecreasing function such that

$$\frac{w(t)}{t^{\frac{3}{2}}} \text{ is non-increasing on } (0, T] \text{ and } \int_0^T \frac{\omega(t)}{t^{\frac{3}{2}}} dt < \infty, \quad (1.5)$$

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see [17] for a similar assumptions. This assumption is fulfilled, for example, if a_{ij} is Hölder continuous with exponent $\frac{1}{2} + \varepsilon$ in time and uniformly continuous in space.

The maximal parabolic regularity for $u_0 \equiv 0$ says that there exists a constant C such that for $f \in L^p(I; L^s(\Omega))$,

$$\|\partial_t u\|_{L^p(I; L^s(\Omega))} + \|A(\cdot)u\|_{L^p(I; L^s(\Omega))} \leq C \|f\|_{L^p(I; L^s(\Omega))}, \quad 1 < s, p < \infty. \quad (1.6)$$

For time-independent coefficients the above result is well understood [4, 6, 7, 18, 19], however for time-dependent coefficients it is still an active area of research [3, 8, 9, 17]. The maximal parabolic regularity is an important analytical tool and has a number of applications, especially to nonlinear problems and optimal control problems when sharp regularity results are required (cf., e.g., [21, 26, 27, 28, 31]).

The main goal of this paper is to establish similar maximal parabolic regularity results for time semidiscrete discontinuous Galerkin solutions as well as for fully discrete Galerkin approximations. Such results are very useful, for example, in a priori error estimates and essential in obtaining error estimates where the spatial mesh size h and the time steps k are independent of each other (cf. [29, 30]).

Previously in [32] we established the corresponding discrete maximal parabolic regularity for discontinuous Galerkin time schemes of arbitrary order for autonomous problems. The extension to non-autonomous problems is not straightforward, especially for the critical values of $s, p = 1$ or $s, p = \infty$. In this paper, we investigate the maximal parabolic regularity for $s = 2$ and arbitrary $1 \leq p \leq \infty$ for the lowest order time discontinuous Galerkin (dG(0)) methods, which can be considered as modified backward Euler (BE) method. The main difference between dG(0) and BE methods lies in the way the time-dependent coefficients and the right-hand side are approximated. In dG(0) formulation they are approximated by averages over each subinterval I_m (see the details below) while in BE methods they are evaluated at time nodes. As a result, dG(0) approximations are weakly consistent, i.e. satisfy the Galerkin orthogonality relation, see Section 2 for details.

Parabolic problems with time-dependent coefficients are important, for example for analyzing quasi-linear problems. Over the years, there have been a considerable number of publications devoted to various numerical methods for problems with time-dependent coefficients [5, 10, 12, 20, 22, 23, 24, 34, 35, 37, 38]. The publication [5] is the most relevant to our presentation since it treats discontinuous Galerkin methods. However, none of the above publications addresses the question of the discrete maximal parabolic regularity and the techniques used in those papers are not immediately applicable for establishing such results even for $p = 2$.

Time discrete maximal parabolic regularity (sometimes called well-posedness or coercivity property in the literature) have been investigated in a number of publications for various time schemes, [1, 2, 14, 15, 16, 25, 32]. However, all the above mentioned publications are dealing with the autonomous case. The extension to non-autonomous situation is not easy. We are only aware of the publication [33], where the discrete maximal parabolic regularity is established for problems with time-dependent coefficients for the backward Euler method. Although the results in [33] are similar in nature, there are some significant differences. The results in [33] require $a_{ij}(t, x) \in W_\infty^1(I \times \Omega)$, smoothness of Ω and treat only uniform time steps, but they are valid in $L^s(\Omega)$ norms for $1 < s < \infty$. Our results, on the other hand, require only a Hölder continuity of $a_{ij}(t, x)$ in t and L^∞ in space, allow Ω to be merely Lipschitz and treat variable time steps, but are valid only in $L^2(\Omega)$ norm in space. Moreover, the discrete maximal parabolic regularity in [33] is shown in $l^p(I; L^s(\Omega))$ norm for

$1 < p, s < \infty$ and since their proof requires Grönwall's inequality, the argument can not be naturally extended to the critical values of $p = 1$ and $p = \infty$ even with the expense of the logarithmic term. We establish our result by a completely different argument, including fully discrete Galerkin approximations, in $L^p(I; L^2(\Omega))$ norm for any $1 \leq p \leq \infty$. For our future applications the inclusion of the critical values of $p = 1$ and $p = \infty$ is essential for error estimates in $L^\infty(I; L^2(\Omega))$ norm. We also want to mention that we went through some technical obstacles in order to incorporate variable time steps. In the case of uniform time steps many arguments can be significantly simplified.

Our presentation is inspired by [17], where the maximal parabolic regularity was established for continuous problems for $s = 2$ and $1 < p < \infty$ with rather weak assumptions on A . Thus, in particular, we show for dG(0) method the semidiscrete solution u_k on any time level m for $u_0 = 0$ and $f \in L^\infty(I; L^2(\Omega))$ satisfies

$$\|A_{k,m}u_{k,m}\|_{L^\infty(I_m; L^2(\Omega))} + \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^2(\Omega)} \leq C \ln \frac{T}{k} \|f\|_{L^\infty(I; L^2(\Omega))}. \quad (1.7)$$

For $p = 1$ with $u_0 \in L^2(\Omega)$ and $f \in L^1(I; L^2(\Omega))$ we also obtain

$$\sum_m \left(\|A_{k,m}u_{k,m}\|_{L^1(I_m; L^2(\Omega))} + \|[u_k]_{m-1}\|_{L^2(\Omega)} \right) \leq C \ln \frac{T}{k} (\|f\|_{L^1(I; L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}), \quad (1.8)$$

where k_m is the time step on subinterval I_m and $A_{k,m}$ is the average of $A(t)$ on I_m (see section 2 for a detailed description). In contrast to the continuous estimate (1.6), the above estimates include the limiting cases of $p = \infty$ and $p = 1$, which explains the logarithmic factor in (1.7) and (1.8).

The corresponding results also hold for the fully discrete approximation u_{kh} . Thus in particular for $1 \leq p \leq \infty$ and $u_0 = 0$, we establish

$$\left[\sum_m \left(\|A_{kh,m}u_{kh,m}\|_{L^p(I_m; L^2(\Omega))}^p + k_m \left\| \frac{[u_{kh}]_{m-1}}{k_m} \right\|_{L^2(\Omega)}^p \right) \right]^{\frac{1}{p}} \leq C \ln \frac{T}{k} \|f\|_{L^p(I; L^2(\Omega))}, \quad (1.9)$$

with corresponding changes for $p = \infty$. We would like to point out that the above fully discrete result is valid on rather general meshes and does not require the mesh to be quasi-uniform or even shape regular, only admissible (no hanging nodes).

As an application of the discrete maximal parabolic regularity we show that if the coefficients $a_{ij}(t, x)$ are sufficiently regular (see Assumption 1) and Ω convex we obtain symmetric error estimate

$$\|u - u_k\|_{L^p(I; L^2(\Omega))} \leq C \ln \frac{T}{k} \|u - \pi_k u\|_{L^p(I; L^2(\Omega))}, \quad 1 \leq p < \infty,$$

where π_k is an interpolation into the space of piecewise constant in time functions defined in (4.5). For $p = \infty$ we can establish even the best approximation type result

$$\|u - u_k\|_{L^\infty(I; L^2(\Omega))} \leq C \ln \frac{T}{k} \|u - \chi\|_{L^\infty(I; L^2(\Omega))},$$

for any χ in the subspace of piecewise constant in time functions, see Theorem 4.2. The corresponding fully discrete versions are

$$\|u - u_{kh}\|_{L^p(I; L^2(\Omega))} \leq C \ln \frac{T}{k} (\|u - \pi_k u\|_{L^p(I; L^2(\Omega))} + \|u - R_h u\|_{L^p(I; L^2(\Omega))})$$

and

$$\|u - u_{kh}\|_{L^\infty(I; L^2(\Omega))} \leq C \ln \frac{T}{k} (\|u - \chi\|_{L^\infty(I; L^2(\Omega))} + \|u - R_h u\|_{L^\infty(I; L^2(\Omega))}),$$

for any χ in the subspace and $R_h(t)$ being the Ritz projection corresponding to $A(t)$. The rate of convergence depends of course on the regularity of u .

The rest of the paper is organized as follows. In section 2 we introduce the discontinuous Galerkin method and some notation. Section 3, which is the central piece of the paper, consists of several parts. First we write the dG(0) approximate solution u_k in a convenient form. Then we introduce a transform function w_k that satisfies a similar equation, but with transform operators. Then in a series of lemmas we show that the resulting operators are bounded in certain norms. Finally in sections 3.2 and 3.3 we establish semidiscrete and fully discrete maximal parabolic regularity in $L^p(I; L^2(\Omega))$ norms, respectively. We conclude our paper with section 4, where we show how the above discrete maximal parabolic regularity results can be used to establish symmetric and best approximation type error estimates for the problems on convex domains with coefficients satisfying some additional assumptions.

2. Preliminaries. First, we introduce the bilinear form $a: \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$a(t; u, v) = \int_{\Omega} \sum_{i,j=1}^d a_{ij}(t, x) \partial_i u(x) \partial_j v(x) dx. \quad (2.1)$$

From $\{a_{ij}(t, x)\}_{i,j=1}^d \subset L^\infty(I \times \Omega)$ one can see that for each $t \in I$ the bilinear form $a(t; \cdot, \cdot)$ is bounded

$$a(t; u, v) \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \quad (2.2)$$

from the uniform ellipticity assumption (1.3), it is coercive

$$a(t; u, u) \geq \alpha \|u\|_{H^1(\Omega)}^2, \quad (2.3)$$

and from (1.4) it follows that

$$|a(t_1; u, v) - a(t_2; u, v)| \leq C \omega(|t_1 - t_2|) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \quad (2.4)$$

In view of the homogeneous Dirichlet boundary conditions the H^1 norm is equivalent to the H^1 seminorm. For each $t \in \bar{I}$ this bilinear form defines an operator $A(t): H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by

$$\langle A(t)u, v \rangle = a(t, u, v) \quad \text{for all } u, v \in H_0^1(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ spaces.

To introduce the time discontinuous Galerkin discretization for the problem, we partition $I = (0, T]$ into subintervals $I_m = (t_{m-1}, t_m]$ of length $k_m = t_m - t_{m-1}$, where $0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T$. The maximal and minimal time steps are denoted by $k = \max_m k_m$ and $k_{\min} = \min_m k_m$, respectively. We impose the following conditions on the time mesh.

(i) There are constants $c, \beta > 0$ independent on k such that

$$k_{\min} \geq ck^\beta.$$

(ii) There is a constant $\kappa > 0$ independent on k such that for all $m = 1, 2, \dots, M-1$

$$\kappa^{-1} \leq \frac{k_m}{k_{m+1}} \leq \kappa.$$

(iii) It holds $k \leq \frac{1}{4}T$.

