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ERROR ANALYSIS FOR A FINITE ELEMENT APPROXIMATION OF ELLIPTIC DIRICHLET BOUNDARY CONTROL PROBLEMS*

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Abstract. We consider the Galerkin finite element approximation of an elliptic Dirichlet boundary control model problem governed by the Laplacian operator. The analytical setting of this problem uses L^2 controls and a "very weak" formulation of the state equation. However, the corresponding finite element approximation uses standard continuous trial and test functions. For this approximation, we derive a priori error estimates of optimal order, which are confirmed by numerical experiments. The proofs employ duality arguments and known results from the L^p error analysis for the finite element Dirichlet projection.

Key words. Dirichlet boundary control, finite elements, a priori error estimates

AMS subject classifications. 65K10, 65N30, 65N21, 49M25, 49K20

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1. Introduction and statement of results.

We consider the following elliptic Dirichlet boundary control problem posed on a convex polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$:

$$\min\ J(u,q) := \tfrac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|q\|^2_{L^2(\Gamma)} \tag{1.1}$$

with respect to $\{u, q\}$, under the constraint

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = q \quad \text{on } \Gamma. \tag{1.2}$$

Here, u_d and f are sufficiently smooth prescribed functions, while $\alpha > 0$ is a regularization parameter. For simplicity, we assume that at least $u_d, f \in L^2(\Omega)$. The natural functional analytic setting of this problem, which is also most convenient for numerical approximation, uses $Q := L^2(\Gamma)$ as a "control space." This prohibits the choice of the associated "state space" to be $H^1(\Omega)$, as the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is not surjective. To overcome this dilemma, we use a "very weak" formulation of the state equation (1.2), allowing for solutions $u \in L^2(\Omega)$ (see Grisvard [11, 12], and Berggren [4]): *For given $q \in L^2(\Gamma)$ find $u \in L^2(\Omega)$ such that*

$$-(u, \Delta\varphi) + \langle q, \partial_n\varphi\rangle = (f, \varphi) \quad \forall \varphi \in H^1_0(\Omega) \cap H^2(\Omega). \tag{1.3}$$

Here, $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$ is the L^2 inner product on the domain Ω, and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ that on its boundary Γ. The corresponding norms are $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $|\cdot| = |\cdot|_{L^2(\Gamma)}$, respectively. There are alternative variational formulations of Dirichlet boundary

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optimal control problems of the type (1.1), (1.2); for a brief survey, we refer to Kunisch and Vexler [17]. However, the formulation considered here appears to be the most attractive one from the computational point of view.

The finite element discretization of this optimization problem uses a standard weak formulation of the state equation, which is possible due to higher regularity of the actual solution pair $\{\hat{u}, \hat{q}\}$. We note that, due to the Galerkin type of discretization, the "optimize-then-discretize" and the "discretize-then-optimize" approaches coincide here. For this approximation the estimate

$$|\hat{q} - \hat{q}_h| + \|\hat{u} - \hat{u}_h\| = \mathcal{O}(h^{1-1/p}) \tag{1.4}$$

has been given by Casas and Raymond [7] for a problem with additional inequality constraints for the control q, which was expected to be only suboptimal for the state. The contribution of this paper consists of the improved L^2 error estimate,

$$\|\hat{u} - \hat{u}_h\| = \mathcal{O}(h^{3/2-1/p}), \tag{1.5}$$

and of "optimal order" error estimates with respect to weaker norms of the form

$$|\hat{q} - \hat{q}_h|_{\tilde{H}^{-1/2}(\Gamma)} = \mathcal{O}(h^{3/2-1/p}), \quad \|\hat{u} - \hat{u}_h\|_{H^{-1}(\Omega)} = \mathcal{O}(h^{2-1/p-1/r}). \tag{1.6}$$

Here, the values of $p \in [2, p_*)$, $p_* := \min\{p_*^\Omega, p_*^d\}$, and $r \in [2, p_*^\Omega)$ depend on the regularity of the data (p_*^d) and the limited elliptic regularity on domains with corners (p_*^Ω), respectively. Further, for the associated adjoint state the error estimate

$$\|\hat{z} - \hat{z}_h\| = \mathcal{O}(h^{2-1/p-1/r}) \tag{1.7}$$

is obtained. In view of the results of the computational experiments presented at the end of this paper, these estimates for the primal state and the control seem to be optimal-order. For maximum regularity, e.g., for smooth data on a rectangular domain, the order $\mathcal{O}(h^2)$ of convergence is obtained for the adjoint state which is best possible for linear or bilinear finite elements. The proofs employ duality arguments based on the KKT system (Karush–Kuhn–Tucker system) associated with the optimization problem (1.1), (1.2) and use various results from the finite element error analysis in L^p for $p \neq 2$.

The content of this paper is organized as follows. Section 2 contains the variational formulation of the Dirichlet boundary control problem and its Galerkin finite element approximation. This includes the derivation of the continuous and discrete KKT systems representing the first order necessary optimality conditions which form the basis of the error analysis. In section 3 several auxiliary results on elliptic regularity and finite element approximation are provided. These are used in section 4 to prove some suboptimal-order error estimates, followed by the final optimal-order ones in section 5. The last section, section 6, contains the results of some test calculations made to check the theoretical predictions.

2. The Dirichlet boundary control problem.

2.1. The state equation in very weak form. For later use, we provide some notation and results from the theory of Sobolev spaces and elliptic boundary value problems, concentrating on results needed in this paper. We refer the interested reader to Appendix A of Casas, Mateos, and Raymond [6] for further regularity results. The standard Sobolev spaces on Ω will be denoted by $H^m(\Omega)$, $H_0^m(\Omega)$, and $W^{m,p}(\Omega)$ for m

being a (nonnegative) integer. The Sobolev spaces of noninteger order s, $H^s(\Omega)$ and $W^{s,p}(\Omega)$, are defined by interpolation (see Adams and Fournier [1]). On a Lipschitz domain, this definition is equivalent to the definition using double-integral norms (see Brenner and Scott [5, Theorem 14.2.3]). On the boundary Γ, we define the norms of $H^s(\Gamma)$ and $W^{s,p}(\Gamma)$, $0 \le s \le 1$ and $1 < p < \infty$, using charts. By Grisvard [11, p. 20], this is equivalent to using double-integral norms on Γ. Moreover, we will use weighted Sobolev spaces $W^{s,p}_\delta(\Omega)$, where the weight depends on the distance to the corner points of Γ; see, e.g., Roßmann [20] for a precise definition.

For functions $v \in H^1(\Omega)$ the "strong" traces $v|_\Gamma \in L^2(\Gamma)$ exist and form the natural "trace space" $H^{1/2}(\Gamma)$ (see Ding [10]) fulfilling

$$|v|_{H^{1/2}(\Gamma)} \le c\|v\|_{H^1(\Omega)}. \tag{2.1}$$

For right-hand side $f \in L^2(\Omega)$ and boundary function $q \in H^{1/2}(\Gamma)$ the boundary value problem (1.2) has a standard "weak" solution $u \in H^1(\Omega)$, which is determined by $u|_\Gamma = q$ and

$$(\nabla u, \nabla\varphi) = (f, \varphi) \quad \forall \varphi \in H^1_0(\Omega). \tag{2.2}$$

For $q = 0$ this weak solution is in $H^2(\Omega)$, as Ω is a convex polygonal domain. Furthermore, it obeys the a priori bound

$$\|u\|_{H^2(\Omega)} \le c\|f\|. \tag{2.3}$$

Let $H^{-1/2}(\Gamma)$ denote the dual space of $H^{1/2}(\Gamma)$ equipped with the natural norm

$$|v|_{H^{-1/2}(\Gamma)} := \sup_{\chi \in H^{1/2}(\Gamma)} \frac{\langle v, \chi\rangle}{|\chi|_{H^{1/2}(\Gamma)}}.$$

For functions $v \in H^2(\Omega)$ the gradient ∇v has a trace $\nabla v|_\Gamma \in H^{1/2}(\Gamma)^2$. On a domain with smooth boundary Γ the outer normal unit vector n is continuous, and thus, the normal derivative $\partial_n v = n \cdot \nabla v \in H^{1/2}(\Gamma)$ is well defined for functions $v \in H^2(\Omega)$ and satisfies

$$|\partial_n v|_{H^{1/2}(\Gamma)} \le c\|v\|_{H^2(\Omega)}, \quad v \in H^2(\Omega). \tag{2.4}$$

This estimate does not make sense if Γ is polygonal, i.e., only Lipschitz continuous. But for $v \in H^2(\Omega)$ we still have $\partial_n v|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$ on each of the straight components $\Gamma_i, i = 1, \dots, m$, of Γ. Accordingly, we introduce the space $\tilde H^{1/2}(\Gamma) := \{q \in L^2(\Gamma),\, q \in H^{1/2}(\Gamma_i),\ i = 1, \dots, m\}$. Further, on $L^2(\Gamma)$, we define the dual norm

$$|v|_{\tilde H^{-1/2}(\Gamma)} := \sup_{\psi \in H^1_0(\Omega)\cap H^2(\Omega)} \frac{\langle v, \partial_n\psi\rangle}{\|\psi\|_{H^2(\Omega)}} \le \sup_{\chi \in \tilde H^{1/2}(\Gamma)} \frac{\langle v, \chi\rangle}{|\chi|_{\tilde H^{1/2}(\Gamma)}}$$

and denote by $\tilde H^{-1/2}(\Gamma)$ the completion of $L^2(\Gamma)$ with respect to this norm. Note that $\tilde H^{-1/2}(\Gamma)$ is in general not the dual space of $\tilde H^{1/2}(\Gamma)$. In the case of a smooth boundary Γ, the mapping $\partial_n : H^1_0(\Omega) \cap H^2(\Omega) \to H^{1/2}(\Gamma)$ is onto, and we therefore have $\tilde H^{-1/2}(\Gamma) = H^{-1/2}(\Gamma)$.

The following lemma states the well-posedness of the general boundary value problem (1.3) in the very weak form. As a special case it also guarantees the existence of the very weak harmonic extension of general boundary data $q \in \tilde H^{-1/2}(\Gamma)$.

LEMMA 2.1. *For any given* $q \in \tilde{H}^{-1/2}(\Gamma)$ *the state equation in its very weak form* (1.3) *possesses a unique solution* $u = u(q) \in L^2(\Omega)$. *There holds the a priori estimate*

$$\|u\| \le c|q|_{\tilde{H}^{-1/2}(\Gamma)} + c\|f\|_{H^{-2}(\Omega)}, \tag{2.5}$$

where $H^{-2}(\Omega)$ *denotes the dual space of* $H_0^1(\Omega) \cap H^2(\Omega)$.