Similar assumptions are made, e.g., in [36]. The semidiscrete space X_k^0 of piecewise constant functions in time is defined by

$$X_k^0 = \{u_k \in L^2(I; H_0^1(\Omega)) : u_k|_{I_m} \in \mathcal{P}_0(I_m; H_0^1(\Omega)), m = 1, 2, \dots, M\},$$

where $\mathcal{P}_0(V)$ is the space of constant functions in time with values in a Banach space V . We will employ the notation

$$v_m^+ := \lim_{t \rightarrow 0^+} v(t_m + t), \quad v_m^- := \lim_{t \rightarrow 0^+} v(t_m - t), \quad \text{and} \quad [v]_m = v_m^+ - v_m^-,$$

if these limits exist. For a function v_k from X_k^0 we denote $v_{k,m} := v_k|_{I_m}$ resulting in

$$v_{k,m}^+ = v_{k,m+1}, \quad v_{k,m}^- = v_{k,m}, \quad \text{and} \quad [v_k]_m = v_{k,m+1} - v_{k,m},$$

for $m = 1, 2, \dots, M-1$.

Next we define the following bilinear form

$$B(u, \varphi) = \sum_{m=1}^M \langle \partial_t u, \varphi \rangle_{I_m \times \Omega} + \sum_{i,j=1}^d (a_{ij} \partial_i u, \partial_j \varphi)_{I \times \Omega} + \sum_{m=2}^M ([u]_{m-1}, \varphi_{m-1}^+)_{\Omega} + (u_0^+, \varphi_0^+)_{\Omega}, \quad (2.5)$$

where $(\cdot, \cdot)_{\Omega}$ and $(\cdot, \cdot)_{I_m \times \Omega}$ are the usual L^2 space and space-time inner-products, $\langle \cdot, \cdot \rangle_{I_m \times \Omega}$ is the duality product between $L^2(I_m; H^{-1}(\Omega))$ and $L^2(I_m; H_0^1(\Omega))$. Rearranging the terms in (2.5), we obtain an equivalent (dual) expression for B ,

$$B(u, \varphi) = - \sum_{m=1}^M \langle u, \partial_t \varphi \rangle_{I_m \times \Omega} + \sum_{i,j=1}^d (a_{ij} \partial_i u, \partial_j \varphi)_{I \times \Omega} - \sum_{m=1}^{M-1} (u_m^-, [\varphi]_m)_{\Omega} + (u_M^-, \varphi_M^-)_{\Omega}. \quad (2.6)$$

We note, that for $u_k, \varphi_k \in X_k^0$ the bilinear form (2.5) simplifies to

$$B(u_k, \varphi_k) = \sum_{i,j=1}^d (a_{ij} \partial_i u_k, \partial_j \varphi_k)_{I \times \Omega} + \sum_{m=2}^M ([u_k]_{m-1}, \varphi_{k,m})_{\Omega} + (u_{k,1}, \varphi_{k,1})_{\Omega}$$

and

$$B(u_k, \varphi_k) = \sum_{i,j=1}^d (a_{ij} \partial_i u_k, \partial_j \varphi_k)_{I \times \Omega} - \sum_{m=1}^{M-1} (u_{k,m}^-, [\varphi_k]_m)_{\Omega} + (u_{k,M}^-, \varphi_{k,M}^-)_{\Omega}.$$

The dG(0) semidiscrete (in time) approximation $u_k \in X_k^0$ of (1.1) is defined as

$$B(u_k, \varphi_k) = (f, \varphi_k)_{I \times \Omega} + (u_0, \varphi_{k,1})_{\Omega} \quad \text{for all } \varphi_k \in X_k^0, \quad (2.7)$$

and by the construction we have the following Galerkin orthogonality

$$B(u - u_k, \varphi_k) = 0 \quad \text{for all } \varphi_k \in X_k^0. \quad (2.8)$$

To rewrite the dG(0) method as a time-stepping scheme we introduce the following notation. We define $f_k \in X_k^0$ by

$$f_{k,m} = \frac{1}{k_m} \int_{I_m} f(t) dt, \quad m = 1, 2, \dots, M \quad (2.9)$$

and $A_{k,m}: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by

$$A_{k,m} = \frac{1}{k_m} \int_{I_m} A(t) dt, \quad m = 1, 2, \dots, M. \quad (2.10)$$

Thus, the dG(0) solution u_k satisfies

$$\begin{aligned} u_{k,1} + k_1 A_{k,1} u_{k,1} &= u_0 + k_1 f_{k,1}, \\ u_{k,m} + k_m A_{k,m} u_{k,m} &= u_{k,m-1} + k_m f_{k,m}, \quad m = 2, 3, \dots, M. \end{aligned} \quad (2.11)$$

To use results known for the autonomous problems we rewrite this formula for some fixed $2 \leq m \leq M$ as

$$\begin{aligned} u_{k,1} + k_1 A_{k,m} u_{k,1} &= u_0 + k_1 f_{k,1} + k_1 (A_{k,m} - A_{k_1}) u_{k,1}, \\ u_{k,l} + k_l A_{k,m} u_{k,l} &= u_{k,l-1} + k_l f_{k,l} + k_l (A_{k,m} - A_{k_l}) u_{k,l}, \quad l = 2, 3, \dots, m. \end{aligned}$$

Then using the representation formula for the dG(0) solution in the autonomous case, (cf. the proof of Theorem 2.1 in [32]), we obtain the following representation

$$u_{k,m} = \sum_{l=1}^{m-1} k_l R_{m,l} (A_{k,m} - A_{k,l}) u_{k,l} + \sum_{l=1}^m k_l R_{m,l} f_{k,l} + R_{m,1} u_0, \quad m = 1, 2, \dots, M, \quad (2.12)$$

where

$$R_{m,l} = \prod_{j=1}^{m-l+1} r(k_{m+1-j} A_{k,m}) \quad \text{and} \quad r(z) = \frac{1}{1+z}. \quad (2.13)$$

Throughout the paper we use a convention $\sum_{l=1}^0 = 0$. Next we define three operators $Q: X_k^0 \rightarrow X_k^0$, $L: X_k^0 \rightarrow X_k^0$, and $D: L^2(\Omega) \rightarrow X_k^0$ by

$$(Qg_k)_m = \sum_{l=1}^{m-1} k_l A_{k,m} R_{m,l} (A_{k,m} - A_{k,l}) A_{k,l}^{-1} g_{k,l}, \quad \text{for } g_k \in X_k^0, \quad (2.14)$$

$$(Lf_k)_m = \sum_{l=1}^m k_l A_{k,m} R_{m,l} f_{k,l}, \quad (2.15)$$

and

$$(Du_0)_m = A_{k,m} R_{m,1} u_0. \quad (2.16)$$

Thus, for $v_k \in X_k^0$ defined by

$$v_{k,l} = A_{k,l} u_{k,l} \quad \text{for } l = 1, 2, \dots, M,$$

we have

$$v_k = Qv_k + Lf_k + Du_0. \quad (2.17)$$

3. Maximal parabolic regularity for time discretization.

3.1. Estimate for the transformed operator. Let $\mu > 0$ be a sufficiently large number to be chosen later. Define $w_{k,m}$ by

$$w_{k,m} = \prod_{l=1}^m (1 + \mu k_l)^{-1} u_{k,m} \quad m = 1, 2, \dots, M.$$

Thus using (2.11) we obtain

$$(1 + \mu k_1)w_{k,1} + k_1(1 + \mu k_1)A_{k,1}w_{k,1} = u_0 + k_1 f_{k,1},$$

$$\prod_{l=1}^m (1 + \mu k_l)w_{k,m} + k_m \prod_{l=1}^m (1 + \mu k_l)A_{k,m}w_{k,m} = \prod_{l=1}^{m-1} (1 + \mu k_l)w_{k,m-1} + k_m f_{k,m},$$

for $m = 2, \dots, M$. Dividing both sides of the last equation by $\prod_{l=1}^{m-1} (1 + \mu k_l)$, we obtain

$$(1 + k_m \mu)w_{k,m} + k_m(1 + k_m \mu)A_{k,m}w_{k,m} = w_{k,m-1} + \prod_{l=1}^{m-1} (1 + \mu k_l)^{-1} k_m f_{k,m}.$$

Hence, we can rewrite (2.11) as

$$w_{k,1} + k_1 \tilde{A}_{k,1} w_{k,1} = u_0 + k_1 \tilde{f}_{k,1},$$

$$w_{k,m} + k_m \tilde{A}_{k,m} w_{k,m} = w_{k,m-1} + k_m \tilde{f}_{k,m}, \quad m = 2, \dots, M,$$

where

$$\tilde{A}_{k,m} = (1 + k_m \mu)A_{k,m} + \mu \text{Id}, \quad \tilde{f}_{k,m} = \prod_{l=1}^{m-1} (1 + \mu k_l)^{-1} f_{k,m}, \quad m = 1, 2, \dots, M. \quad (3.1)$$

Here we use a convention $\prod_{l=1}^0 = 1$. Similarly to (2.17), for $\tilde{v}_k \in X_k^0$ defined by

$$\tilde{v}_{k,l} = \tilde{A}_{k,l} w_{k,l} \quad \text{for } l = 1, \dots, M,$$

we have

$$\tilde{v}_k = \tilde{Q} \tilde{v}_k + \tilde{L} \tilde{f}_k + \tilde{D} u_0, \quad (3.2)$$

where similarly to the definitions of Q , L , and D above,

$$\tilde{R}_{m,l} = \prod_{j=1}^{m-l+1} r(k_{m+1-j} \tilde{A}_{k,m}), \quad (3.3)$$

$$(\tilde{Q} g_k)_m = \sum_{l=1}^{m-1} k_l \tilde{A}_{k,m} \tilde{R}_{m,l} (\tilde{A}_{k,m} - \tilde{A}_{k,l}) \tilde{A}_{k,l}^{-1} g_{k,l} \quad (3.4)$$

and

$$(\tilde{L} \tilde{f}_k)_m = \sum_{l=1}^m k_l \tilde{A}_{k,m} \tilde{R}_{m,l} \tilde{f}_{k,l}, \quad (\tilde{D} u_0)_m = \tilde{A}_{k,m} \tilde{R}_{m,1} u_0. \quad (3.5)$$

Using the ellipticity of $A_{k,m}$ we obtain the following resolvent estimate for $\tilde{A}_{k,m}$. For a given $\gamma \in (0, \pi/2)$ we define

$$\Sigma_\gamma = \{z \in \mathbb{C} : |\arg(z)| \leq \gamma\}. \quad (3.6)$$

Moreover we introduce the complex spaces $\mathbb{H} = L^2(\Omega) + iL^2(\Omega)$ and $\mathbb{V} = H_0^1(\Omega) + iH_0^1(\Omega)$.