Proof. First, suppose that $q \in H^{1/2}(\Gamma)$ and $f \in H^{-1}(\Omega)$. Then, there exists a unique weak solution $u = u(q) \in H^1(\Omega)$ of the boundary value problem (2.2). By integration by parts, we find that this solution fulfills

$$-(u, \Delta\varphi) + \langle q, \partial_n \varphi\rangle = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega).$$

To prove the a priori estimate, we use a duality argument. Let $w \in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = u \quad \text{in } \Omega, \quad w|_\Gamma = 0.$$

By elliptic regularity, we have $w \in H^2(\Omega)$ and conclude

$$\|u\|^2 = (u, -\Delta w) = (f, w) - \langle q, \partial_n w\rangle.$$

Hence, using the dual norms defined above gives us

$$\|u\|^2 \le c\big\{\|f\|_{H^{-2}(\Omega)} + |q|_{\tilde{H}^{-1/2}(\Gamma)}\big\}\|w\|_{H^2(\Omega)}.$$

Since $\|w\|_{H^2(\Omega)} \le c\|u\|$, the a priori estimate (2.5) follows. Now, since the subspaces $H^{-1}(\Omega) \subset H^{-2}(\Omega)$ and $H^{1/2}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$ are dense, the existence of a solution to the very weak variational problem for given data $f \in H^{-2}(\Omega)$ and $q \in \tilde{H}^{-1/2}(\Gamma)$ follows by a standard continuation argument. The a priori bound (2.5) carries over to these solutions by continuity and therefore also implies uniqueness. ∎

We will call a function $v \in L^2(\Omega)$ "very weakly harmonic" if it satisfies

$$(v, \Delta\varphi) - \langle q, \partial_n \varphi\rangle = 0 \quad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.6}$$

with some function $q \in \tilde{H}^{-1/2}(\Gamma)$. Then, the function q is the very weak trace of v on Γ. For this, almost by definition, we have the following trace estimate:

$$|q|_{\tilde{H}^{-1/2}(\Gamma)} = \sup_{\psi \in H_0^1(\Omega)\cap H^2(\Omega)} \frac{(v, \Delta\psi)}{\|\psi\|_{H^2(\Omega)}} \le c\|v\|. \tag{2.7}$$

We also call $Bq := v$ the "harmonic extension" of the boundary data $q \in \tilde{H}^{-1/2}(\Gamma)$ to Ω. For $q \in H^{1/2}(\Gamma)$ there holds $Bq \in H^1(\Omega)$ and

$$(\nabla Bq, \nabla\varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega), \quad Bq|_\Gamma = q. \tag{2.8}$$

2.2. Properties of the harmonic extension and trace estimates. In the course of the further analysis, we will frequently use the following a priori bounds for the harmonic extension Bq and trace inequalities for the normal derivative $\partial_n v$.

LEMMA 2.2. *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a bounded convex polygonal domain with boundary* Γ. *For* $0 \le s \le 1$ *the harmonic extension is continuously defined from* $H^s(\Gamma)$ *into* $H^{s+1/2}(\Omega)$ *and satisfies*

$$\|Bq\|_{H^{s+1/2}(\Omega)} \le c|q|_{H^s(\Gamma)}. \tag{2.9}$$

If $q \in W^{\sigma,t}(\Gamma)$ *with* $1 < t < 2$ *and* $1/t < \sigma < 1$*, there holds* $Bq \in W^{\sigma+1/t,t}(\Omega)$ *and*

$$\|Bq\|_{W^{\sigma+1/t,t}(\Omega)} \le c\,|q|_{W^{\sigma,t}(\Gamma)}. \tag{2.10}$$

In view of (2.9)*, in the limit case* $t = 2$ *the estimate* (2.10) *holds true even for* $1/2 \le \sigma \le 1$.

Proof. (i) The assertion is proved in Jerison and Kenig [13, 14] (see also Jerison and Kenig [15, p. 165]. Another way to show it is to view the cases $s = 0$ and $s = 1$ as special cases of Theorem 5.15 in Jerison and Kenig [15] and then use operator interpolation theory.

(ii) Due to $\sigma - \frac{1}{t} \ge \frac{1}{2} - \frac{1}{2} = 0$, we have $q \in W^{\sigma,t}(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$ and therefore $Bq \in H^1(\Omega)$. By Theorem 3 in Roßmann [20], we obtain $q \in W^{\sigma,t}(\Gamma) \subset W_\delta^{1-1/t,t}(\Gamma)$ with $\delta = 1-\sigma-1/t$. A regularity result for weighted spaces, see Mazya and Roßmann [18], implies $Bq \in W_\delta^{1,t}(\Omega)$ since $\delta + \frac{2}{t} = 1 - \sigma + \frac{1}{t} < 1$. Using the arguments from the proof of Theorem 8 in Roßmann [20] (also referring to Roßmann [21]), we obtain $Bq \in W^{\sigma+1/t,t}(\Omega)$ and the desired estimate. □

LEMMA 2.3. *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a bounded convex polygonal domain with boundary* Γ*. Let* $2 \le p < p_*^\Omega$*. If* $w \in L^p(\Omega)$*, then the solution* v *to the equation*

$$-\Delta v = w \quad \text{in } \Omega, \quad v|_\Gamma = 0, \tag{2.11}$$

belongs to $W^{2,p}(\Omega)$*, and we have*

$$\|v\|_{W^{2,p}(\Omega)} + |\partial_n v|_{W^{1-1/p,p}(\Gamma)} \le c\|w\|_{L^p(\Omega)}. \tag{2.12}$$

Proof. This lemma is a special case of Lemma A.2 in Casas, Mateos, and Raymond [6]. □

2.3. The optimization problem. For deriving existence results for the optimization problem considered, we define the "solution operator" $S\colon L^2(\Gamma) \to L^2(\Omega)$ by $Sq = u(q) = u$ and

$$-(u, \Delta\varphi) + \langle q, \partial_n\varphi\rangle = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega).$$

This operator is affine linear and continuous due to the a priori estimate (2.5). Then, the optimization problem can be rephrased in the following compact form:

$$j(q) := J(Sq, q) \to \min \quad \text{for } q \in L^2(\Gamma). \tag{2.13}$$

LEMMA 2.4. *The optimization problem* (1.1) *together with the very weak formulation* (1.3) *of the state equation possesses a uniquely determined solution* $\{\hat{u}, \hat{q}\} \in L^2(\Omega) \times L^2(\Gamma)$*. This solution satisfies the necessary and in this case also the sufficient optimality condition*

$$\langle j'(\hat{q}), \chi\rangle = 0 \quad \forall \chi \in L^2(\Gamma), \tag{2.14}$$

with the derivative $j'(\hat{q}) : L^2(\Gamma) \to L^2(\Gamma)^* \simeq L^2(\Gamma)$.

Proof. (i) The proof of existence is by the direct method of variational calculus. The reduced functional $j(\cdot)$ is bounded from below on $L^2(\Gamma)$ and strictly convex since the solution operator S is affine linear. Therefore, it is weakly lower semicontinuous in $L^2(\Gamma)$. Hence, there exists a minimizing sequence $(q_k)_{k\in\mathbb{N}}$, $\inf_{q\in L^2(\Gamma)} j(q) = \lim_{k\to\infty} j(q_k)$, which is bounded in $L^2(\Gamma)$ due to the coercivity of $j(\cdot)$ on $L^2(\Gamma)$. For any of its weak accumulation points $\hat{q}$ there holds $j(\hat{q}) \le \lim_{k\to\infty} j(q_k)$. Such an accumulation point is a unique global minimum of the reduced functional.

(ii) To prove the necessary optimality condition, we note that

$$\frac{1}{\varepsilon}\big(j(\hat{q}+\varepsilon\chi) - j(\hat{q})\big) \ge 0$$

for any $\chi \in L^2(\Gamma)$. Then, letting $\varepsilon \to 0$ yields $\langle j'(\hat{q}), \chi\rangle \ge 0$ for all $\chi \in L^2(\Gamma)$, which implies (2.14). As the reduced cost functional is strictly convex, the condition (2.14) is not only necessary but also sufficient. ∎

LEMMA 2.5. *The directional derivative of $j(\cdot)$ at some point $q \in L^2(\Gamma)$ is given by*

$$j'(q)(\chi) = \alpha\langle q, \chi\rangle - \langle \partial_n z, \chi\rangle \tag{2.15}$$

for $\chi \in L^2(\Gamma)$, where $z = z(q) \in H_0^1(\Omega) \cap H^2(\Omega)$ is the solution of the associated "adjoint problem"

$$-(\psi, \Delta z) = (Sq - u_d, \psi) \quad \forall \psi \in L^2(\Omega). \tag{2.16}$$

Proof. We introduce the Lagrangian functional $\mathcal{L}\colon L^2(\Omega) \times L^2(\Gamma) \times \{H_0^1(\Omega) \cap H^2(\Omega)\} \to \mathbb{R}$ by $\mathcal{L}(u,q,z) := J(u,q) + (f,z) + (u,\Delta z) - \langle q, \partial_n z\rangle$. Then, with $u(q) = Sq$, there holds $j(q) = J(u(q),q) = \mathcal{L}(u(q),q,z(q))$, and for $\chi \in L^2(\Gamma)$

$$\begin{aligned} j'(q)(\chi) &= \mathcal{L}'_u(u(q),q,z(q))(u'_q(q)(\chi)) + \mathcal{L}'_q(u(q),q,z(q))(\chi) \\ &\quad + \mathcal{L}'_z(u(q),q,z(q))(z'_q(q)(\chi)). \end{aligned}$$

Since by construction $\mathcal{L}'_u(u(q),q,z(q))(\cdot) = 0$ and $\mathcal{L}'_z(u(q),q,z(q))(\cdot) = 0$, we obtain

$$j'(q)(\chi) = \mathcal{L}'_q(u(q),q,z(q))(\chi) = \alpha\langle \chi, q\rangle - \langle \chi, \partial_n z\rangle,$$

which proves the asserted representation. ∎

As a consequence of the foregoing lemmas, we have the following result.

LEMMA 2.6. *The solution of the optimization problem* (1.1), (1.3) *is characterized by the Euler–Lagrange principle stating that the pair $\{\hat{u},\hat{q}\} \in L^2(\Omega) \times L^2(\Gamma)$ is a solution if and only if there exists an "adjoint state" $\hat{z} \in H_0^1(\Omega) \cap H^2(\Omega)$ such that the triplet $\{\hat{u},\hat{q},\hat{z}\}$ solves the "optimality system" (KKT system)*

$$-(\hat{u}, \Delta\varphi) + \langle \hat{q}, \partial_n\varphi\rangle = (f,\varphi) \qquad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.17}$$

$$\alpha\langle \hat{q}, \chi\rangle - \langle \partial_n \hat{z}, \chi\rangle = 0 \qquad \forall \chi \in L^2(\Gamma), \tag{2.18}$$

$$-(\psi, \Delta\hat{z}) - (\hat{u}, \psi) = -(u_d, \psi) \qquad \forall \psi \in L^2(\Omega). \tag{2.19}$$

Proof. Let $\{\hat{u},\hat{q}\} \in L^2(\Omega) \times L^2(\Gamma)$ be a solution of the optimization problem. Then, by definition, (2.17) and (2.19) hold. The necessary condition (2.14) and the representation (2.15) of $j'(\cdot)$ imply (2.18). In turn, for each solution $\{\hat{u},\hat{q},\hat{z}\} \in L^2(\Omega) \times L^2(\Gamma) \times \{H_0^1(\Omega) \cap H^2(\Omega)\}$ of the KKT system, the necessary (and sufficient) optimality condition (2.14) is satisfied, implying that $\hat{q}$ is a minimum. ∎

2.4. Galerkin finite element approximation. For the approximation of the optimization problem (1.1), (1.3), we consider a finite element method. Let $V_h \subset H^1(\Omega)$ be finite element subspaces defined on meshes $\mathbb{T}_h = \{K\}$ consisting of (closed) triangles or quadrilaterals and satisfying the usual conditions of shape and size regularity. These meshes are characterized by the mesh-width parameter $h \in (0,1]$, where

$h \approx \max\{\mathrm{diam}(K),\, K \in \mathbb{T}_h\}$. Further, we set $V_{h,0} := V_h \cap H_0^1(\Omega)$ and let V_h^∂ be the trace space corresponding to V_h. For simplicity, we consider only lowest-order finite elements, i.e., piecewise linear or bilinear trial and test functions. The discrete solutions $\{\hat{u}_h, \hat{q}_h\} \in V_h \times V_h^\partial$ are determined by the discrete optimization problems

$$J(\hat{u}_h, \hat{q}_h) = \min_{u_h \in V_h, q_h \in V_h^\partial} J(u_h, q_h), \tag{2.20}$$

under the constraints

$$(\nabla u_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_{h,0}, \quad u_h|_\Gamma = q_h. \tag{2.21}$$

By arguments analogous to these used for the continuous optimization problem (1.1), (1.3), we see that their discrete counterparts (2.20), (2.21) possess uniquely determined solutions. These solutions may be computed by applying the gradient or the Newton method for the discrete reduced functional $j_h(q_h) := J(S_h q_h, q_h)$, which requires the evaluation of first and second directional derivatives of $j_h(\cdot)$. Here, $S_h : V_h^\partial \to V_h$ denotes the discrete solution operator defined by $S_h q_h = u_h(q_h) = u_h$ and (2.21). This equation may be rewritten in the form

$$(\nabla v_h, \nabla \varphi_h) + (\nabla B_h q_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_{h,0} \tag{2.22}$$

for the function $v_h := u_h - B_h q_h \in V_{h,0}$, where $B_h : V_h^\partial \to V_h$ is an arbitrary extension operator of discrete boundary data to all of $\bar{\Omega}$. (Later on, for simplicity we will specify B_h to be the "discrete harmonic extension" defined in (2.44) below.)

LEMMA 2.7. *With the foregoing notation the first directional derivative of $j_h(\cdot)$ at some point $q_h \in V_h^\partial$ is given by*

$$j_h'(q_h)(\chi_h) = \alpha\langle \chi_h, q_h\rangle + (B_h \chi_h, u_h - u_d) - (\nabla B_h \chi_h, \nabla z_h) \tag{2.23}$$

for $\chi_h \in V_h^\partial$, where $u_h = S_h q_h$ and $z_h = z_h(q_h) \in V_{h,0}$ is the solution of the "discrete adjoint problem"

$$(\nabla \psi_h, \nabla z_h) = (u_h - u_d, \psi_h) \quad \forall \psi_h \in V_{h,0}. \tag{2.24}$$

Proof. The argument is analogous to that used in Lemma 2.5 on the continuous level. For the discrete Lagrangian functional $\mathcal{L}_h : V_{h,0} \times V_h^\partial \times V_{h,0} \to \mathbb{R}$, defined by

$$\mathcal{L}_h(v_h, q_h, z_h) := J(v_h + B_h q_h, q_h) + (f, z_h) - (\nabla v_h, \nabla z_h) - (\nabla B_h q_h, \nabla z_h),$$

we have $j_h(q_h) = J(v_h + B_h q_h, q_h) = \mathcal{L}_h(v_h, q_h, z_h(q_h))$. Further,

$$\begin{aligned} j_h'(q_h)(\chi_h) &= \mathcal{L}'_{h,v}(v_h, q_h, z_h(q_h))(v'_{h,q}(\chi_h)) + \mathcal{L}'_{h,q}(v_h, q_h, z_h(q_h))(\chi_h) \\ &\quad + \mathcal{L}'_{h,z}(v_h, q_h, z_h(q_h))(z'_{h,q}(q_h)(\chi_h)) \\ &= \mathcal{L}'_{h,q}(v_h, q_h, z_h(q_h))(\chi_h) \\ &= (v_h + B_h q_h - u_d, B_h \chi_h) + \alpha\langle \chi_h, q_h\rangle - (\nabla B_h \chi_h, \nabla z_h) \\ &= (u_h - u_d, B_h \chi_h) + \alpha\langle \chi_h, q_h\rangle - (\nabla B_h \chi_h, \nabla z_h), \end{aligned}$$

which proves the asserted representation. ∎

As a consequence of the foregoing lemma, we obtain the following result, which is analogous to the corresponding one on the continuous level, Lemma 2.6.

Lemma 2.8. *The solution of the discrete optimization problem* (2.20), (2.21) *is characterized by the Euler–Lagrange principle stating that the pair* $\{\hat u_h, \hat q_h\} \in V_h \times V_h^\partial$ *is a solution if and only if there exists an adjoint state* $\hat z_h \in V_{h,0}$ *such that the triplet* $\{\hat u_h, \hat q_h, \hat z_h\}$ *solves the discrete KKT system*

$$\hat u_h|_\Gamma = \hat q_h, \quad (\nabla \hat u_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.25}$$

$$\alpha\langle \hat q_h, \chi_h\rangle + (\hat u_h, B_h\chi_h) - (\nabla B_h\chi_h, \nabla \hat z_h) = (u_d, B_h\chi_h) \qquad \forall \chi_h \in V_h^\partial, \tag{2.26}$$

$$(\nabla\psi_h, \nabla\hat z_h) - (\hat u_h, \psi_h) = -(u_d, \psi_h) \qquad \forall \psi_h \in V_{h,0}. \tag{2.27}$$

The solution $\{\hat u_h, \hat q_h, \hat z_h\}$ *is independent of the particular choice of the extension operator* B_h.

Proof. We show the independence of the extension operator. The unique solvability of the system can then be established using the special choice of B_h as the discrete harmonic extension. The statement of the theorem then follows by arguments analogous to Lemma 2.6. To prove the independence of the extension operator, we assume that $\{\hat u_h, \hat q_h, \hat z_h\}$ is the solution to extension operator B_h^1 and denote by B_h^2 another extension operator. We need to show that $\{\hat u_h, \hat q_h, \hat z_h\}$ fulfill (2.26) with $B_h = B_h^2$. Using that (2.26) is true for B_h^1, we get

$$\begin{aligned}\alpha\langle \hat q_h, \chi_h\rangle &+ (\hat u_h, B_h^2\chi_h) - (\nabla B_h^2\chi_h, \nabla\hat z_h) - (u_d, B_h^2\chi_h) - 0\\ &= (\hat u_h, (B_h^2\chi_h - B_h^1\chi_h)) - (\nabla(B_h^2\chi_h - B_h^1\chi_h), \nabla\hat z_h) - (u_d, (B_h^2\chi_h - B_h^1\chi_h)).\end{aligned}$$

As $B_h^2\chi_h - B_h^1\chi_h \in V_{h,0}$, this evaluates to zero by the adjoint equation (2.27). ∎

The numerical results for this approximation presented in section 6 suggest the following partially "optimal" rates of convergence under generic assumptions on the regularity of the solution:

$$h^{1-1/r}|\hat q - \hat q_h| + h^{1/2-1/r}\|\hat u - \hat u_h\| + \|\hat z - \hat z_h\| = \mathcal{O}(h^{2-1/p-1/r}). \tag{2.28}$$

Here, $2 \le r < p_*^\Omega \le \infty$ and $2 \le p < p_* := \min\{p_*^d, p_*^\Omega\}$ are essentially determined by the regularity $\hat z \in H_0^1(\Omega) \cap W^{2,p_*}(\Omega)$, depending on the maximum interior angle $\omega_{\max}$ of the polygonal domain Ω like

$$p_*^\Omega = \frac{2\omega_{\max}}{2\omega_{\max} - \pi}, \tag{2.29}$$

including the special case $p_*^\Omega = \infty$ for $\omega = \frac{\pi}{2}$, and the regularity of the data $u_d \in L^{p_*^d}(\Omega)$, $p_*^d > 2$. These convergence rates turn out to be better for weaker error measures; e.g., for the mean values,

$$|\langle \hat q - \hat q_h, 1\rangle| + |(\hat u - \hat u_h, 1)| = \mathcal{O}(h^{2-1/p-1/r}). \tag{2.30}$$

It is the main goal of the following analysis to provide theoretical support for these practically observed convergence rates. This will also cover the case of solutions with reduced regularity induced by irregular data.

2.5. The KKT systems of the optimization problems. The error analysis of the finite element approximation (2.20), (2.21) of the optimization problem (1.1), (1.3) is based on its equivalent formulation in terms of the corresponding KKT systems. We begin by recasting the KKT system (2.17), (2.18), (2.19) in a form which can be approximated by a standard finite element method using only continuous trial and test functions. This is possible, since the solution of the very weak optimality system (2.17), (2.18), (2.19) turns out to be more regular than required for its definition. For later use, we determine its degree of regularity, which is guaranteed in general on a convex polygonal domain. Actually, the regularity of the solution pair $\{\hat{u}, \hat{q}\}$ is essentially determined by that of the adjoint state $\hat{z}$.

LEMMA 2.9. *Suppose that* $f \in L^2(\Omega)$ *and* $u_d \in L^{p_*^d}(\Omega)$, $p_*^d > 2$. *Let* $p_*^\Omega \geq 2$ *be defined by* (2.29) *and* $p_* := \min\{p_*^d, p_*^\Omega\}$. *Then, the solution* $\{\hat{u}, \hat{q}\} \in L^2(\Omega) \times L^2(\Gamma)$ *of the optimization problem* (1.1), (1.3) *and the associated adjoint state* $\hat{z} \in H_0^1(\Omega) \cap H^2(\Omega)$ *determined by* (2.16) *have the additional regularity properties*

$$\{\hat{u}, \hat{q}\} \in H^{3/2-1/p}(\Omega) \times H^{1-1/p}(\Gamma), \quad \hat{z} \in W^{2,p}(\Omega), \quad 2 \leq p < p_*. \tag{2.31}$$

Proof. A similar regularity statement was shown in Casas and Raymond [7, Theorem 3.4], but for completeness we give a slightly different proof here. By Lemma 2.6, the triplet $\{\hat{u}, \hat{q}, \hat{z}\}$ satisfies (2.17), (2.18), and (2.19). Since $\hat{q} \in L^2(\Gamma)$, we have $\hat{u} \in H^{1/2}(\Omega)$ by Lemma 2.2. Therefore $\hat{u}$ lies in $L^p(\Omega)$ for some $p > 2$. Let this p be chosen such that $p < p_*$. This in turn implies by Lemma 2.3 that $\hat{z} \in W^{2,p}(\Omega)$ and $\partial_n \hat{z} \in W^{1-1/p,p}(\Gamma)$ for this $p > 2$. Then, from (2.18), we infer that $\hat{q} \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma) \subset H^{1/2}(\Gamma)$. Using this in (2.17) yields $\hat{u} \in H^1(\Omega)$; i.e., $\hat{u}$ is the usual weak H^1 solution of the boundary value problem (1.2). By elliptic regularity theory, this implies $\hat{z} \in W^{2,p}(\Omega)$ for $2 \leq p < p_*$. Finally, in view of Lemma 2.2, by elliptic regularity theory, $\hat{u}|_\Gamma = \hat{q} \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma)$ implies that $\hat{u} \in H^{3/2-1/p}(\Omega)$, which completes the proof. □

Remark 1. In view of Lemma 2.9, the regularity to be expected for the optimal solution $\{\hat{u}, \hat{q}, \hat{z}\}$ is limited by the parameter $p_* := \min\{p_*^d, p_*^\Omega\}$, depending on the regularity of the data and the domain. The right-hand side in the equation for $\hat{z}$ is $\hat{u} - u_d$. Hence, under the mere assumption that $u_d \in L^\infty(\Omega)$, in general we cannot expect $\hat{z} \in W^{2,\infty}(\Omega)$ or higher regularity, even on a rectangle. This restricts all our results to the case $2 \leq p < \infty$ with constants blowing up as $p \to \infty$. However, in the special case $\omega_{\max} = \frac{\pi}{2}$, we have (2.31) with $p = \infty$, provided that $u_d \in C^\gamma(\bar{\Omega})$ for some $\gamma > 0$ and $u_d(x_i) = 0$ in all cornerpoints x_i of Ω. This follows from Grisvard [11] due to the fact that $\hat{u}(x_i) = \hat{q}(x_i) = \alpha^{-1}\partial_n \hat{z}(x_i) = 0$.