LEMMA 3.1. *For any $\gamma > 0$, there exists a constant C independent of k and μ such that*

$$\|(z - \tilde{A}_{k,m})^{-1}v\|_{L^2(\Omega)} \leq \frac{C}{|z| + \mu} \|v\|_{L^2(\Omega)}, \quad \forall z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \forall v \in \mathbb{H}$$

and

$$\|\tilde{A}_{k,m}(z - \tilde{A}_{k,m})^{-1}v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}, \quad \forall z \in \mathbb{C} \setminus \Sigma_\gamma, \quad \forall v \in \mathbb{H}.$$

Proof. For an arbitrary $v \in \mathbb{H}$ we define

$$g = -(z - \tilde{A}_{k,m})^{-1}v \in \mathbb{V},$$

or equivalently

$$-z(g, \varphi) + (1 + k_m\mu)(A_{k,m}g, \varphi) + \mu(g, \varphi) = (v, \varphi), \quad \forall \varphi \in \mathbb{V}, \quad (3.7)$$

which existence and uniqueness follow from the Fredholm alternative. In this proof (\cdot, \cdot) denotes the Hermitian inner product, i.e. $(v, \varphi) = \int_\Omega v \bar{\varphi} \, dx$.

Taking $\varphi = g$ we obtain

$$-z\|g\|_{L^2(\Omega)}^2 + (1 + k_m\mu)(A_{k,m}g, g) + \mu\|g\|_{L^2(\Omega)}^2 = (v, g). \quad (3.8)$$

Since $\gamma \leq |\arg z| \leq \pi$ and $\alpha\|g\|_{H^1(\Omega)}^2 \leq (A_{k,m}g, g)$ and is real, this equation is of the form

$$e^{i\delta}a + b = c, \quad \text{with } a, b > 0, \quad \gamma \leq |\delta| \leq \pi,$$

by multiplying it by $e^{-\frac{i\delta}{2}}$ and taking real parts, we have

$$a + b \leq \left(\cos\left(\frac{\delta}{2}\right)\right)^{-1} |(v, g)| \leq \left(\sin\left(\frac{\gamma}{2}\right)\right)^{-1} |(v, g)| = C_\gamma |(v, g)|.$$

From (3.8) we therefore conclude

$$(|z| + \mu)\|g\|_{L^2(\Omega)}^2 + \alpha(1 + k_m\mu)\|g\|_{H^1(\Omega)}^2 \leq C_\gamma\|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \quad \text{for } z \in \mathbb{C} \setminus \Sigma_\gamma.$$

Thus, we have

$$\|g\|_{L^2(\Omega)} \leq \frac{C_\gamma}{|z| + \mu} \|v\|_{L^2(\Omega)},$$

which establishes the first result. The second result follows from the identity

$$\tilde{A}_{k,m}(z - \tilde{A}_{k,m})^{-1} = -\text{Id} + z(z - \tilde{A}_{k,m})^{-1}.$$

□

LEMMA 3.2. *There exists a constant C independent of k and μ such that*

$$\|(\tilde{A}_{k,m})^{-1}v\|_{L^2(\Omega)} \leq \frac{1}{\mu}\|v\|_{L^2(\Omega)}$$

and

$$\|(\tilde{A}_{k,m})^{-1}v\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\mu}(1+k_m\mu)^{\frac{1}{2}}}\|v\|_{L^2(\Omega)}.$$

Proof. For an arbitrary $v \in L^2(\Omega)$, we define

$$g = (\tilde{A}_{k,m})^{-1}v,$$

or equivalently

$$(\tilde{A}_{k,m}g, \varphi) = (1+k_m\mu)(A_{k,m}g, \varphi) + \mu(g, \varphi) = (v, \varphi), \quad \forall \varphi \in H_0^1(\Omega). \quad (3.9)$$

Taking $\varphi = g$ and using the coercivity (2.3), we obtain

$$\alpha(1+k_m\mu)\|g\|_{H^1(\Omega)}^2 + \mu\|g\|_{L^2(\Omega)}^2 \leq \|v\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}. \quad (3.10)$$

From the estimate above, we immediately conclude that

$$\|g\|_{L^2(\Omega)} \leq \frac{1}{\mu}\|v\|_{L^2(\Omega)} \quad (3.11)$$

and using (3.11), we also obtain

$$\alpha(1+k_m\mu)\|g\|_{H^1(\Omega)}^2 \leq \|v\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)} \leq \frac{1}{\mu}\|v\|_{L^2(\Omega)}^2,$$

from which the second estimate of the lemma follows. □

We will also require the following result that estimates the difference $\tilde{A}_{k,m} - \tilde{A}_{k,l}$.

LEMMA 3.3. *There exists a constant C independent of k and μ such that for $m \geq l$*

$$\|(\tilde{A}_{k,m} - \tilde{A}_{k,l})v\|_{H^{-1}(\Omega)} \leq C((1+\mu \min\{k_l, k_m\})\omega(t_m - t_{l-1}) + \mu|k_m - k_l|)\|v\|_{H^1(\Omega)}.$$

Proof. By duality we have

$$\|(\tilde{A}_{k,m} - \tilde{A}_{k,l})v\|_{H^{-1}(\Omega)} = \sup_{\substack{w \in H_0^1(\Omega), \\ \|w\|_{H^1(\Omega)} \leq 1}} ((\tilde{A}_{k,m} - \tilde{A}_{k,l})v, w)_\Omega.$$

For each such w , we have

$$((\tilde{A}_{k,m} - \tilde{A}_{k,l})v, w)_\Omega = \mu((k_m A_{k,m} - k_l A_{k,l})v, w)_\Omega + ((A_{k,m} - A_{k,l})v, w)_\Omega = J_1 + J_2.$$

Using the definitions of $A_{k,m}$ and $A_{k,l}$, changing variables, and using (1.4) and that ω is nondecreasing, we have we have

$$\begin{aligned} J_2 &= \left(\left(\frac{1}{k_m} \int_{t_{m-1}}^{t_m} A(t)dt - \frac{1}{k_l} \int_{t_{l-1}}^{t_l} A(t)dt \right) v, w \right)_\Omega \\ &= \int_0^1 \left((A(sk_m + t_{m-1}) - A(sk_l + t_{l-1}))v, w \right)_\Omega ds \\ &\leq \int_0^1 \omega(|sk_m + t_{m-1} - sk_l - t_{l-1}|) \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} ds \\ &\leq \omega(t_m - t_{l-1}) \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}. \end{aligned} \quad (3.12)$$

To estimate J_1 we use

$$k_m A_{k,m} - k_l A_{k,l} = k_m (A_{k,m} - A_{k,l}) + (k_m - k_l) A_{k,l}$$

or

$$k_m A_{k,m} - k_l A_{k,l} = k_l (A_{k,m} - A_{k,l}) + (k_m - k_l) A_{k,m}.$$

Then using (3.12), we obtain

$$J_1 \leq C\mu (\min\{k_l, k_m\} \omega(t_m - t_{l-1}) \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} + |k_m - k_l| \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}).$$

Combining the estimates for J_1 and J_2 , we obtain the lemma. \square

LEMMA 3.4. *There exists a constant C independent of k and μ such that*

$$\|\tilde{A}_{k,m} \tilde{R}_{m,l} v\|_{L^2(\Omega)} \leq \frac{C}{t_m - t_{l-1}} \|v\|_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

Moreover, for $m - l \geq 1$ there holds

$$\|\tilde{A}_{k,m}^2 \tilde{R}_{m,l} v\|_{L^2(\Omega)} \leq \frac{C}{(t_m - t_{l-1})^2} \|v\|_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

Proof. First we observe that each term $\tilde{R}_{m,l} v$ can be thought of as $m - l + 1$ time steps of dG(0) method of the homogeneous problem

$$\partial_t u + \tilde{A}_{k,m} u = 0$$

with the initial condition $u(t_{l-1}) = v$. Then, using Lemma 3.1 the first estimate follows from [11], (cf. also [32, Theorem 1]). To prove the second estimate we use a representation

$$\tilde{A}_{k,m}^2 \tilde{R}_{m,l} v = g(\tilde{A}_{k,m}) v$$

with the function

$$g(\lambda) = \lambda^2 \prod_{j=l}^m r(k_j \lambda).$$

Using the fact that the spectrum $\sigma(\tilde{A}_{k,m})$ of $\tilde{A}_{k,m}$ is real and positive we obtain by the Parseval's identity (cf. [39, Chap. 7])

$$\|\tilde{A}_{k,m}^2 \tilde{R}_{m,l} v\|_{L^2(\Omega)} \leq \sup_{\lambda \in \sigma(\tilde{A}_{k,m})} |g(\lambda)| \|v\|_{L^2(\Omega)}.$$

To estimate $|g(\lambda)|$ we proceed similar to the proof of Theorem 5.1 in [11] and observe

$$\prod_{j=l}^m (1 + k_j \lambda) \geq 1 + \lambda \sum_{j=l}^m k_j + \frac{\lambda^2}{2} \left(\sum_{\substack{i,j=l \\ i \neq j}}^m k_i k_j \right) = 1 + \lambda(t_m - t_{l-1}) + \frac{\lambda^2}{2} \left(\sum_{\substack{i,j=l \\ i \neq j}}^m k_i k_j \right).$$

Let $k_{\max} = \max_{l \leq j \leq m} k_j$ and first consider the case $k_{\max} < (t_m - t_{l-1})/2$. We have

$$(t_m - t_{l-1})^2 = \left(\sum_{j=l}^m k_j \right)^2 = \sum_{j=l}^m k_j^2 + \sum_{\substack{i,j=l \\ i \neq j}}^m k_i k_j \leq k_{\max} (t_m - t_{l-1}) + \sum_{\substack{i,j=l \\ i \neq j}}^m k_i k_j,$$

and with the assumption $k_{\max} < (t_m - t_{l-1})/2$,

$$\sum_{\substack{i,j=l \\ i \neq j}}^m k_i k_j \geq \frac{(t_m - t_{l-1})^2}{2}.$$

This results in

$$\prod_{j=l}^m (1 + k_j \lambda) \geq 1 + \lambda(t_m - t_{l-1}) + \frac{\lambda^2}{4}(t_m - t_{l-1})^2$$

and therefore we have

$$|g(\lambda)| \leq \frac{\lambda^2}{1 + \lambda(t_m - t_{l-1}) + \frac{\lambda^2}{4}(t_m - t_{l-1})^2} \leq \frac{4}{(t_m - t_{l-1})^2},$$

which proves the assertion in this case. In the case $k_{\max} \geq (t_m - t_{l-1})/2$ let $l \leq m_0 \leq m$ be such that $k_{m_0} = k_{\max}$. Due to $m - l \geq 1$ we can choose m'_0 as either $m_0 - 1$ or $m_0 + 1$ such that $l \leq m'_0 \leq m$. Then we obtain

$$\begin{aligned} \sup_{\lambda \in \sigma(\tilde{A}_{k,m})} |\lambda^2 \prod_{j=l}^m r(k_j \lambda)| &\leq \sup_{\lambda \in \sigma(\tilde{A}_{k,m})} \left| \frac{\lambda^2}{(1 + k_{m_0} \lambda)(1 + k_{m'_0} \lambda)} \right| \\ &\leq \frac{1}{k_{m_0} k_{m'_0}} \leq \frac{C}{k_{\max}^2} \leq \frac{C}{(t_m - t_{l-1})^2}, \end{aligned}$$

where we have used our assumption (ii) on the time steps. This completes the proof for this case. \square