Next, we rewrite (2.18) using (2.19) and the harmonic extension $B\chi \in H^1(\Omega)$ of the boundary function $\chi \in H^{1/2}(\Gamma)$ defined by (2.8), to obtain

$$\begin{aligned} \alpha\langle \hat{q}, \chi\rangle - \langle \partial_n \hat{z}, \chi\rangle &= \alpha\langle \hat{q}, \chi\rangle - (\Delta \hat{z}, B\chi) - (\nabla \hat{z}, \nabla B\chi) \\ &= \alpha\langle \hat{q}, \chi\rangle + (\hat{u} - u_d, B\chi). \end{aligned}$$

Hence, in view of Lemma 2.9, the solution $\{\hat{u}, \hat{q}, \hat{z}\} \in H^{3/2-1/p}(\Omega) \times H^{1-1/p}(\Gamma) \times [H_0^1(\Omega) \cap W^{2,p}(\Omega)]$ of (2.17), (2.18), (2.19) also satisfies the following set of equations:

$$\hat{u}|_\Gamma = \hat{q}, \quad (\nabla \hat{u}, \nabla \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.32}$$

$$\alpha\langle \hat{q}, \chi\rangle + (\hat{u}, B\chi) = (u_d, B\chi) \qquad \forall \chi \in H^{1/2}(\Gamma), \tag{2.33}$$

$$(\nabla \psi, \nabla \hat{z}) - (\hat{u}, \psi) = -(u_d, \psi) \qquad \forall \psi \in H_0^1(\Omega). \tag{2.34}$$

We note that the additional regularity of the optimal pair $(\hat{u}, \hat{q})$ allows for this formulation, avoiding the use of very weak solutions. In order to remove the nonhomogeneous boundary condition, we introduce the function $\hat{v} := \hat{u} - B\hat{q} \in H_0^1(\Omega)$. Then, the triplet

$\{\hat{v}, \hat{q}, \hat{z}\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ satisfies the system

$$(\nabla \hat{v}, \nabla \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.35}$$

$$\alpha \langle \hat{q}, \chi \rangle + (\hat{v} + B\hat{q}, B\chi) = (u_d, B\chi) \qquad \forall \chi \in H^{1/2}(\Gamma), \tag{2.36}$$

$$(\nabla \psi, \nabla \hat{z}) - (\hat{v} + B\hat{q}, \psi) = -(u_d, \psi) \qquad \forall \psi \in H_0^1(\Omega). \tag{2.37}$$

The corresponding finite element approximation $\{\hat{u}_h, \hat{q}_h, \hat{z}_h\} \in V_h \times V_h^\partial \times V_{h,0}$ is characterized by the discrete KKT system (see Lemma 2.8)

$$\hat{u}_h|_\Gamma = \hat{q}_h, \quad (\nabla \hat{u}_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.38}$$

$$\alpha \langle \hat{q}_h, \chi_h \rangle + (\hat{u}_h, B_h\chi_h) - (\nabla \hat{z}_h, \nabla B_h \chi_h) = (u_d, B_h\chi_h) \qquad \forall \chi_h \in V_h^\partial, \tag{2.39}$$

$$(\nabla \psi_h, \nabla \hat{z}_h) - (\hat{u}_h, \psi_h) = -(u_d, \psi_h) \qquad \forall \psi_h \in V_{h,0}, \tag{2.40}$$

or, incorporating the nonhomogeneous boundary condition for $\hat{u}_h$ into the variational formulation,

$$(\nabla \hat{v}_h, \nabla \varphi_h) + (\nabla B_h \hat{q}_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.41}$$

$$\alpha \langle \hat{q}_h, \chi_h \rangle + (\hat{v}_h + B_h\hat{q}_h, B_h\chi_h) - (\nabla \hat{z}_h, \nabla B_h \chi_h) = (u_d, B_h\chi_h) \quad \forall \chi_h \in V_h^\partial, \tag{2.42}$$

$$(\nabla \psi_h, \nabla \hat{z}_h) - (\hat{v}_h + B_h\hat{q}_h, \psi_h) = -(u_d, \psi_h) \qquad \forall \psi_h \in V_{h,0}, \tag{2.43}$$

where $\hat{v}_h := \hat{u}_h - B_h\hat{q}_h \in V_{h,0}$. The solution of this system is independent of the particular choice of the extension operator $B_h : V_h^\partial \to V_h$.

From now on, we choose B_h to be the "discrete harmonic extension" defined by

$$(\nabla B_h q_h, \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \quad B_h q_h|_\Gamma = q_h. \tag{2.44}$$

Then, the system (2.41), (2.42), (2.43) reduces to

$$(\nabla \hat{v}_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.45}$$

$$\alpha \langle \hat{q}_h, \chi_h \rangle + (\hat{v}_h + B_h\hat{q}_h, B_h\chi_h) = (u_d, B_h\chi_h) \qquad \forall \chi_h \in V_h^\partial, \tag{2.46}$$

$$(\nabla \psi_h, \nabla \hat{z}_h) - (\hat{v}_h + B_h\hat{q}_h, \psi_h) = -(u_d, \psi_h) \qquad \forall \psi_h \in V_{h,0}. \tag{2.47}$$

Writing the equations for $\{\hat{v}, \hat{q}, \hat{z}\}$ for discrete test functions and subtracting the corresponding discrete equations (2.45), (2.46), (2.47) yields the following equations for the errors $e_v := \hat{v} - \hat{v}_h$, $e_q := \hat{q} - \hat{q}_h$, and $e_z := \hat{z} - \hat{z}_h$:

$$(\nabla e_v, \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \tag{2.48}$$

$$\alpha \langle e_q, \chi_h \rangle + (\hat{v} + B\hat{q} - u_d, B\chi_h) - (\hat{v}_h + B_h\hat{q}_h - u_d, B_h\chi_h) = 0 \quad \forall \chi_h \in V_h^\partial, \tag{2.49}$$

$$(\nabla e_z, \nabla \psi_h) - (e_v + B\hat{q} - B_h\hat{q}_h, \psi_h) = 0 \quad \forall \psi_h \in V_{h,0}. \tag{2.50}$$

Remark 2. Notice that, since $B_h \neq B$, the system (2.45), (2.46), (2.47) is *not* the Galerkin approximation of (2.35), (2.36), (2.37). In this situation the general paradigm that "Galerkin discretization" and "optimization" (i.e., forming the necessary optimality condition) commute does not hold. This essentially complicates the error analysis, as several additional terms need to be estimated, which originate from the lacking Galerkin orthogonality of the approximation.

Remark 3. The choice of $B\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ and $B_h : V_h^\partial \to V_h$ as the harmonic (respectively, discrete harmonic) extension operators is for the convenience of the argument used in the following error analysis. Actually, the continuous as well as the discrete optimal solutions are independent of this particular choice. For practical computations B_h is usually chosen to satisfy $B_h q_h(a_i) = 0$ in all interior nodal points a_i of the finite element mesh.

3. Auxiliary estimates.

3.1. Auxiliary error and stability estimates. Next, we collect some known results on the approximation behavior of finite element methods.

(I) We will use the following "inverse estimate" for finite element functions $\chi_h \in V_h^\partial$ (notice that Γ is one-dimensional):

$$|\chi_h|_{W^{r,t}(\Gamma)} \le ch^{s-r}|\chi_h|_{W^{s,t}(\Gamma)} \tag{3.1}$$

for $0 \le s \le r \le 1$, $1 \le t < \infty$. This can be proven by combining estimates in Ciarlet [9] and Brenner and Scott [5] with standard results from interpolation theory.

(II) Let $I_h : C(\bar{\Omega}) \to V_h$ and $I_h : C(\Gamma) \to V_h^\partial$ denote the natural nodal interpolation operators which satisfy (see Ciarlet [9] and Brenner and Scott [5])

$$\|\nabla(v - I_h v)\|_{L^p(\Omega)} \le ch\|v\|_{W^{2,p}(\Omega)} \tag{3.2}$$

for $v \in W^{2,p}(\Omega)$, $1 \le p \le \infty$. The interpolation operator I_h is local but only defined for continuous functions. A locally defined quasi-interpolation operator $\tilde{I}_h : L^2(\Omega) \to V_h$ has been devised by Scott and Zhang [22]. This operator preserves (polynomial) boundary conditions and satisfies $\tilde{I}_h|_{V_h} = id$. For this, the same estimate (3.2) holds as for the standard nodal interpolation I_h, and in addition the estimate

$$\|\nabla(v - \tilde{I}_h v)\|_{L^p(\Omega)} \le ch^s\|v\|_{W^{1+s,p}(\Omega)} \tag{3.3}$$

for $v \in W^{1+s,p}(\Omega)$, $0 \le s \le 1$, $1 \le p \le \infty$.

(III) The L^2 projection $P_h^\partial : L^2(\Gamma) \to V_h^\partial$ is defined by

$$\langle q - P_h^\partial q, \chi_h\rangle = 0 \quad \forall \chi_h \in V_h^\partial.$$

By standard results for finite elements we have the error estimate (see Ciarlet [9], Brenner and Scott [5], and Casas and Raymond [8])

$$|q - P_h^\partial q|_{L^p(\Gamma)} + h^s|P_h^\partial q|_{W^{s,p}(\Gamma)} \le ch^s|q|_{W^{s,p}(\Gamma)} \tag{3.4}$$

for $q \in W^{s,p}(\Gamma)$, $0 \le s \le 1$, $1 < p < \infty$.

(IV) For a function $u \in H_0^1(\Omega)$ let $R_h^D u \in V_{h,0}$ denote the corresponding "Ritz projection" ("Dirichlet projection") defined by

$$(\nabla(u - R_h^D u), \nabla\varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}.$$

We recall the following standard results from the literature.

LEMMA 3.1. *On a convex polygonal domain, the Ritz projection $R_h^D : H_0^1(\Omega) \to V_{h,0}$ satisfies the stability estimate*

$$\|\nabla R_h^D u\|_{L^p(\Omega)} \le c\|\nabla u\|_{L^p(\Omega)}, \quad 1 < p \le \infty, \tag{3.5}$$

for $u \in W^{1,p}(\Omega)$. Furthermore, the following error estimate holds:

$$\|u - R_h^D u\|_{L^p(\Omega)} + h\|u - R_h^D u\|_{W^{1,p}(\Omega)} \le ch^2\|u\|_{W^{2,p}(\Omega)}, \quad 2 \le p < p_*^\Omega. \tag{3.6}$$

Proof. For a proof, we refer to Brenner and Scott [5]; see also Rannacher and Scott [19]. ∎

3.2. Properties of discrete harmonic extension. In this section, we provide some a priori bounds and error estimates for the discrete harmonic extension. We note that here the parameter $p \in [2, p_*^\Omega)$ may be limited only by the interior angles of the polygonal domain, i.e., the size of $p_*^\Omega \in [2, \infty)$ (including the case $p_*^\Omega = \infty$) determined by (2.29). It is independent of the assumed regularity of the data.

LEMMA 3.2. *For the discrete harmonic extension $B_h q_h \in V_h$ of the boundary data $q_h \in V_h^\partial$ the following a priori estimates hold:*

$$\|\nabla B_h q_h\| \le c|q_h|_{H^{1/2}(\Gamma)}, \tag{3.7}$$

$$\|B_h q_h\| \le c|q_h|, \tag{3.8}$$

$$\|\nabla B_h q_h\|_{L^p(\Omega)} \le c h^{1/p-1} |q_h|_{L^p(\Gamma)}, \quad 1 < p < \infty. \tag{3.9}$$

Proof. (i) The poof proceeds by referring back to the corresponding estimates for B. For the modified interpolation operator $\tilde{I}_h$ introduced in section 3.1(II), $(B_h - \tilde{I}_h B)q_h|_\Gamma = 0$. Hence by the properties of B_h and the estimate (3.3) for $s = 0$, we have

$$\begin{aligned}\|\nabla B_h q_h\|^2 &= (\nabla B_h q_h, \nabla(B_h - \tilde{I}_h B)q_h) + (\nabla B_h q_h, \nabla \tilde{I}_h B q_h)\\ &\le \|\nabla B_h q_h\| \|\nabla \tilde{I}_h B q_h\| \le c\|\nabla B_h q_h\| \|B q_h\|_{H^1(\Omega)}.\end{aligned}$$

Then, the stability estimate (2.9) yields (3.7),

$$\|\nabla B_h q_h\| \le c\|B q_h\|_{H^1(\Omega)} \le c|q_h|_{H^{1/2}(\Gamma)}.$$

(ii) Next, let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = B_h q_h \quad \text{in } \Omega, \quad w|_\Gamma = 0.$$

Then, using several of the foregoing estimates in a standard way, we obtain

$$\begin{aligned}\|B_h q_h\|^2 &= (B_h q_h, -\Delta w) = (\nabla B_h q_h, \nabla w) - \langle q_h, \partial_n w\rangle\\ &= (\nabla B_h q_h, \nabla(w - R_h^D w)) - \langle q_h, \partial_n w\rangle\\ &\le \|\nabla B_h q_h\| \|\nabla(w - R_h^D w)\| + |q_h| \|w\|_{H^2(\Omega)}\\ &\le c\big\{h|q_h|_{H^{1/2}(\Gamma)} + |q_h|\big\}\|w\|_{H^2(\Omega)}\\ &\le c|q_h| \|B_h q_h\|.\end{aligned}$$

This yields (3.8).

(iii) To prove (3.9), we recall the estimate

$$\|\nabla v\|_{L^p(\Omega)} \le c \sup_{w \in W_0^{1,q}(\Omega)} \frac{(\nabla v, \nabla w)}{\|\nabla w\|_{L^q(\Omega)}}, \tag{3.10}$$

which holds for $1 < p < \infty$ and $q = p(p-1)^{-1}$, particularly on convex polygonal domains. This follows from a result in Alkhutov and Kondratev [2], which states that the boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u|_\Gamma = 0,$$

possesses for $f \in W^{-1,p}(\Omega)$ a uniquely determined solution $u \in W_0^{1,p}(\Omega)$ satisfying

$$\|u\|_{W^{1,p}(\Omega)} \le c\|f\|_{W^{-1,p}(\Omega)}.$$

For given $q_h \in V_h^\partial$ let $\tilde{B}_h q_h \in V_h$ denote that extension which coincides with q_h at each nodal point $a_i \in \Gamma$ but vanishes at each interior nodal point $a_i \in \Omega$. Then, for $v_h := B_h q_h - \tilde{B}_h q_h \in V_{h,0}$ there holds

$$(\nabla v_h, \nabla\varphi_h) = (\nabla B_h q_h, \nabla\varphi_h) - (\nabla\tilde{B}_h q_h, \nabla\varphi_h) = -(\nabla\tilde{B}_h q_h, \nabla\varphi_h), \quad \varphi_h \in V_{h,0}.$$

Now, let $v \in H_0^1(\Omega)$ be defined by the equation

$$(\nabla v, \nabla\varphi) = -(\nabla\tilde{B}_h q_h, \nabla\varphi) \quad \forall\varphi \in H_0^1(\Omega).$$

Then, v_h can be viewed as the Ritz projection of v. The estimate (3.10) implies that $\|\nabla v\|_{L^p(\Omega)} \le c\|\nabla\tilde{B}_h q_h\|_{L^p(\Omega)}$. Then, by Lemma 3.1, we obtain the estimate

$$\|\nabla v_h\|_{L^p(\Omega)} \le c\|\nabla\tilde{B}_h q_h\|_{L^p(\Omega)}, \quad 1 < p < \infty.$$

This implies that

$$\|\nabla B_h q_h\|_{L^p(\Omega)} \le \|\nabla v_h\|_{L^p(\Omega)} + \|\nabla\tilde{B}_h q_h\|_{L^p(\Omega)} \le c\|\nabla\tilde{B}_h q_h\|_{L^p(\Omega)}. \tag{3.11}$$

Therefore it remains to estimate the norm $\|\nabla\tilde{B}_h q_h\|_{L^p(\Omega)}$, which is localized to a strip S_h along the boundary of width h. For this purpose let $T \in \mathbb{T}_h$ be a cell of the decomposition of $\bar{\Omega}$, which nontrivially intersects the boundary: $\Gamma_T := T \cap \Gamma \neq \emptyset$. Then, by a standard argument employing transformations to a reference unit cell, we find

$$\|\nabla\tilde{B}_h q_h\|_{L^p(T)}^p \le ch^{1-p}\|q_h\|_{L^p(\Gamma_T)}^p,$$

and summing this over all such cells belonging to the strip S_h,

$$\|\nabla\tilde{B}_h q_h\|_{L^p(S_h)} \le ch^{1/p-1}|q_h|_{L^p(\Gamma)}.$$

This proves the asserted estimate. ∎

Lemma 3.3. *Let $B\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ be the harmonic extension defined in* (2.8), *$B_h\colon V_h^\partial \to V_h$ its discrete analogue from* (2.44), *and $q_h \in V_h^\partial$.*

(i) *For $2 \le p \le \infty$ there holds*

$$\|\nabla(B - B_h)q_h\| \le ch^{1/2-1/p}|q_h|_{H^{1-1/p}(\Gamma)}. \tag{3.12}$$

(ii) *For $1 < t < 2$ and $\frac{2}{t} - 1 \le s < \frac{1}{t}$ there holds*

$$\|\nabla(B - B_h)q_h\|_{L^t(\Omega)} \le ch^s|q_h|_{W^{s+1-1/t,t}(\Gamma)}. \tag{3.13}$$

In view of (3.12), *in the limit case $t = 2$, the estimate* (3.13) *holds true also for $s = 1/2$.*

(iii) *For $\psi \in L^\tau(\Omega)$, $2 \le \tau \le \theta < p_*^\Omega$, there holds*

$$(\psi, (B - B_h)q_h) \le ch^{2-1/\tau-1/\theta}|q_h|_{H^{1-1/\theta}(\Gamma)}\|\psi\|_{L^\tau(\Omega)}. \tag{3.14}$$

(iv) *Particularly, for $2 \le p < p_*^\Omega$ there holds*

$$\|(B - B_h)q_h\| \le ch^{3/2-1/p}|q_h|_{H^{1-1/p}(\Gamma)}. \tag{3.15}$$

Proof. Notice that $B_h q_h \in V_h$ is just the Ritz projection of Bq_h corresponding to the same boundary values $q_h \in V_h^\partial$,

$$(\nabla(B - B_h)q_h, \nabla\varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \quad (B - B_h)q_h|_\Gamma = 0.$$

(i) For the modified interpolation operator $\tilde{I}_h$ introduced in section 3.1(II), $(\tilde{I}_h B - B_h)q_{h|\Gamma} = 0$ holds. Hence,

$$\begin{aligned}\|\nabla(B - B_h)q_h\|^2 &= (\nabla(B - B_h)q_h, \nabla(B - \tilde{I}_h B)q_h)\\ &\le \|\nabla(B - B_h)q_h\|\|\nabla(B - \tilde{I}_h B)q_h\|.\end{aligned}$$

The interpolation estimate (3.3) together with the stability estimate (2.9) yields

$$\begin{aligned}\|\nabla(B - \tilde{I}_h B)q_h\| &\le ch^{1/2-1/p}\|Bq_h\|_{H^{3/2-1/p}(\Omega)}\\ &\le ch^{1/2-1/p}|q_h|_{H^{1-1/p}(\Gamma)},\end{aligned}$$

which proves the estimate (3.12).

(ii) For $e = (B - B_h)q_h \in H_0^1(\Omega)$, we have

$$(\nabla e, \nabla\varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}.$$

Using the quasi interpolation $\tilde{I}_h$ defined above, we split

$$e = (Bq_h - \tilde{I}_h Bq_h) + (\tilde{I}_h Bq_h - B_h q_h) = \xi + \eta.$$

Now $\eta = -R_h^D \xi$. Using the stability of R_h^D from Lemma 3.1 with respect to the $W^{1,t}$-seminorm, we get

$$\|\nabla e\|_{L^t(\Omega)} \le c\|\nabla\xi\|_{L^t(\Omega)}.$$

Then, by the interpolation estimate (3.3),

$$\|\nabla e\|_{L^t(\Omega)} \le ch^s\|Bq_h\|_{W^{s+1,t}(\Omega)}.$$

Finally, using estimate (2.10) in Lemma 2.2 with $\sigma = s + 1 - \frac{1}{t} \in (1/t, 1)$ yields

$$\|Bq_h\|_{W^{s+1,t}(\Omega)} \le c|q_h|_{W^{s+1-1/t,t}(\Gamma)}.$$

This proves (3.13) for the parameter ranges $1 < t < 2$ and $\frac{2}{t} - 1 < s < \frac{1}{t}$. In order to extend the validity of (3.13) to the lower limit case $s = \frac{2}{t} - 1$, let $s = \frac{2}{t} - 1 + \delta$ with some $\delta > 0$. Then, by the inverse relation for finite elements (3.1),

$$\|\nabla(B - B_h)q_h\|_{L^t(\Omega)} \le ch^{\frac{2}{t}-1+\delta}|q_h|_{W^{\frac{1}{t}+\delta,t}(\Gamma)} \le ch^{\frac{2}{t}-1}|q_h|_{W^{\frac{1}{t},t}(\Gamma)}.$$

(iii) For an arbitrary but fixed $\psi \in L^\tau(\Omega)$ let $w \in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = \psi \quad \text{in } \Omega, \quad w|_\Gamma = 0$$

satisfying $w \in W^{2,\tau}(\Omega)$ for $2 \le \tau \le \theta < p_*^\Omega$ and $\|w\|_{W^{2,\tau}(\Omega)} \le c\|\psi\|_{L^\tau(\Omega)}$ (see Grisvard [11]). We obtain

$$\begin{aligned}(\psi, (B - B_h)q_h) &= (\nabla w, \nabla(B - B_h)q_h)\\ &= (\nabla(w - R_h^D w), \nabla(B - B_h)q_h)\\ &\le \|\nabla(w - R_h^D w)\|_{L^\tau(\Omega)}\|\nabla(B - B_h)q_h\|_{L^{\tau'}(\Omega)}\end{aligned}$$

with $1/\tau + 1/\tau' = 1$. Due to Lemma 3.1, we have

$$\|\nabla(w - R_h^D w)\|_{L^\tau(\Omega)} \le ch\|\nabla^2 w\|_{L^\tau(\Omega)} \le ch\|\psi\|_{L^\tau(\Omega)}.$$

For the term $\|\nabla(B - B_h)q_h)\|_{L^{\tau'}(\Omega)}$, we use the estimate (3.13) with $t = \tau'$ and $s = 1 - 1/\theta - 1/\tau$. Since, by assumption, $2 \le \tau \le \theta < p_*^\Omega$, we have

$$s - \frac{2}{t} + 1 = 1 - \frac{1}{\theta} - \frac{1}{\tau} - \frac{2}{\tau'} + 1 = 2 - \frac{1}{\theta} - \frac{1}{\tau} - 2 + \frac{2}{\tau} = \frac{1}{\tau} - \frac{1}{\theta} \ge 0$$

and

$$\frac{1}{t} - s = \frac{1}{\tau'} - 1 + \frac{1}{\theta} + \frac{1}{\tau} = \frac{1}{\theta} > 0.$$

Therefore, we obtain

$$\|\nabla(B - B_h)q_h)\|_{L^{\tau'}(\Omega)} \le ch^{1-1/\theta-1/\tau}|q_h|_{W^{1+s-1/\tau',\tau'}(\Gamma)}.$$

There holds

$$1 + s - \frac{1}{\tau'} = 2 - \frac{1}{\theta} - \frac{1}{\tau} - \frac{1}{\tau'} = 1 - \frac{1}{\theta}.$$

Observing $\tau' \le 2$, we obtain

$$\|\nabla(B - B_h)q_h)\|_{L^{\tau'}(\Omega)} \le ch^{1-1/\theta-1/\tau}|q_h|_{H^{1-1/\theta}(\Gamma)},$$

which completes the proof of (3.14).