LEMMA 3.5. *There exists a constant C independent of k and μ such that for $m - l \geq 1$,*

$$\|\tilde{A}_{k,m} \tilde{R}_{m,l} v\|_{H^1(\Omega)} \leq \frac{C}{(t_m - t_{l-1})^{\frac{3}{2}} (1 + \mu k_m)^{\frac{1}{2}}} \|v\|_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

Proof. By the coercivity of the operator A for any $w \in H_0^1(\Omega)$, we have

$$(\tilde{A}_{k,m} w, w) \geq (1 + \mu k_m)(A_{k,m} w, w) \geq (1 + \mu k_m) \alpha \|w\|_{H^1(\Omega)}^2.$$

Thus, with $w = \tilde{A}_{k,m} \tilde{R}_{m,l} v$, we have by the previous lemma

$$\begin{aligned} (1 + \mu k_m) \alpha \|\tilde{A}_{k,m} \tilde{R}_{m,l} v\|_{H^1(\Omega)}^2 &\leq (\tilde{A}_{k,m} \tilde{A}_{k,m} \tilde{R}_{m,l} v, \tilde{A}_{k,m} \tilde{R}_{m,l} v) \\ &\leq \|\tilde{A}_{k,m}^2 \tilde{R}_{m,l} v\|_{L^2(\Omega)} \|\tilde{A}_{k,m} \tilde{R}_{m,l} v\|_{L^2(\Omega)} \\ &\leq \frac{C}{(t_m - t_{l-1})^3} \|v\|_{L^2(\Omega)}. \end{aligned}$$

This completes the proof. \square

LEMMA 3.6. *There exists a constant C independent of k and μ such that for $m - l \geq 1$*

$$\|\tilde{A}_{k,m} \tilde{R}_{m,l} v\|_{L^2(\Omega)} \leq \frac{C}{(t_m - t_{l-1})^{\frac{3}{2}} (1 + \mu k_m)^{\frac{1}{2}}} \|v\|_{H^{-1}(\Omega)}, \quad \forall v \in H^{-1}(\Omega).$$

Proof. By duality

$$\|\tilde{A}_{k,m}\tilde{R}_{m,l}v\|_{L^2(\Omega)} = \sup_{\substack{w \in L^2(\Omega), \\ \|w\|_{L^2(\Omega)} \leq 1}} (\tilde{A}_{k,m}\tilde{R}_{m,l}v, w)_\Omega.$$

Since A is a symmetric operator, we have $\tilde{A}_{k,m} = \tilde{A}_{k,m}^*$ and as a result $\tilde{R}_{m,l} = \tilde{R}_{m,l}^*$. Moreover $\tilde{A}_{k,m}$ and $\tilde{R}_{m,l}$ commute. Thus,

$$(\tilde{A}_{k,m}\tilde{R}_{m,l}v, w)_\Omega = (v, \tilde{A}_{k,m}^*\tilde{R}_{m,l}^*w)_\Omega \leq \|v\|_{H^{-1}(\Omega)}\|\tilde{A}_{k,m}^*\tilde{R}_{m,l}^*w\|_{H^1(\Omega)}.$$

Since $\tilde{A}_{k,m}^* = \tilde{A}_{k,m}$, by Lemma 3.5, we obtain

$$\|\tilde{A}_{k,m}^*\tilde{R}_{m,l}^*w\|_{H^1(\Omega)} \leq \frac{C}{(t_m - t_{l-1})^{\frac{3}{2}}(1 + \mu k_m)^{\frac{1}{2}}} \|w\|_{L^2(\Omega)},$$

which establishes the lemma. \square

Combining the above lemmas we obtain the following result.

LEMMA 3.7. *There exist constants C_1 and C_2 independent of μ and k such that for any $g_k \in X_k^0$*

$$\|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \max_{1 \leq j \leq m} \|g_{k,j}\|_{L^2(\Omega)} \left(\frac{C_1}{\sqrt{\mu}} + C_2\sqrt{\mu} \sum_{l=1}^{m-1} \frac{k_l|k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} \right), \quad m = 1, \dots, M$$

and

$$\sum_{m=1}^M k_m \|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \left(\sum_{l=1}^M k_l \|g_{k,l}\|_{L^2(\Omega)} \right) \left(\frac{C_1}{\sqrt{\mu}} + C_2\sqrt{\mu} \max_{1 \leq l \leq M} \sum_{m=l+1}^M \frac{k_m|k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} \right),$$

where \tilde{Q} is the operator defined in (3.4).

Proof. Using that

$$(\tilde{Q}g_k)_m = \sum_{l=1}^{m-1} k_l \tilde{A}_{k,m} \tilde{R}_{m,l} (\tilde{A}_{k,m} - \tilde{A}_{k,l}) \tilde{A}_{k,l}^{-1} g_{k,l},$$

we have

$$\|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \sum_{l=1}^{m-1} k_l \|\tilde{A}_{k,m} \tilde{R}_{m,l}\|_{H^{-1} \rightarrow L^2} \|\tilde{A}_{k,m} - \tilde{A}_{k,l}\|_{H^1 \rightarrow H^{-1}} \|(\tilde{A}_{k,l})^{-1}\|_{L^2 \rightarrow H^1} \|g_{k,l}\|_{L^2(\Omega)}.$$

Combining estimates from Lemma 3.2, Lemma 3.3 and Lemma 3.6, and using that

$$\frac{(1 + \mu \min\{k_l, k_m\})}{(1 + \mu k_l)^{\frac{1}{2}}(1 + \mu k_m)^{\frac{1}{2}}} \leq 1,$$

for any $m = 1, 2, \dots, M$ we have

$$\|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq C \sum_{l=1}^{m-1} \left(\frac{k_l \omega(t_m - t_{l-1})}{\sqrt{\mu} (t_m - t_{l-1})^{\frac{3}{2}}} + \sqrt{\mu} k_l \frac{|k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} \right) \|g_{k,l}\|_{L^2(\Omega)}. \quad (3.13)$$

From the condition (1.5) and properties of the Riemann sums, we obtain

$$\sum_{l=1}^{m-1} k_l \frac{\omega(t_m - t_{l-1})}{(t_m - t_{l-1})^{\frac{3}{2}}} \leq \int_0^{t_m} \frac{\omega(t_m - s)}{(t_m - s)^{\frac{3}{2}}} ds \leq C,$$

and taking the maximum over l of $\|g_{k,l}\|_{L^2(\Omega)}$, we obtain the first estimate of the lemma.

From (3.13) we also obtain

$$\sum_{m=1}^M k_m \|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq C \sum_{m=1}^M k_m \sum_{l=1}^{m-1} \left(\frac{k_l \omega(t_m - t_{l-1})}{\sqrt{\mu} (t_m - t_{l-1})^{\frac{3}{2}}} + \frac{\sqrt{\mu} k_l |k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} \right) \|g_{k,l}\|_{L^2(\Omega)}.$$

Changing the order of summation we have

$$\sum_{m=1}^M k_m \|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \sum_{l=1}^{M-1} k_l \|g_{k,l}\|_{L^2(\Omega)} \sum_{m=l+1}^M \left(\frac{k_m \omega(t_m - t_{l-1})}{\sqrt{\mu} (t_m - t_{l-1})^{\frac{3}{2}}} + \frac{\sqrt{\mu} k_m |k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} \right).$$

Again similar to the above, by (1.5) we have,

$$\sum_{m=l+1}^M k_m \frac{\omega(t_m - t_{l-1})}{(t_m - t_{l-1})^{\frac{3}{2}}} \leq \int_{t_{l-1}}^T \frac{w(s - t_{l-1})}{(s - t_{l-1})^{\frac{3}{2}}} ds \leq C.$$

Taking maximum over l in the sum $\sum_{m=l+1}^M \frac{k_m |k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}}$ completes the proof. \square

PROPOSITION 3.8. *There exists $\mu > 0$ sufficiently large and $\delta_0 > 0$ such that for $k - k_{\min} \leq \delta_0$ the following estimates hold for all $g_k \in X_k^0$,*

$$\|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \frac{3}{4} \max_{1 \leq l \leq m} \|g_{k,l}\|_{L^2(\Omega)}, \quad m = 1, 2, \dots, M, \quad (3.14)$$

and

$$\sum_{m=1}^M k_m \|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \frac{3}{4} \sum_{l=1}^M k_l \|g_{k,l}\|_{L^2(\Omega)}. \quad (3.15)$$

Proof. Using the first estimate from Lemma 3.7 and choosing $\mu = 4C_1^2$ we obtain

$$\|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \max_{1 \leq j \leq m} \|g_{k,j}\|_{L^2(\Omega)} \left(\frac{1}{2} + C_2 \sqrt{\mu} \sum_{l=1}^{m-1} \frac{k_l |k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} \right).$$

Using $|k_m - k_l| \leq t_m - t_{l-1}$ we get for some $0 < \varepsilon < 1$

$$\begin{aligned} C_2 \sqrt{\mu} \sum_{l=1}^{m-1} \frac{k_l |k_m - k_l|}{(t_m - t_{l-1})^{\frac{3}{2}}} &\leq 2C_1 C_2 \sum_{l=1}^{m-1} \frac{k_l |k_m - k_l|^{\frac{1}{2}-\varepsilon}}{(t_m - t_{l-1})^{1-\varepsilon}} \\ &\leq 2C_1 C_2 (k - k_{\min})^{\frac{1}{2}-\varepsilon} \sum_{l=1}^{m-1} \frac{k_l}{(t_m - t_{l-1})^{1-\varepsilon}}. \end{aligned}$$

Using the properties of the Riemann sums we can estimate

$$\sum_{l=1}^{m-1} \frac{k_l}{(t_m - t_{l-1})^{1-\varepsilon}} \leq \int_0^{t_{m-1}} \frac{1}{t_m - s} ds \leq C_\varepsilon.$$

Choosing for example $\varepsilon = \frac{1}{4}$ we get with $C_3 = 2C_1C_2C_\varepsilon$

$$\|(\tilde{Q}g_k)_m\|_{L^2(\Omega)} \leq \max_{1 \leq j \leq m} \|g_{k,j}\|_{L^2(\Omega)} \left(\frac{1}{2} + C_3(k - k_{\min})^{\frac{1}{4}} \right).$$

The estimate (3.14) follows then with the choice $\delta_0 = \frac{1}{(4C_3)^4}$. The estimate (3.15) follows from Lemma 3.7 similarly. \square

REMARK 3.9. *The condition $k - k_{\min} \leq \delta_0$ trivially holds in the case of uniform time steps. For non-uniform time steps it is sufficient to assume $k \leq \frac{1}{2}\delta_0$.*

The above proposition shows that under certain conditions, the operator $\text{Id} - \tilde{Q}$ is invertible with a bounded inverse with respect to both the $L^\infty(I; L^2(\Omega))$ and $L^1(I; L^2(\Omega))$ norms on X_k^0 . This is the central piece of our argument. In order to obtain a discrete maximal parabolic regularity, we will also require estimates for \tilde{L} and \tilde{D} , which we will show next.