(iv) The L^2-norm estimate (3.15) follows from (3.14) by setting $\psi := (B - B_h)q_h$, $\tau = \theta := 2$. This completes the proof of the lemma. ∎

4. Basic error estimates. In the following, we will use the quantity

$$\Sigma_p := |\hat{q}|_{H^{1-1/p}(\Gamma)} + \|f\| + \|\hat{z}\|_{W^{2,p}(\Omega)},$$

which involves the data of the optimization problem and is bounded for $2 \le p < p_*$, according to Lemma 2.9, with $p_* := \min\{p_*^d, p_*^\Omega\}$. The range of the parameter $r \in [2, p_*^\Omega)$ is limited only by the angles of the polygonal boundary Γ. Below, we will use a generic constant $c_\alpha \approx 1 + \alpha^{-1}$ depending on the regularization parameter α, which may blow up as $p \to p_*$ or $r \to p_*^\Omega$. We begin with the estimate of the reduced state error.

LEMMA 4.1. *For the reduced state error $e_v = \hat{v} - \hat{v}_h$,*

$$\|e_v\| \le ch^2\|f\|. \tag{4.1}$$

Proof. The function $\hat{v} = \hat{u} - B\hat{q} \in H_0^1(\Omega)$ is the solution of the boundary value problem

$$-\Delta\hat{v} = f \quad \text{in } \Omega, \quad \hat{v}|_\Gamma = 0,$$

and $\hat{v}_h \in V_{h,0}$ denotes its Ritz projection satisfying

$$(\nabla e_v, \nabla\varphi_h) = 0, \quad \varphi_h \in V_{h,0}.$$

Since $f \in L^2(\Omega)$, we have $\hat{v} \in H^2(\Omega)$, and the estimate (3.6) of Lemma 3.1 yields the asserted error estimate. ∎

As starting point for the estimate of the control, we recall the perturbed Galerkin orthogonality equation (2.49),

$$\alpha\langle e_q, \chi_h\rangle + (\hat{u} - u_d, B\chi_h) - (\hat{u}_h - u_d, B_h\chi_h) = 0, \quad \chi_h \in V_h^\partial, \tag{4.2}$$

where $\hat{u} = \hat{v} + B\hat{q}$ and $\hat{u}_h = \hat{v}_h + B_h\hat{q}_h$. This can be rearranged into the form

$$\alpha\langle e_q, \chi_h\rangle = -(B\hat{q} - B_h\hat{q}_h, B_h\chi_h) - (\hat{v} - \hat{v}_h, B_h\chi_h) - (\hat{u} - u_d, (B - B_h)\chi_h). \tag{4.3}$$

THEOREM 4.2. *For the control error $e_q := \hat{q} - \hat{q}_h$ and the state error $e_u := \hat{u} - \hat{u}_h$ we have the estimate*

$$|e_q| + \|e_u\| \le c_\alpha h^{1-1/p}\Sigma_p, \quad 2 \le p < p_*. \tag{4.4}$$

Proof. (i) We begin with the relation

$$\alpha|e_q|^2 = \alpha\langle e_q, \hat{q} - P_h^\partial\hat{q}\rangle + \alpha\langle e_q, P_h^\partial e_q\rangle.$$

Setting $\chi_h := P_h^\partial e_q = P_h^\partial\hat{q} - \hat{q}_h$ in (4.3) and rearranging terms, we obtain

$$\begin{aligned}\alpha\langle e_q, P_h^\partial e_q\rangle &= -(B\hat{q} - B_h\hat{q}_h, B_hP_h^\partial e_q) - (\hat{v} - \hat{v}_h, B_hP_h^\partial e_q)\\ &\quad - (\hat{u} - u_d, (B - B_h)P_h^\partial e_q)\\ &= (B\hat{q} - B_h\hat{q}_h, (B - B_h)P_h^\partial\hat{q}) - (B\hat{q} - B_h\hat{q}_h, B(P_h^\partial\hat{q} - \hat{q}))\\ &\quad - (B\hat{q} - B_h\hat{q}_h, B\hat{q} - B_h\hat{q}_h) - (\hat{v} - \hat{v}_h, B_hP_h^\partial e_q)\\ &\quad - (\hat{u} - u_d, (B - B_h)P_h^\partial e_q).\end{aligned}$$

Combining this with the first equation results in

$$\begin{aligned}\alpha|e_q|^2 + \|B\hat{q} - B_h\hat{q}_h\|^2 &= \alpha\langle e_q, \hat{q} - P_h^\partial\hat{q}\rangle + (B\hat{q} - B_h\hat{q}_h, (B - B_h)P_h^\partial\hat{q})\\ &\quad - (B\hat{q} - B_h\hat{q}_h, B(P_h^\partial\hat{q} - \hat{q})) - (\hat{v} - \hat{v}_h, B_hP_h^\partial e_q)\\ &\quad - (\hat{u} - u_d, (B - B_h)P_h^\partial e_q).\end{aligned} \tag{4.5}$$

The five terms on the right-hand side of (4.5) will be treated separately.

First term: By the error estimate (3.4) for P_h^∂,

$$\begin{aligned}\alpha\langle e_q, \hat{q} - P_h^\partial\hat{q}\rangle &\le \alpha|e_q|\,|\hat{q} - P_h^\partial\hat{q}| \le c\alpha h^{1-1/p}|e_q|\,|\hat{q}|_{H^{1-1/p}(\Gamma)}\\ &\le \tfrac{\alpha}{4}|e_q|^2 + c\alpha h^{2-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}.\end{aligned}$$

Second term: By the L^2 error estimate (3.15) and the stability estimate (3.4),

$$\begin{aligned}(B\hat{q} - B_h\hat{q}_h, (B - B_h)P_h^\partial\hat{q}) &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + \|(B - B_h)P_h^\partial\hat{q}\|^2\\ &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + ch^{3-2/p}|P_h^\partial\hat{q}|^2_{H^{1-1/p}(\Gamma)}\\ &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + ch^{3-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}.\end{aligned}$$

Third term: By the stability estimate (2.9) and the error estimate (3.4),

$$\begin{aligned}-(B\hat{q} - B_h\hat{q}_h, B(P_h^\partial\hat{q} - \hat{q})) &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + \|B(P_h^\partial\hat{q} - \hat{q})\|^2\\ &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + c|P_h^\partial\hat{q} - \hat{q}|^2\\ &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + ch^{2-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}.\end{aligned}$$

Fourth term: By the estimate (3.8) of Lemma 3.2, the L^2 stability of the projection P_h^∂, and the estimate (4.1) of Lemma 4.1,

$$\begin{aligned}-(\hat{v}-\hat{v}_h, B_hP_h^\partial e_q) &\le \|\hat{v}-\hat{v}_h\|\|B_hP_h^\partial e_q\| \le c\|\hat{v}-\hat{v}_h\|\,|P_h^\partial e_q| \\ &\le c\|\hat{v}-\hat{v}_h\|\,|e_q| \le \tfrac{c}{\alpha}\|\hat{v}-\hat{v}_h\|^2+\tfrac{\alpha}{4}|e_q|^2 \\ &\le \tfrac{c}{\alpha}h^4\|f\|^2+\tfrac{\alpha}{4}|e_q|^2.\end{aligned}$$

Fifth term: We recall that $\hat{z}\in H_0^1(\Omega)\cap H^2(\Omega)$ satisfies $-\Delta\hat{z}=\hat{u}-u_d$ in Ω. Hence, observing that $(B-B_h)P_h^\partial e_q|_\Gamma=0$ and using the properties of the harmonic extension operators B and B_h, we deduce that

$$\begin{aligned}-(\hat{u}-u_d,(B-B_h)P_h^\partial e_q) &= (\Delta\hat{z},(B-B_h)P_h^\partial e_q) \\ &= -(\nabla\hat{z},\nabla(B-B_h)P_h^\partial e_q)+\langle\partial_n z,(B-B_h)P_h^\partial e_q\rangle \\ &= (\nabla(\hat{z}-I_h\hat{z}),\nabla B_hP_h^\partial e_q).\end{aligned}$$

Further, using the interpolation error estimate (3.2), the a priori estimate (3.9) of Lemma 3.2, and the L^2 stability of the projection P_h^∂, with $p'=p(p-1)^{-1}\le 2$,

$$\begin{aligned}-(\hat{u}-u_d,(B-B_h)P_h^\partial e_q) &\le \|\nabla(\hat{z}-I_h\hat{z})\|_{L^p(\Omega)}\|\nabla B_hP_h^\partial e_q\|_{L^{p'}(\Omega)} \\ &\le ch\|\hat{z}\|_{W^{2,p}(\Omega)}h^{1/p'-1}|P_h^\partial e_q| \\ &= ch^{1-1/p}\|\hat{z}\|_{W^{2,p}(\Omega)}|P_h^\partial e_q| \\ &\le c\alpha^{-1}h^{2-2/p}\|\hat{z}\|^2_{W^{2,p}(\Omega)}+\tfrac{1}{4}\alpha|e_q|^2.\end{aligned}$$

Collecting the foregoing estimates, we obtain

$$\begin{aligned}\alpha|e_q|^2+\|B\hat{q}-B_h\hat{q}_h\|^2 \le{}& \tfrac{3}{4}\alpha|e_q|^2+c(1+\alpha)h^{2-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}+\tfrac{1}{2}\|B\hat{q}-B_h\hat{q}_h\|^2 \\ &+c\alpha^{-1}h^4\|f\|^2+c\alpha^{-1}h^{2-2/p}\|\hat{z}\|^2_{W^{2,p}(\Omega)},\end{aligned}$$

and absorbing terms into the left-hand side,

$$\alpha|e_q|^2\le ch^{2-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}+c\alpha^{-1}\big\{h^4\|f\|^2+h^{2-2/p}\|\hat{z}\|^2_{W^{2,p}(\Omega)}\big\}.$$

This proves the asserted estimate of $|e_q|$.

(ii) Further, observing that $e_u=e_v+B\hat{q}-B_h\hat{q}_h$ and using the estimate (4.1) of Lemma 4.1 for $\|e_v\|$, we obtain the estimate of $\|e_u\|$. ∎

Remark 4. The estimate of the control error provided by Theorem 4.2 is optimal-order with respect to the regularity of the solution, but that for the state error is only suboptimal. The latter will be improved in the next section.