LEMMA 3.10. *For the operator \tilde{L} defined in (3.5) there exists a constant C independent of k such that for all $f_k \in X_k^0$ the following estimates hold:*

$$\|(\tilde{L}f_k)_m\|_{L^2(\Omega)} \leq C \ln \frac{T}{k} \max_{1 \leq l \leq m} \|f_{k,l}\|_{L^2(\Omega)}. \quad (3.16)$$

and

$$\sum_{m_1}^M k_m \|(\tilde{L}f_k)_m\|_{L^2(\Omega)} \leq C \ln \frac{T}{k} \sum_{l=1}^M k_l \|f_{k,l}\|_{L^2(\Omega)}. \quad (3.17)$$

Proof. From the definition of \tilde{L} and Lemma 3.4 we obtain

$$\begin{aligned} \|(\tilde{L}f_k)_m\|_{L^2(\Omega)} &\leq \sum_{l=1}^m k_l \|\tilde{A}_{k,m} \tilde{R}_{m,l}\|_{L^2 \rightarrow L^2} \|f_{k,l}\|_{L^2(\Omega)} \\ &\leq C \max_{1 \leq l \leq m} \|f_{k,l}\|_{L^2(\Omega)} \sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \leq C \ln \frac{T}{k} \max_{1 \leq l \leq m} \|f_{k,l}\|_{L^2(\Omega)}, \end{aligned}$$

where in the last step we used that

$$\sum_{l=1}^m \frac{k_l}{t_m - t_{l-1}} \leq 1 + \int_0^{t_{m-1}} \frac{dt}{t_m - t} = 1 + \ln \frac{t_m}{k_m} \leq C \ln \frac{T}{k}.$$

This completes the proof of (3.16). The estimate (3.17) can be shown similarly by changing the order of summation. \square

LEMMA 3.11. *For the operator \tilde{D} defined in (3.5) there exists a constant C independent of k such that for all $u_0 \in L^2(\Omega)$ the following estimate holds,*

$$\sum_{m=1}^M k_m \|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \leq C \ln \frac{T}{k} \|u_0\|_{L^2(\Omega)}. \quad (3.18)$$

If in addition $u_0 \in H_0^1(\Omega)$ with $A_{k,m}u_0 \in L^2(\Omega)$ for all $m = 1, 2, \dots, M$ then

$$\max_{1 \leq m \leq M} \|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \leq C\mu \max_{1 \leq m \leq M} \|A_{k,m}u_0\|_{L^2(\Omega)}. \quad (3.19)$$

Proof. There holds by Lemma 3.4

$$\|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \leq \frac{C}{t_m} \|u_0\|_{L^2(\Omega)}.$$

This results in

$$\sum_{m=1}^M k_m \|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \leq C \|u_0\|_{L^2(\Omega)} \sum_{m=1}^M \frac{k_m}{t_m} \leq C \ln \frac{T}{k} \|u_0\|_{L^2(\Omega)}.$$

To prove (3.19) we use the fact that

$$\|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \leq C \|\tilde{A}_{k,m} u_0\|_{L^2(\Omega)} \leq C \mu \|A_{k,m} u_0\|_{L^2(\Omega)}.$$

□

3.2. Semidiscrete in time maximal parabolic regularity. Combining the above results we can finally establish the maximal parabolic regularity with respect to the $L^\infty(I; L^2(\Omega))$ and the $L^1(I; L^2(\Omega))$ norms in the following two theorems.

THEOREM 3.12 (Discrete maximal parabolic regularity for $p = \infty$). *Let $f \in L^\infty(I; L^2(\Omega))$, let $u_0 \in H_0^1(\Omega)$ with $A_{k,m} u_0 \in L^2(\Omega)$ for all $m = 1, 2, \dots, M$. Let moreover u_k be the $dG(0)$ semidiscrete solution to (2.7). There exists $\mu > 0$ sufficiently large and $\delta_0 > 0$ such that for $k - k_{\min} \leq \delta_0$*

$$\begin{aligned} \max_{1 \leq m \leq M} \left(\|A_{k,m} u_{k,m}\|_{L^2(\Omega)} + \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^2(\Omega)} \right) \\ \leq C e^{\mu T} \left(\ln \frac{T}{k} \|f\|_{L^\infty(I; L^2(\Omega))} + \mu T \max_{1 \leq m \leq M} \|A_{k,m} u_0\|_{L^2(\Omega)} \right). \end{aligned}$$

Proof. Recalling the definitions of \tilde{v}_k and w_k , namely

$$\tilde{v}_{k,m} = \tilde{A}_{k,m} w_{k,m}, \quad \text{and} \quad u_{k,m} = \prod_{l=1}^m (1 + \mu k_l) w_{k,m},$$

we have

$$\tilde{A}_{k,m} u_{k,m} = \prod_{l=1}^m (1 + \mu k_l) \tilde{A}_{k,m} w_{k,m} = \prod_{l=1}^m (1 + \mu k_l) \tilde{v}_{k,m}. \quad (3.20)$$

By (3.2) we have

$$\max_{1 \leq m \leq M} \|\tilde{v}_{k,m}\|_{L^2(\Omega)} \leq \max_{1 \leq m \leq M} \left(\|(\tilde{Q}\tilde{v}_k)_m\|_{L^2(\Omega)} + \|(\tilde{L}\tilde{f}_k)_m\|_{L^2(\Omega)} + \|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \right).$$

For the first term we use estimate (3.14) from Proposition 3.8, for the second one estimate (3.16) from Lemma 3.10 and for the third one estimate (3.19) from Lemma 3.11. This results in

$$\begin{aligned} \max_{1 \leq m \leq M} \|\tilde{v}_{k,m}\|_{L^2(\Omega)} \leq \frac{3}{4} \max_{1 \leq m \leq M} \|\tilde{v}_{k,m}\|_{L^2(\Omega)} + C \ln \frac{T}{k} \max_{1 \leq m \leq M} \|\tilde{f}_{k,m}\|_{L^2(\Omega)} \\ + C \mu T \max_{1 \leq m \leq M} \|A_{k,m} u_0\|_{L^2(\Omega)}. \end{aligned}$$

Absorbing the first term on the right-hand side we obtain

$$\max_{1 \leq m \leq M} \|\tilde{v}_{k,m}\|_{L^2(\Omega)} \leq C \ln \frac{T}{k} \|\tilde{f}_k\|_{L^\infty(I; L^2(\Omega))} + C\mu T \max_{1 \leq m \leq M} \|A_{k,m} u_0\|_{L^2(\Omega)}.$$

Now using that

$$\|\tilde{f}_k\|_{L^\infty(I; L^2(\Omega))} \leq C \|f\|_{L^\infty(I; L^2(\Omega))},$$

we obtain

$$\max_{1 \leq m \leq M} \|\tilde{A}_{k,m} u_{k,m}\|_{L^2(\Omega)} \leq C e^{\mu T} \left(\ln \frac{T}{k} \|f\|_{L^\infty(I; L^2(\Omega))} + \mu T \max_{1 \leq m \leq M} \|A_{k,m} u_0\|_{L^2(\Omega)} \right),$$

where we used that $\prod_{l=1}^m (1 + \mu k_l) \leq e^{\mu t_m}$. Since $\tilde{A}_{k,m}$ is invertible for each m , using Lemma 3.2 we also have

$$\|u_{k,m}\|_{L^2(\Omega)} = \|\tilde{A}_{k,m}^{-1} \tilde{A}_{k,m} u_{k,m}\|_{L^2(\Omega)} \leq \frac{1}{\mu} \|\tilde{A}_{k,m} u_{k,m}\|_{L^2(\Omega)}.$$

Thus from (3.20), the definition of $\tilde{A}_{k,m}$, namely

$$\tilde{A}_{k,m} = (1 + k_m \mu) A_{k,m} + \mu \text{Id}$$

and by the triangle inequality and the estimates above we obtain

$$\|A_{k,m} u_{k,m}\|_{L^2(\Omega)} \leq \mu \|u_{k,m}\|_{L^2(\Omega)} + \|\tilde{A}_{k,m} u_{k,m}\|_{L^2(\Omega)} \leq 2 \|\tilde{A}_{k,m} u_{k,m}\|_{L^2(\Omega)}$$

and therefore

$$\max_{1 \leq m \leq M} \|A_{k,m} u_{k,m}\|_{L^2(\Omega)} \leq C e^{\mu T} \left(\ln \frac{T}{k} \|f\|_{L^\infty(I; L^2(\Omega))} + \mu T \max_{1 \leq m \leq M} \|A_{k,m} u_0\|_{L^2(\Omega)} \right).$$

The estimate for the second term in the statement of the theorem follows from the observation that (2.11) is just

$$\frac{[u_k]_{m-1}}{k_m} = -A_{k,m} u_{k,m} + f_{k,m}.$$

□

Next we establish the maximal parabolic regularity in $L^1(I; L^2(\Omega))$ norm.

THEOREM 3.13 (Discrete maximal parabolic regularity for $p = 1$). *Let $f \in L^1(I; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$. Let moreover u_k be the $dG(0)$ semidiscrete solution to (2.7). There exists $\mu > 0$ sufficiently large and $\delta_0 > 0$ such that for $k - k_{\min} \leq \delta_0$ there holds*

$$\sum_{m=1}^M (k_m \|A_{k,m} u_{k,m}\|_{L^2(\Omega)} + \|[u_k]_{m-1}\|_{L^2(\Omega)}) \leq C e^{\mu T} \ln \frac{T}{k} (\|f\|_{L^1(I; L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}).$$

Proof. The proof is very similar to the proof of the previous theorem. By (3.2) we have

$$\sum_{m=1}^M k_m \|\tilde{v}_{k,m}\|_{L^2(\Omega)} \leq \sum_{m=1}^M k_m \left(\|(\tilde{Q}\tilde{v}_k)_m\|_{L^2(\Omega)} + \|(\tilde{L}\tilde{f}_k)_m\|_{L^2(\Omega)} + \|(\tilde{D}u_0)_m\|_{L^2(\Omega)} \right).$$

For the first term we use estimate (3.15) from Proposition 3.8, for the second one estimate (3.17) from Lemma 3.10 and for the third one estimate (3.18) from Lemma 3.11. This results in

$$\begin{aligned} \sum_{m=1}^M k_m \|\tilde{v}_{k,m}\|_{L^2(\Omega)} &\leq \frac{3}{4} \sum_{m=1}^M k_m \|\tilde{v}_{k,m}\|_{L^2(\Omega)} + C \ln \frac{T}{k} \sum_{m=1}^M k_m \|\tilde{f}_{k,m}\|_{L^2(\Omega)} \\ &\quad + C \ln \frac{T}{k} \|u_0\|_{L^2(\Omega)}. \end{aligned}$$

The rest of the proof goes along the lines of the proof of the previous theorem. \square

COROLLARY 3.14. *For $u_0 = 0$ interpolating between the results of Theorem 3.12 and Theorem 3.13 we obtain the discrete maximal parabolic regularity for $1 \leq p < \infty$, namely*

$$\left[\sum_{m=1}^M \left(\|A_{k,m} u_{k,m}\|_{L^p(I_m; L^2(\Omega))}^p + k_m \left\| \frac{[u_k]_{m-1}}{k_m} \right\|_{L^2(\Omega)}^p \right) \right]^{\frac{1}{p}} \leq C e^{\mu T} \ln \frac{T}{k} \|f\|_{L^p(I; L^2(\Omega))}.$$

3.3. Fully discrete maximal parabolic regularity. In this section, we consider the fully discrete approximation of the equation (1.1). We will establish the corresponding results for fully discrete approximations.

Let Ω be a polygonal/polyhedral domain and let \mathcal{T} denote an admissible triangulation of Ω , i.e., $\mathcal{T} = \{\tau\}$ is a conformal partition of Ω into simplices (line segments, triangles, tetrahedrons, and etc.) τ of diameter h_τ . Let $h = \max_\tau h_\tau$ and V_h be the set of all functions in $H_0^1(\Omega)$ that are polynomials of degree $r \geq 1$ on each τ , i.e., V_h is the usual space of continuous finite elements. We would like to point out that we do not make any assumptions on shape regularity or quasi-uniformity of the meshes. To obtain the fully discrete approximation we consider the space-time finite element space

$$X_{k,h}^{0,r} = \{v_{kh} : v_{kh}|_{I_m} \in \mathcal{P}_0(I_m; V_h), m = 1, 2, \dots, M\}. \quad (3.21)$$

We define a fully discrete analog $u_{kh} \in X_{k,h}^{0,r}$ of u_k introduced in (2.7) by

$$B(u_{kh}, \varphi_{kh}) = (f, \varphi_{kh})_{I \times \Omega} + (u_0, \varphi_{kh}^+)_{\Omega} \quad \text{for all } \varphi_{kh} \in X_{k,h}^{0,r}. \quad (3.22)$$

Moreover, we introduce two operators $A_h(t) : V_h \rightarrow V_h$ defined by

$$(A_h(t)v_h, \chi)_{\Omega} = \sum_{i,j=1}^d (a_{ij}(t, \cdot) \partial_i v_h, \partial_j \chi)_{\Omega}, \quad \forall \chi \in V_h \quad (3.23)$$

and the orthogonal L^2 projection $P_h : V_h \rightarrow V_h$ defined by

$$(P_h v_h, \chi)_{\Omega} = (v_h, \chi)_{\Omega}, \quad \forall \chi \in V_h.$$

Similarly to $A_{k,m}$ in (2.10) we also define $A_{kh,m} : X_{k,h}^{0,r} \rightarrow X_{k,h}^{0,r}$

$$A_{kh,m} = \frac{1}{k_m} \int_{I_m} A_h(t) dt, \quad m = 1, 2, \dots, M. \quad (3.24)$$

With the help of the above operators, the fully discrete approximation $u_{kh} \in X_{k,h}^{0,r}$ defined in (3.22) satisfies

$$\begin{aligned} u_{kh,1} + k_1 A_{kh,1} u_{kh,1} &= P_h u_0 + k_1 P_h f_{k,1}, \\ u_{kh,m} + k_m A_{kh,m} u_{kh,m} &= u_{kh,m-1} + k_m P_h f_{k,m}, \quad m = 2, 3, \dots, M, \end{aligned} \quad (3.25)$$

where $f_{k,m}$ is defined in (2.9). Hence the same formulas for u_k , namely (2.17) also hold for u_{kh} with the difference that A_k is replaced by A_{kh} and f_k by $P_h f_k$. The analysis from section 3 of the paper translates almost immediately to the fully discrete setting since all arguments are energy based arguments and $\|P_h\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq 1$ on any mesh. Thus we directly establish the following results.

THEOREM 3.15 (Discrete maximal parabolic regularity for $p = \infty$). *Let the conditions of Theorem 3.12 be fulfilled and let u_{kh} be fully discrete solution to (1.1) defined by (3.22) on any conformal triangulation of Ω . Then there exists a constant C independent of k and h such that the following estimate holds:*

$$\begin{aligned} \max_{1 \leq m \leq M} \left(\|A_{kh,m} u_{kh,m}\|_{L^2(\Omega)} + \left\| \frac{[u_{kh}]_{m-1}}{k_m} \right\|_{L^2(\Omega)} \right) \\ \leq C e^{\mu T} \left(\ln \frac{T}{k} \|f\|_{L^\infty(I; L^2(\Omega))} + \mu T \max_{1 \leq m \leq M} \|A_{kh,m} P_h u_0\|_{L^2(\Omega)} \right). \end{aligned}$$

The corresponding result for the $L^1(I; L^2(\Omega))$ norm is formulated in the following theorem.

THEOREM 3.16 (Discrete maximal parabolic regularity for $p = 1$). *Under the conditions of Theorem 3.15 there exists a constant C independent of k and h such that*

$$\sum_{m=1}^M (k_m \|A_{kh,m} u_{kh,m}\|_{L^2(\Omega)} + \|[u_{kh}]_{m-1}\|_{L^2(\Omega)}) \leq C e^{\mu T} \ln \frac{T}{k} (\|f\|_{L^1(I; L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}).$$

COROLLARY 3.17. *For $u_0 = 0$ interpolating between Theorem 3.15 and Theorem 3.16 we obtain discrete maximal parabolic regularity for $1 \leq p < \infty$*

$$\left[\sum_{m=1}^M \left(\|A_{kh,m} u_{kh,m}\|_{L^p(I_m; L^2(\Omega))}^p + k_m \left\| \frac{[u_{kh}]_{m-1}}{k_m} \right\|_{L^2(\Omega)}^p \right) \right]^{\frac{1}{p}} \leq C e^{\mu T} \ln \frac{T}{k} \|f\|_{L^p(I; L^2(\Omega))}.$$

4. Applications to error estimates. In this section we illustrate how the discrete maximal parabolic results from the previous section can be applied to error estimates. For the rest of the section we assume that Ω is convex and in addition the following assumption holds.

ASSUMPTION 1.

$$a_{ij}(t, \cdot) \in W^{1,\infty}(\Omega) \quad \text{for all } t \in \bar{I},$$

and

$$L := \max_{1 \leq i, j \leq d} \sup_{t \in \bar{I}} \|a_{ij}(t)\|_{W^{1,\infty}} < \infty.$$

4.1. Time semidiscrete error estimates. Using the convexity of Ω and Assumption 1 we establish the following preliminary result.

LEMMA 4.1. *There exists a constant C independent of k such that*

$$\sup_{t \in I_m} \|A(t)A_{k,m}^{-1}\|_{L^2 \rightarrow L^2} \leq C, \quad m = 1, 2, \dots, M.$$

Proof. Take an arbitrary $v \in L^2(\Omega)$ and set $w = A_{k,m}^{-1}v$, where $A_{k,m}$ is an elliptic operator with coefficients $a_{k,m}^{ij}$ defined by

$$a_{k,m}^{ij}(x) = \frac{1}{k_m} \int_{I_m} a_{ij}(t, x) dt. \quad (4.1)$$

By Assumption 1 we have $a_{k,m}^{ij} \in W^{1,\infty}(\Omega)$ with $\|a_{k,m}^{ij}\|_{W^{1,\infty}} \leq L$ for all $1 \leq i, j \leq d$. From Theorems 2.2.2.3 and 3.2.1.2 in [13], we can conclude that $w \in H^2(\Omega)$ and

$$\|w\|_{H^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}, \quad (4.2)$$

where the constant C depends on Ω and L only. Again by Assumption 1, $A(t)$ is a bounded operator from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$ and as a result

$$\|A(t)A_{k,m}^{-1}v\|_{L^2(\Omega)} = \|A(t)w\|_{L^2(\Omega)} \leq C\|w\|_{H^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}.$$

Taking supremum over v concludes the proof. \square

As a first application of the discrete maximum regularity we establish semidiscrete best approximation result in the case of $p = \infty$.

THEOREM 4.2. *Let the coefficients $a_{ij}(t, x)$ satisfy the Assumption 1 and let u be the solution to (1.1) with $u \in C(\bar{I}; L^2(\Omega))$ and u_k be the dG(0) semidiscrete solution to (2.7). Then under the conditions of Theorem 3.13 there exists a constant C independent of k such that*

$$\|u - u_k\|_{L^\infty(I; L^2(\Omega))} \leq Ce^{\mu T} \ln \frac{T}{k} \inf_{\chi \in X_k^0} \|u - \chi\|_{L^\infty(I; L^2(\Omega))}.$$

Proof. Let $\tilde{t} \in (0, T]$ be an arbitrary but fixed point in time. Without loss of generality we assume $\tilde{t} \in I_M = (t_{M-1}, T]$. We consider the following dual problem

$$\begin{aligned} \partial_t g(t, x) - A(t, x)g(t, x) &= \tilde{\theta}(\tilde{t})u_k(\tilde{t}, x), & (t, x) \in I \times \Omega, \\ g(t, x) &= 0, & (t, x) \in I \times \partial\Omega, \\ g(T, x) &= 0, & x \in \Omega, \end{aligned} \quad (4.3)$$

where $\tilde{\theta} \in C^\infty(0, T)$ is the regularized Delta function in time with the properties $\text{supp}(\tilde{\theta}) \subset I_M$, $\|\tilde{\theta}\|_{L^1(I_M)} \leq C$ and

$$(\tilde{\theta}, \varphi_k)_{I_M} = \varphi_k(\tilde{t}), \quad \forall \varphi_k \in X_k^0.$$

Let $g_k \in X_k^0$ be dG(0) approximation of g , i.e., $B(g - g_k, \varphi_k) = 0$ for any $\varphi_k \in X_k^0$. Then

$$\begin{aligned} \|u_k(\tilde{t})\|_{L^2(\Omega)}^2 &= (u_k(\cdot, \cdot), \tilde{\theta}(\cdot)u_k(\tilde{t}, \cdot))_{I \times \Omega} \\ &= B(u_k, g) = B(u_k, g_k) = B(u, g_k) \\ &= \sum_{m=1}^M (u, A(t)g_{k,m})_{I_m \times \Omega} - \sum_{m=1}^M (u(t_m), [g_k]_m)_\Omega := J_1 + J_2. \end{aligned}$$

Using the Hölder inequality in time and the Cauchy-Schwarz inequality in space, Theorem 3.13 and Lemma 4.1, we have

$$\begin{aligned}
J_1 &\leq \|u\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^M k_m \sup_{t \in I_m} \|A(t)A_{k,m}^{-1}A_{k,m}g_{k,m}\|_{L^2(\Omega)} \\
&\leq C\|u\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^M k_m \sup_{t \in I_m} \|A(t)A_{k,m}^{-1}\|_{L^2 \rightarrow L^2} \|A_{k,m}g_{k,m}\|_{L^2(\Omega)} \\
&\leq Ce^{\mu T} \ln \frac{T}{k} \|u\|_{L^\infty(I;L^2(\Omega))} \|\tilde{\theta}\|_{L^1(I)} \|u_k(\tilde{t})\|_{L^2(\Omega)} \\
&\leq Ce^{\mu T} \ln \frac{T}{k} \|u\|_{L^\infty(I;L^2(\Omega))} \|u_k(\tilde{t})\|_{L^2(\Omega)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
J_2 &\leq \sum_{m=1}^M \|u(t_m)\|_{L^2(\Omega)} \| [g_k]_m \|_{L^2(\Omega)} \\
&\leq \|u\|_{L^\infty(I;L^2(\Omega))} \sum_{m=1}^M \| [g_k]_m \|_{L^2(\Omega)} \\
&\leq Ce^{\mu T} \ln \frac{T}{k} \|u\|_{L^\infty(I;L^2(\Omega))} \|u_k(\tilde{t})\|_{L^2(\Omega)}.
\end{aligned}$$