COROLLARY 4.3. *The discrete controls admit the uniform bound*

$$|\hat{q}_h|_{H^{1-1/p}(\Gamma)}\le c_\alpha\Sigma_p,\quad 2\le p<p_*. \tag{4.6}$$

Proof. Using the foregoing results together with the estimate (3.4) for the L^2 projection P_h^∂, we estimate as follows:

$$\begin{aligned}|\hat{q}_h|_{H^{1-1/p}(\Gamma)} &\le |\hat{q}_h-P_h^\partial\hat{q}|_{H^{1-1/p}(\Gamma)}+|P_h^\partial\hat{q}-\hat{q}|_{H^{1-1/p}(\Gamma)}+|\hat{q}|_{H^{1-1/p}(\Gamma)} \\ &\le ch^{-1+1/p}|\hat{q}_h-P_h^\partial\hat{q}|+c|\hat{q}|_{H^{1-1/p}(\Gamma)} \\ &\le ch^{-1+1/p}\big\{|e_q|+|\hat{q}-P_h^\partial\hat{q}|\big\}+c|\hat{q}|_{H^{1-1/p}(\Gamma)} \\ &\le ch^{-1+1/p}|e_q|+c|\hat{q}|_{H^{1-1/p}(\Gamma)} \\ &\le c_\alpha\big\{|\hat{q}|_{H^{1-1/p}(\Gamma)}+\|f\|+\|\hat{z}\|_{W^{2,p}(\Omega)}\big\}.\end{aligned}$$

This implies the asserted estimate. (A similar proof is given by Casas and Raymond for [7, Theorem 6.2].) ∎

5. Improved error estimates. The orders of convergence derived in the preceding section for the state $\hat{u}$ may not be optimal, as is demonstrated by the numerical results presented in section 6 below. The key to improved error estimates for the state is the proof of higher-order error estimates for the control in norms weaker than the L^2-norm. To this end, we will employ duality arguments based on the KKT system. Below, the ranges of the parameters $p \in [2, p_*)$ and $r \in [2, p_*^\Omega)$ are limited by the regularity of the data and the regularity of the domain, respectively. As before, we use a generic constant $c_\alpha \approx 1 + \alpha^{-1}$, which may blow up as $p \to p_*$ or $r \to p_*^\Omega$.

The modified KKT system (2.35), (2.36), (2.37) can be written in compact form for the triplet $\hat{X} := \{\hat{v}, \hat{q}, \hat{z}\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ as follows:

$$A(\hat{X}, \Phi) = F(\Phi) \tag{5.1}$$

for all $\Phi = \{\varphi^z, \varphi^q, \varphi^v\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$, with the bilinear form

$$A(\hat{X}, \Phi) := (\nabla \hat{v}, \nabla \varphi^v) + \alpha \langle \hat{q}, \varphi^q \rangle + (\hat{v} + B\hat{q}, B\varphi^q) + (\nabla \hat{z}, \nabla \varphi^z) - (\hat{v} + B\hat{q}, \varphi^z)$$

and the right-hand side

$$F(\Phi) := (f, \varphi^v) - (u_d, \varphi^z) + (u_d, B\varphi^q).$$

For a given linear functional $J(\cdot)$ let $W = \{w^v, w^q, w^z\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ be the solution of the dual problem

$$A(\Psi, W) = J(\Psi) \quad \forall \Psi = \{\psi^z, \psi^q, \psi^v\}. \tag{5.2}$$

For $J(\Psi) = J_u(\psi^v) + J_q(\psi^q) + J_z(\psi^z)$ this is equivalent to the system

$$(\nabla \psi^v, \nabla w^v) = J_u(\psi^v) \qquad \forall \psi^v \in H_0^1(\Omega), \tag{5.3}$$

$$\alpha \langle \psi^q, w^q \rangle + (B\psi^q, Bw^q) - (B\psi^q, w^v) = J_q(\psi^q) \qquad \forall \psi^q \in H^{1/2}(\Gamma), \tag{5.4}$$

$$(\nabla \psi^z, \nabla w^z) - (\psi^z, w^v) + (\psi^z, Bw^q) = J_z(\psi^z) \qquad \forall \psi^z \in H_0^1(\Omega). \tag{5.5}$$

In the special case $J(\Psi) = J_q(\psi^q)$, equation (5.3) has the unique solution $w^v = 0$, and we obtain the triangular system

$$\alpha \langle \psi^q, w^q \rangle + (B\psi^q, Bw^q) = J_q(\psi^q) \qquad \forall \psi^q \in H^{1/2}(\Gamma), \tag{5.6}$$

$$(\nabla \psi^z, \nabla w^z) + (\psi^z, Bw^q) = 0 \qquad \forall \psi^z \in H_0^1(\Omega). \tag{5.7}$$

By coercivity arguments it is easily seen that for $J_q(\cdot) \in L^2(\Gamma)^*$ there is a unique solution $\{w^q, w^z\} \in L^2(\Gamma) \times H_0^1(\Omega)S$.

LEMMA 5.1. *For $J(\Psi) = J_q(\psi^q) = (B\psi^q, \psi)$ with a fixed $\psi \in L^r(\Omega)$, $2 \le r < p_*^\Omega$, the solution $\{w^q, w^z\}$ of the dual system* (5.6), (5.7) *has the regularity $\{w^q, w^z\} \in H^{1-1/r}(\Gamma) \times W^{2,r}(\Omega)$, and there holds the a priori estimate*

$$|w^q|_{H^{1-1/r}(\Gamma)} + \|w^z\|_{W^{2,r}(\Omega)} \le c\alpha^{-1} \|\psi\|_{L^r(\Omega)}. \tag{5.8}$$

Proof. For $Bw^q \in L^2(\Omega)$, we infer that $w^z \in H_0^1(\Omega) \cap H^2(\Omega)$. Since (5.6) also holds for $\psi^q \in L^2(\Gamma)$, we can test with $\psi^q = w^q$, which gives us

$$\alpha |w^q| + \|Bw^q\| \le c\|\psi\|.$$

Further, by elliptic regularity, we have $\|w^z\|_{H^2(\Omega)} \le c\|\psi\|$. Next, we employ a duality argument. Let $\eta \in H_0^1(\Omega)$ be the weak solution of the auxiliary problem

$$-\Delta\eta = \psi - Bw^q \quad \text{in } \Omega.$$

Since $Bw^q - \psi \in L^2(\Omega)$, we have $\eta \in H^2(\Omega)$ and $\partial_n\eta|_\Gamma \in H^{1/2}(\Gamma)$ by Lemma 2.3, as well as

$$\|\eta\|_{H^2(\Omega)} + |\partial_n\eta|_{H^{1/2}(\Gamma)} \le c\|\psi\|.$$

Then, from

$$\alpha\langle\chi, w^q\rangle = (B\chi, \psi - Bw^q) = -(B\chi, \Delta\eta) = (\nabla B\chi, \nabla\eta) - \langle\chi, \partial_n\eta\rangle = -\langle\chi, \partial_n\eta\rangle$$

we infer that also $w^q = -\alpha^{-1}\partial_n\eta \in H^{1/2}(\Gamma)$, which implies $Bw^q \in H^1(\Omega)$. By the Sobolev embedding theorem we conclude that $Bw^q \in L^r(\Omega)$ for $r \ge 2$ as considered and $\|Bw^q\|_{L^r(\Omega)} \le c\|\psi\|$. Now, this implies that $\eta \in H_0^1(\Omega) \cap W^{2,r}(\Omega)$ and consequently, in virtue of Lemma 2.3, $w^q = -\alpha^{-1}\partial_n\eta \in W^{1-1/r,r}(\Gamma) \subset H^{1-1/r}(\Gamma)$ and $\|w^q\|_{H^{1-1/r}(\Gamma)} \le c\alpha^{-1}\|\psi\|_{L^r(\Omega)}$. Finally, again by elliptic regularity theory, we obtain $w^z \in H_0^1(\Omega) \cap W^{2,r}(\Omega)$ and $\|w^z\|_{W^{2,r}(\Omega)} \le c\alpha^{-1}\|\psi\|_{L^r(\Omega)}$. This completes the proof. ◻

THEOREM 5.2. *For the control error $e_q := \hat{q} - \hat{q}_h$ and any $\psi \in L^r(\Omega)$,*

$$(Be_q, \psi) \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p\|\psi\|_{L^r(\Omega)} \tag{5.9}$$

for $2 \le p < p_*$ and $2 \le r < p_*^\Omega$.

Proof. We use the dual problem described above with $\psi \in L^p(\Omega)$. Taking the test pair $\{\psi^q, \psi^z\} = \{e_q, e_v\}$ in the (5.6) and (5.7) gives us

$$\alpha\langle e_q, w^q\rangle + (Be_q, Bw^q) = (Be_q, \psi), \tag{5.10}$$
$$(\nabla e_v, \nabla w^z) + (e_v, Bw^q) = 0. \tag{5.11}$$

Then, subtracting the L^2 projection $P_h^\partial w^q \in V_h^\partial$ in (5.10) yields

$$\begin{aligned}(Be_q, \psi) = {} & \alpha\langle e_q, w^q - P_h^\partial w^q\rangle + (Be_q, B(w^q - P_h^\partial w^q)) \\ & + \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q).\end{aligned} \tag{5.12}$$

We recall the Galerkin orthogonality relations (2.48), (2.49), (2.50):

$$(\nabla e_v, \nabla\varphi_h) = 0 \quad \forall\varphi_h \in V_{h,0}, \tag{5.13}$$
$$\alpha\langle e_q, \chi_h\rangle + (\hat{v} + B\hat{q} - u_d, B\chi_h) - (\hat{v}_h + B_h\hat{q}_h - u_d, B_h\chi_h) = 0 \quad \forall\chi_h \in V_h^\partial, \tag{5.14}$$
$$(\nabla e_z, \nabla\psi_h) - (e_v + B\hat{q} - B_h\hat{q}_h, \psi_h) = 0 \quad \forall\psi_h \in V_{h,0}. \tag{5.15}$$

We also set $\chi_h = P_h^\partial w^q$ in (5.14) and rearrange terms to obtain

$$\begin{aligned}
0 &= \alpha\langle e_q, P_h^\partial w^q\rangle + (\hat v + B\hat q - u_d, BP_h^\partial w^q) - (\hat v_h + B_h\hat q_h - u_d, B_hP_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q) - (Be_q, B_hP_h^\partial w^q)\\
&\quad + (\hat v + B\hat q - u_d, BP_h^\partial w^q) - (\hat v_h + B_h\hat q_h - u_d, B_hP_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q) - (Be_q, B_hP_h^\partial w^q)\\
&\quad + (\hat v + B\hat q - u_d, (B-B_h)P_h^\partial w^q) + (\hat v + B\hat q - u_d, B_hP_h^\partial w^q)\\
&\quad - (\hat v_h + B_h\hat q_h - u_d, B_hP_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q)\\
&\quad + (e_v + B\hat q - B_h\hat q_h - Be_q, B_hP_h^\partial w^q) + (\hat u - u_d, (B-B_h)P_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q)\\
&\quad + (e_v, B_hP_h^\partial w^q) + ((B-B_h)\hat q_h, B_hP_h^\partial w^q) + (\hat u - u_d, (B-B_h)P_h^\partial w^q).
\end{aligned}$$

Combining this with (5.12) gives us

$$\begin{aligned}
(Be_q, \psi) &= \alpha\langle e_q, w^q - P_h^\partial w^q\rangle + (Be_q, B(w^q - P_h^\partial w^q))\\
&\quad + (Be_q, (B-B_h)P_h^\partial w^q) - (e_v, B_hP_h^\partial w^q)\\
&\quad - ((B-B_h)\hat q_h, B_hP_h^\partial w^q) - (\hat u - u_d, (B-B_h)P_h^\partial w^q).
\end{aligned} \tag{5.16}$$

In the following, the six terms on the right-hand side of (5.16) will be treated separately for $2 \le p \le p_*$, $2 \le r \le p_*^\Omega$.