Canceling by $\|u_k(\tilde{t})\|_{L^2(\Omega)}$ and taking the supremum over \tilde{t} , we establish

$$\|u_k\|_{L^\infty(I;L^2(\Omega))} \leq Ce^{\mu T} \ln \frac{T}{k} \|u\|_{L^\infty(I;L^2(\Omega))}. \quad (4.4)$$

Using that the dG(0) method is invariant on X_k^0 , by replacing u and u_k with $u - \chi$ and $u_k - \chi$ for any $\chi \in X_k^0$, and using the triangle inequality we obtain the theorem. \square

For $1 \leq p < \infty$, we can obtain the following result which is similar to the result was obtained for time independent coefficients in [32] for the $L^p(I;L^s(\Omega))$ norm. To state the result we define a projection π_k for $u \in C(I, L^2(\Omega))$ with $\pi_k u|_{I_m} \in P_0(I_m; L^2(\Omega))$ for $m = 1, 2, \dots, M$ on each subinterval I_m by

$$\pi_k u(t) = u(t_m), \quad t \in I_m. \quad (4.5)$$

THEOREM 4.3. *Let the coefficients $a_{ij}(t, x)$ satisfy the Assumption 1 and let u be the solution to (1.1) with $u \in C(\bar{I}; L^2(\Omega))$ and u_k be the dG(0) semidiscrete solution to (2.7). Then under the conditions of Theorem 3.13 there exists a constant C independent of k such that*

$$\|u - u_k\|_{L^p(I;L^2(\Omega))} \leq Ce^{\mu T} \ln \frac{T}{k} \|u - \pi_k u\|_{L^p(I;L^2(\Omega))}, \quad 1 \leq p < \infty,$$

where the projection π_k is defined above in (4.5).

Proof. The proof uses the result of Corollary 3.14 and goes along the lines of the proof of Theorem 9 in [32] and Theorem 4.2 above. \square

4.2. Applications to fully discrete error estimates. Similarly to the semidiscrete case, as an application of the fully discrete maximal parabolic regularity, we show optimal convergence rates for the dG(0)cG(r) solution. As in the semidiscrete case, first using the convexity of Ω and Assumption 1 and in addition that the triangulation \mathcal{T} is quasi-uniform we establish the space discrete version of the Lemma 4.1. Thus, for rest of the paper we assume

ASSUMPTION 2. *There exists a constant C independent of h such that*

$$\text{diam}(\tau) \leq h \leq C|\tau|^{\frac{1}{d}}, \quad \forall \tau \in \mathcal{T},$$

where $d = 2, 3$ is the dimension on Ω . For the results below we will require one Ritz projection $R_h(t): H_0^1(\Omega) \rightarrow V_h$, which is for every $t \in \bar{I}$ defined by

$$\sum_{i,j=1}^d (a_{ij}(t) \partial_i (R_h(t)v), \partial_j \chi)_\Omega = \sum_{i,j=1}^d (a_{ij}(t) \partial_i v, \partial_j \chi)_\Omega, \quad \forall \chi \in V_h \quad (4.6)$$

and another Ritz projection $R_{kh,m}: H_0^1(\Omega) \rightarrow V_h$, which is for every $m = 1, 2, \dots, M$ defined by

$$\sum_{i,j=1}^d \left(a_{k,m}^{ij} \partial_i R_{kh,m} v, \partial_j \chi \right)_\Omega = \sum_{i,j=1}^d \left(a_{k,m}^{ij} \partial_i v, \partial_j \chi \right)_\Omega, \quad \forall \chi \in V_h, \quad (4.7)$$

where $a_{k,m}^{ij}$ are defined in (4.1).

LEMMA 4.4. *There exists a constant C independent of k and h such that*

$$\sup_{t \in I_m} \|A_h(t) A_{kh,m}^{-1}\|_{L^2 \rightarrow L^2} \leq C, \quad m = 1, 2, \dots, M.$$

Proof. Take an arbitrary $v \in L^2(\Omega)$ and define $w_h = A_{kh,m}^{-1} P_h v$. In addition we also define $w = A_{k,m}^{-1} v$. Notice that $R_{kh,m} w = w_h$. By the definition of A_h in (3.23),

$$(A_h(t) w_h, \varphi_h)_\Omega = \sum_{i,j=1}^d (a_{ij}(t) \partial_i w_h, \partial_j \varphi_h)_\Omega, \quad \forall \varphi_h \in V_h.$$

Put $z_h(t) = A_h(t) w_h$. Then adding and subtracting w , we have

$$\begin{aligned} \|z_h(t)\|_{L^2(\Omega)}^2 &= \|A_h(t) w_h\|_{L^2(\Omega)}^2 = \sum_{i,j=1}^d (a_{ij}(t) \partial_i w_h, \partial_j z_h(t))_\Omega \\ &= \sum_{i,j=1}^d (a_{ij}(t) \partial_i w, \partial_j z_h(t))_\Omega + \sum_{i,j=1}^d (a_{ij}(t) \partial_i (w_h - w), \partial_j z_h(t))_\Omega := J_1 + J_2. \end{aligned}$$

To estimate J_1 we integrate by parts and use the Cauchy-Schwarz inequality

$$J_1 = - \sum_{i,j=1}^d (\partial_j (a_{ij}(t) \partial_i w), z_h(t))_\Omega \leq L \|w\|_{H^2(\Omega)} \|z_h(t)\|_{L^2(\Omega)}.$$

Using that $w_h = R_{kh,m} w$, the standard elliptic error estimates and the inverse inequality, we obtain

$$\begin{aligned} J_2 &\leq \sup_{i,j} \|a_{ij}(t)\|_{L^\infty(\Omega)} \|w - w_h\|_{H^1(\Omega)} \|z_h(t)\|_{H^1(\Omega)} \leq Ch \|w\|_{H^2(\Omega)} \|z_h(t)\|_{H^1(\Omega)} \\ &\leq C \|w\|_{H^2(\Omega)} \|z_h(t)\|_{L^2(\Omega)}. \end{aligned}$$

Combining the estimates for J_1 and J_2 we can conclude that

$$\|z_h(t)\|_{L^2(\Omega)} \leq C\|w\|_{H^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}, \quad (4.8)$$

where we used that the definition of w is identical to the definition of w in Lemma 4.1 and from (4.2) we know that $\|w\|_{H^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}$. Taking supremum over v concludes the proof. \square

Similar to the semidiscrete case, we also establish a corresponding result for $p = \infty$ in the fully discrete case.

THEOREM 4.5. *Let the coefficients $a_{ij}(t, x)$ satisfy the Assumption 1 and let u be the solution to (1.1) with $u \in C(\bar{I}; L^2(\Omega))$ and u_{kh} be the $dG(0)cG(r)$ solution for $r \geq 1$ on a quasi-uniform triangulation \mathcal{T} with the coefficients $a_{ij}(t, x)$ satisfying the Assumption 1. Then under the assumption of Theorem 3.15 there exists a constant C independent of k and h such that for $1 \leq p < \infty$,*

$$\|u - u_{kh}\|_{L^\infty(I; L^2(\Omega))} \leq Ce^{\mu T} \ln \frac{T}{k} \left(\min_{\chi \in X_{k,h}^{0,r}} \|u - \chi\|_{L^\infty(I; L^2(\Omega))} + \|u - R_h u\|_{L^\infty(I; L^2(\Omega))} \right).$$

Proof. As in the proof of Theorem 4.2, let $\tilde{t} \in (0, T]$ be an arbitrary but fixed point in time. Without loss of generality we assume $\tilde{t} \in I_M = (t_{M-1}, T]$. Consider the following dual problem

$$\begin{aligned} \partial_t g(t, x) - A(t, x)g(t, x) &= \tilde{\theta}(t)u_{kh}(\tilde{t}, x), & (t, x) \in I \times \Omega, \\ g(t, x) &= 0, & (t, x) \in I \times \partial\Omega, \\ g(T, x) &= 0, & x \in \Omega, \end{aligned} \quad (4.9)$$

where $\tilde{\theta} \in C^\infty(0, T)$ is the regularized Delta function in time with properties $\text{supp}(\tilde{\theta}) \subset I_M$, $\|\tilde{\theta}\|_{L^1(I_M)} \leq C$ and

$$(\tilde{\theta}, \varphi_{kh})_{I_M} = \varphi_{kh}(\tilde{t}), \quad \forall \varphi_{kh} \in X_{k,h}^{0,r}.$$

Let g_{kh} be $dG(0)cG(r)$ approximation of g , i.e., $B(g - g_{kh}, \varphi_{kh}) = 0$ for any $\varphi_{kh} \in X_{k,h}^{0,r}$. Then

$$\begin{aligned} \|u_{kh}(\tilde{t})\|_{L^2(\Omega)}^2 &= (u_{kh}, \tilde{\theta}u_{kh}(\tilde{t}))_{I \times \Omega} \\ &= B(u_{kh}, g) = B(u_{kh}, g_{kh}) = B(u, g_{kh}) \\ &= \sum_{m=1}^M (R_h(t)u, A_h(t)g_{kh,m})_{I_m \times \Omega} - \sum_{m=1}^M (u(t_m), [g_{kh}]_m)_\Omega = J_1 + J_2. \end{aligned}$$

Using the Hölder inequality in time and the Cauchy-Schwarz inequality in space, Lemma 4.4 and Theorem 3.16, we have

$$\begin{aligned} J_1 &\leq \|R_h u\|_{L^\infty(I; L^2(\Omega))} \sum_{m=1}^M k_m \sup_{t \in I_m} \|A_h(t)A_{kh,m}^{-1}A_{kh,m}g_{k,m}\|_{L^2(\Omega)} \\ &\leq \|R_h u\|_{L^\infty(I; L^2(\Omega))} \sum_{m=1}^M k_m \sup_{t \in I_m} \|A_h(t)A_{kh,m}^{-1}\|_{L^2 \rightarrow L^2} \|A_{kh,m}g_{k,m}\|_{L^2(\Omega)} \\ &\leq Ce^{\mu T} \ln \frac{T}{k} \|R_h u\|_{L^\infty(I; L^2(\Omega))} \|\tilde{\theta}\|_{L^1(I)} \|u_{kh}(\tilde{t}, \cdot)\|_{L^2(\Omega)} \\ &\leq Ce^{\mu T} \ln \frac{T}{k} \|R_h u\|_{L^\infty(I; L^2(\Omega))} \|u_{kh}(\tilde{t})\|_{L^2(\Omega)}. \end{aligned}$$

Exactly as in the estimate of J_2 in Theorem 4.5, we obtain

$$J_2 \leq Ce^{\mu T} \ln \frac{T}{k} \|u\|_{L^\infty(I; L^2(\Omega))} \|u_{kh}(\tilde{t})\|_{L^2(\Omega)}.$$

Thus canceling $\|u_{kh}(\tilde{t})\|_{L^2(\Omega)}$ and taking supremum over \tilde{t} , we establish

$$\|u_{kh}\|_{L^\infty(I; L^2(\Omega))} \leq Ce^{\mu T} \ln \frac{T}{k} (\|R_h u\|_{L^\infty(I; L^2(\Omega))} + \|u\|_{L^\infty(I; L^2(\Omega))}). \quad (4.10)$$

Using that dG(0)cG(r) method is invariant on $X_{k,h}^{0,r}$, by replacing u and u_{kh} with $u - \chi$ and $u_{kh} - \chi$ for any $\chi \in X_{k,h}^{0,r}$, and using the triangle inequality we obtain the theorem. \square

THEOREM 4.6. *Let the coefficients $a_{ij}(t, x)$ satisfy the Assumption 1 and let u be the solution to (1.1) with $u \in C(\bar{I}; L^2(\Omega))$ and u_{kh} be the dG(0)cG(r) solution for $r \geq 1$ on a quasi-uniform triangulation \mathcal{T} with the coefficients $a_{ij}(t, x)$ satisfying the Assumption 1. Then under the assumption of Theorem 3.15 there exists a constant C independent of k and h such that for $1 \leq p < \infty$,*

$$\|u - u_{kh}\|_{L^p(I; L^2(\Omega))} \leq C \ln \frac{T}{k} (\|u - \pi_k u\|_{L^p(I; L^2(\Omega))} + \|u - R_h u\|_{L^p(I; L^2(\Omega))}),$$

where the projection π_k is defined in (4.5) and R_h in (4.6).

Proof. The proof uses the result of Corollary (3.17) and goes along the lines of the proof of Theorem 12 in [32] and Theorem 4.5 above. \square

For sufficiently regular solutions, combining the above two theorems and using the approximation theory we immediately obtain an optimal order convergence result.

COROLLARY 4.7. *Under the assumptions of Theorem 4.6 and the regularity $u \in W^{1,p}(I; L^2(\Omega)) \cap L^p(I; H^{r+1}(\Omega))$ for some $1 \leq p \leq \infty$, there exists a constant C independent of k and h such that*

$$\|u - u_{kh}\|_{L^p(I; L^2(\Omega))} \leq C \ln \frac{T}{k} (k \|u\|_{W^{1,p}(I; L^2(\Omega))} + h^{r+1} \|u\|_{L^p(I; H^{r+1}(\Omega))}).$$

REMARK 4.8. *The results of Theorems 4.2, 4.3, 4.5, and 4.6 also hold for the elliptic operator of the form $A(t, x) = b(t)A(x)$, where $b(t) \in C^{\frac{1}{2}+\varepsilon}(\bar{I})$ and $A(x)$ is the second order elliptic operator with bounded coefficients. In view of the uniform ellipticity condition (1.3), we have $b(t) \geq b_0 > 0$ for some $b_0 \in \mathbb{R}^+$, and as a result Lemmas 4.1 and 4.4 trivially hold without any additional assumptions, such as Assumptions 1 and 2. Thus, the results of the theorems hold for non-convex polygonal/polyhedral domains and on graded meshes.*

Acknowledgments. The authors would like to thank Dominik Meidner and Lucas Bonifacius for the careful reading of the manuscript and providing valuable suggestions that help to improve the presentation of the paper.

REFERENCES

- [1] A. ASHYRALYEV, S. PISKAREV, AND L. WEIS, *On well-posedness of difference schemes for abstract parabolic equations in $L^p([0, T]; E)$ spaces*, Numer. Funct. Anal. Optim., 23 (2002), pp. 669–693.
- [2] A. ASHYRALYEV AND P. E. SOBOLEVSKIĬ, *Well-posedness of parabolic difference equations*, vol. 69 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1994. Translated from the Russian by A. Iacob.

- [3] P. AUSCHER AND M. EGERT, *On non-autonomous maximal regularity for elliptic operators in divergence form*, Arch. Math. (Basel), 107 (2016), pp. 271–284.
- [4] P. CANNARSA AND V. VESPRI, *On maximal L^p regularity for the abstract Cauchy problem*, Boll. Un. Mat. Ital. B (6), 5 (1986), pp. 165–175.
- [5] K. CHRYSAFINOS AND N. J. WALKINGTON, *Error estimates for the discontinuous Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., 44 (2006), pp. 349–366 (electronic).
- [6] T. COULHON AND X. T. DUONG, *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss*, Adv. Differential Equations, 5 (2000), pp. 343–368.
- [7] R. DENK, M. HIEBER, AND J. PRÜSS, *R -boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., 166 (2003), pp. viii+114.
- [8] D. DIER, *Non-autonomous maximal regularity for forms of bounded variation*, J. Math. Anal. Appl., 425 (2015), pp. 33–54.
- [9] K. DISSER, A. F. M. TER ELST, AND J. REHBERG, *On maximal parabolic regularity for non-autonomous parabolic operators*, J. Differential Equations, 262 (2017), pp. 2039–2072.
- [10] M. DOBROWOLSKI, *Optimale Konvergenz der Methode der finiten Elemente bei parabolischen Anfangs-, Randwertaufgaben*, Bonner Mathematische Schriften [Bonn Mathematical Publications], 112, Universität Bonn, Mathematisches Institut, Bonn, 1978. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1978.
- [11] K. ERIKSSON, C. JOHNSON, AND S. LARSSON, *Adaptive finite element methods for parabolic problems. VI. Analytic semigroups*, SIAM J. Numer. Anal., 35 (1998), pp. 1315–1325.
- [12] C. GONZÁLEZ AND C. PALENCIA, *Stability of time-stepping methods for abstract time-dependent parabolic problems*, SIAM J. Numer. Anal., 35 (1998), pp. 973–989 (electronic).
- [13] P. GRISVARD, *Elliptic problems in nonsmooth domains*, vol. 69 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.
- [14] D. GUIDETTI, *Backward Euler scheme, singular Hölder norms, and maximal regularity for parabolic difference equations*, Numer. Funct. Anal. Optim., 28 (2007), pp. 307–337.
- [15] D. GUIDETTI AND S. PISKAREV, *Stability of Rothe’s scheme and maximal regularity for parabolic equations in $C^\theta(\bar{\Omega})$* , in Progress in partial differential equations, Vol. 1 (Pont-à-Mousson, 1997), vol. 383 of Pitman Res. Notes Math. Ser., Longman, Harlow, 1998, pp. 167–180.
- [16] ———, *Stability of the Crank-Nicolson scheme and maximal regularity for parabolic equations in $C^\theta(\bar{\Omega})$ spaces*, Numer. Funct. Anal. Optim., 20 (1999), pp. 251–277.
- [17] B. H. HAAK AND E. M. OUHABAZ, *Maximal regularity for non-autonomous evolution equations*, Math. Ann., 363 (2015), pp. 1117–1145.
- [18] R. HALLER-DINTELMANN AND J. REHBERG, *Maximal parabolic regularity for divergence operators including mixed boundary conditions*, J. Differential Equations, 247 (2009), pp. 1354–1396.
- [19] M. HIEBER AND J. PRÜSS, *Heat kernels and maximal L^p - L^q estimates for parabolic evolution equations*, Comm. Partial Differential Equations, 22 (1997), pp. 1647–1669.
- [20] D. HIPPEL, M. HOCHBRUCK, AND A. OSTERMANN, *An exponential integrator for non-autonomous parabolic problems*, Electron. Trans. Numer. Anal., 41 (2014), pp. 497–511.
- [21] D. HÖMBERG, C. MEYER, J. REHBERG, AND W. RING, *Optimal control for the thermistor problem*, SIAM J. Control Optim., 48 (2009/10), pp. 3449–3481.
- [22] M. Y. HUANG AND V. THOMÉE, *Some convergence estimates for semidiscrete type schemes for time-dependent nonselfadjoint parabolic equations*, Math. Comp., 37 (1981), pp. 327–346.
- [23] O. A. KARAKASHIAN, *On Runge-Kutta methods for parabolic problems with time-dependent coefficients*, Math. Comp., 47 (1986), pp. 77–101.
- [24] S. L. KEELING, *Galerkin/Runge-Kutta discretizations for parabolic equations with time-dependent coefficients*, Math. Comp., 52 (1989), pp. 561–586.
- [25] B. KOVÁCS, B. LI, AND C. LUBICH, *A-Stable Time Discretizations Preserve Maximal Parabolic Regularity*, SIAM J. Numer. Anal., 54 (2016), pp. 3600–3624.
- [26] K. KRUMBIEGEL AND J. REHBERG, *Second order sufficient optimality conditions for parabolic optimal control problems with pointwise state constraints*, SIAM J. Control Optim., 51 (2013), pp. 304–331.
- [27] K. KUNISCH, K. PIEPER, AND B. VEXLER, *Measure valued directional sparsity for parabolic optimal control problems*, SIAM J. Control Optim., 52 (2014), pp. 3078–3108.
- [28] D. LEYKEKHMAN AND B. VEXLER, *Optimal a priori error estimates of parabolic optimal control problems with pointwise control*, SIAM J. Numer. Anal., 51 (2013), pp. 2797–2821.
- [29] D. LEYKEKHMAN AND B. VEXLER, *Global and interior pointwise best approximation results for the gradient of Galerkin solutions for parabolic problems*, ArXiv e-prints, (2016).
- [30] D. LEYKEKHMAN AND B. VEXLER, *Pointwise best approximation results for Galerkin finite element solutions of parabolic problems*, SIAM J. Numer. Anal., 54 (2016), pp. 1365–1384.

- [31] ———, *A priori error estimates for three dimensional parabolic optimal control problems with pointwise control*, SIAM J. Control Optim., 54 (2016), pp. 2403–2435.
- [32] ———, *Discrete maximal parabolic regularity for Galerkin finite element methods*, Numerische Mathematik, 135 (2017), pp. 923–952.
- [33] B. LI AND W. SUN, *Maximal Regularity of Fully Discrete Finite Element Solutions of Parabolic Equations*, SIAM J. Numer. Anal., 55 (2017), pp. 521–542.
- [34] M. LUSKIN AND R. RANNACHER, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal., 19 (1982), pp. 93–113.
- [35] ———, *On the smoothing property of the Crank-Nicolson scheme*, Applicable Anal., 14 (1982/83), pp. 117–135.
- [36] D. MEIDNER, R. RANNACHER, AND B. VEXLER, *A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time*, SIAM J. Control Optim., 49 (2011), pp. 1961–1997.
- [37] P. SAMMON, *Fully discrete approximation methods for parabolic problems with nonsmooth initial data*, SIAM J. Numer. Anal., 20 (1983), pp. 437–470.
- [38] P. H. SAMMON, *Approximations for parabolic equations with time-dependent coefficients*, ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)—Cornell University.
- [39] V. THOMÉE, *Galerkin finite element methods for parabolic problems*, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, second ed., 2006.