First term: By the properties of the L^2 projection P_h^∂ and the error estimate (3.4),

$$\begin{aligned}
\alpha\langle e_q, w^q - P_h^\partial w^q\rangle &= \alpha\langle \hat q - P_h^\partial \hat q, w^q - P_h^\partial w^q\rangle\\
&\le \alpha|\hat q - P_h^\partial \hat q|\,|w^q - P_h^\partial w^q|\\
&\le c\alpha h^{1-1/p}|\hat q|_{H^{1-1/p}(\Gamma)} h^{1-1/r}|w^q|_{H^{1-1/r}(\Gamma)}\\
&\le c\alpha h^{2-1/p-1/r}\Sigma_p|w^q|_{H^{1-1/r}(\Gamma)}.
\end{aligned}$$

Second term: By the stability estimate (2.9), the error estimate (3.4), and the estimate (4.4) of Theorem 4.2, for $2 \le p \le p_*^d$, $2 \le r \le p_*^\Omega$,

$$\begin{aligned}
(Be_q, B(w^q - P_h^\partial w^q)) &\le \|Be_q\|\|B(w^q - P_h^\partial w^q)\|\\
&\le c|e_q|\,|w^q - P_h^\partial w^q|\\
&\le c_\alpha h^{1-1/p}\Sigma_p h^{1-1/r}|w^q|_{H^{1-1/r}(\Gamma)}\\
&= c_\alpha h^{2-1/p-1/r}\Sigma_p|w^q|_{H^{1-1/r}(\Gamma)}.
\end{aligned}$$

Third term: By the stability estimate (2.9), the error estimate (3.15) of Lemma 3.3, the result (4.4) of Theorem 4.2, and the stability estimate (3.4), for $2 \le p < p_*$, $2 \le r < p_*^\Omega$,

$$\begin{aligned}
(Be_q, (B-B_h)P_h^\partial w^q) &\le \|Be_q\|\|(B-B_h)P_h^\partial w^q\|\\
&\le c|e_q|h^{3/2-1/r}|P_h^\partial w^q|_{H^{1-1/r}(\Gamma)}\\
&\le c_\alpha h^{1-1/p}\Sigma_p h^{3/2-1/r}|P_h^\partial w^q|_{H^{1-1/r}(\Gamma)}\\
&\le c_\alpha h^{5/2-1/p-1/r}\Sigma_p|w^q|_{H^{1-1/r}(\Gamma)}.
\end{aligned}$$

Fourth term: By the error estimate (4.1) of Lemma 4.1, the stability estimates (3.8) of Lemma 3.2 and (3.4), for $2 \le p \le p_*$, $2 \le r \le p_*^\Omega$,

$$\begin{aligned} -(e_v, B_h P_h^\partial w^q) &\le \|e_v\| \|B_h P_h^\partial w^q\| \\ &\le ch^2 \|f\| \, |P_h^\partial w^q| \\ &\le ch^2 \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)}. \end{aligned}$$

Fifth term: First, we use the error estimate (3.14) of Lemma 3.3 with parameters $\tau := \max\{r, p\} \le \theta < p_*^\Omega$ to obtain

$$-((B - B_h)\hat{q}_h, B_h P_h^\partial w^q) \le ch^{2-1/\theta-1/\tau} |\hat{q}_h|_{H^{1-1/\theta}(\Gamma)} \|B_h P_h^\partial w^q\|_{L^\tau(\Omega)}.$$

Further, by the stability estimates (3.7), (3.8) of Lemma 3.2 for B_h and the stability estimate (3.4) for P_h^∂

$$\begin{aligned} -((B - B_h)\hat{q}_h, B_h P_h^\partial w^q) &\le ch^{2-1/\theta-1/\tau} |\hat{q}_h|_{H^{1-1/\theta}(\Gamma)} \|B_h P_h^\partial w^q\|_{H^1(\Omega)} \\ &\le ch^{2-1/\theta-1/\tau} |\hat{q}_h|_{H^{1-1/\theta}(\Gamma)} |P_h^\partial w^q|_{H^{1/2}(\Gamma)} \\ &\le ch^{2-1/\theta-1/\tau} |\hat{q}_h|_{H^{1-1/\theta}(\Gamma)} |w^q|_{H^{1/2}(\Gamma)}. \end{aligned}$$

Finally, using the inverse relation (3.1) and the stability estimate (4.6) of Corollary 4.3, we conclude that for $2 \le p < p_*$ and $2 \le r < p_*^\Omega$

$$\begin{aligned} -((B - B_h)\hat{q}_h, B_h P_h^\partial w^q) &\le ch^{2-1/\theta-1/\tau} h^{1/\theta-1/p} |\hat{q}_h|_{H^{1-1/p}(\Gamma)} |w^q|_{H^{1-1/r}(\Gamma)} \\ &\le c_\alpha h^{2-1/p-1/r} \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)}. \end{aligned}$$

Sixth term: To estimate the sixth term, we recall that $\hat{z} \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies $-\Delta \hat{z} = \hat{u} - u_d$ in Ω. Hence, we can use the error estimate (3.14) of Lemma 3.3, with $\tau := p$, $\max\{r, p\} \le \theta < p_*^\Omega$, to obtain

$$-(\hat{u} - u_d, (B - B_h) P_h^\partial w^q) \le ch^{2-1/p-1/\theta} \|\hat{z}\|_{W^{2,p}} |P_h^\partial w^q|_{H^{1-1/\theta}(\Gamma)}.$$

Then, again by the inverse relation (3.1) and the stability estimate (3.4),

$$\begin{aligned} -(\hat{u} - u_d, (B - B_h) P_h^\partial w^q) &\le ch^{2-1/p-1/\theta} h^{1/\theta-1/r} \|\hat{z}\|_{W^{2,p}} |P_h^\partial w^q|_{H^{1-1/r}(\Gamma)} \\ &\le ch^{2-1/p-1/r} \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)} \end{aligned}$$

for $2 \le p < p_*$, $2 \le r < p_*^\Omega$.

Combining all the above estimates gives us

$$(Be_q, \psi) \le c_\alpha h^{2-1/p-1/r} \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)},$$

and hence in view of Lemma 5.1,

$$(Be_q, \psi) \le c_\alpha^2 h^{2-1/p-1/r} \Sigma_p \|\psi\|_{L^r(\Omega)}.$$

This completes the proof. ∎

COROLLARY 5.3. *For the control error $e_q := \hat{q} - \hat{q}_h$ and the state error $e_u := \hat{u} - \hat{u}_h$ there holds*

$$|e_q|_{\tilde{H}^{-1/2}(\Gamma)} + \|e_u\| \le c_\alpha^2 h^{3/2-1/p} \Sigma_p \tag{5.17}$$

for $2 \le p < p_*$.

Proof. (i) Observing that Be_q is harmonic, by the trace estimate (2.7), we have

$$|e_q|_{\tilde H^{-1/2}(\Gamma)} \le c\|Be_q\|.$$

Taking $\psi := Be_q$ in the estimate (5.9) of Theorem 5.2 for $r = 2$, we obtain

$$\|Be_q\| \le c_\alpha^2 h^{3/2-1/p}\Sigma_p, \tag{5.18}$$

which implies the first part of the assertion.

(ii) From the identity

$$(e_u, \psi) = (e_v, \psi) + ((B - B_h)\hat q_h, \psi) + (Be_q, \psi), \tag{5.19}$$

we conclude that

$$\|e_u\| \le \|e_v\| + \|(B - B_h)\hat q_h\| + \|Be_q\|.$$

Hence, by the estimate (4.1) of Lemma 4.1, the estimate (3.15) of Lemma 3.3, and the just proven estimate (5.18),

$$\|e_u\| \le ch^2\|f\| + ch^{3/2-1/p}|\hat q_h|_{H^{1-1/p}(\Gamma)} + c_\alpha^2 h^{3/2-1/p}\Sigma_p.$$

In view of the estimate (4.6), this implies the second part of the assertion. ∎

COROLLARY 5.4. *For the primal state error $e_u := \hat u - \hat u_h$ and the adjoint state error $e_z := \hat z - \hat z_h$ there holds*

$$\|e_u\|_{H^{-1}(\Omega)} + \|e_z\| \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p \tag{5.20}$$

for $2 \le p < p_*$ and $2 \le r < p_*^\Omega$.

Proof. (i) We recall the identity (5.19). By Lemma 4.1, Lemma 3.3(iii), with $\tau := r$, $\max\{p, r\} \le \theta < p_*^\Omega$, and Theorem 5.2, we obtain

$$(e_u, \psi) \le c\big\{h^2\|f\| + h^{2-1/r-1/\theta}|\hat q_h|_{H^{1-1/\theta}(\Gamma)} + c_\alpha^2 h^{2-1/p-1/r}\Sigma_p\big\}\|\psi\|_{L^r(\Omega)}$$

for $2 \le p \le p_*$ and $2 \le r < p_*^\Omega$. Using again the inverse relation for finite elements (3.1), this results in

$$(e_u, \psi) \le c\big\{h^2\|f\| + h^{2-1/r-1/p}|\hat q_h|_{H^{1-1/p}(\Gamma)} + c_\alpha^2 h^{2-1/p-1/r}\Sigma_p\big\}\|\psi\|_{L^r(\Omega)}.$$

Hence, by arguments already used before,

$$\|e_u\|_{H^{-1}(\Omega)} = \sup_{\psi\in H^1(\Omega)} \frac{(e_u, \psi)}{\|\psi\|_{H^1(\Omega)}} \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p.$$

(ii) For proving the error estimate of the adjoint state, we recall (2.50) in the form

$$(\nabla e_z, \nabla\psi_h) = (e_u, \psi_h), \quad \psi_h \in V_{h,0}. \tag{5.21}$$

The adjoint state $\hat z \in H_0^1(\Omega)$ is determined by the boundary value problem

$$-\Delta\hat z = \hat u - u_d \ \text{ in } \ \Omega, \quad \hat z|_\Gamma = 0,$$

and has the regularity $\hat z \in W^{2,p}(\Omega)$, with $2 \le p < p_*$. Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = e_z \quad \text{in } \Omega, \quad w|_\Gamma = 0,$$

satisfying $\|w\|_{H^2(\Omega)} \le c\|e_z\|$. Then, using (2.50), we conclude

$$\begin{aligned}
\|e_z\|^2 &= (\nabla e_z, \nabla(w - R_h^D w)) + (\nabla e_z, \nabla R_h^D w)\\
&= (\nabla(\hat z - I_h \hat z), \nabla(w - R_h^D w)) + (e_u, R_h^D w)\\
&\le \|\nabla(\hat z - I_h \hat z)\|\|\nabla(w - R_h^D w)\| + \|e_u\|_{H^{-1}(\Omega)}\|R_h^D w\|_{H^1(\Omega)}\\
&\le ch^2\|\hat z\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)} + c_\alpha^2 h^{2-1/p-1/r}\Sigma_p\|w\|_{H^2(\Omega)}\\
&\le c\big\{h^2 + c_\alpha^2 h^{2-1/p-1/r}\big\}\Sigma_p\|e_z\|.
\end{aligned}$$

This proves the asserted estimate. ∎

COROLLARY 5.5. *For the primal state error* $e_u := \hat u - \hat u_h$ *and the control error* $e_q := \hat q - \hat q_h$,

$$|(e_u, 1)| + |\langle e_q, 1\rangle| \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p \tag{5.22}$$

for $2 \le p < p_*$ *and* $2 \le r < p_*^\Omega$.

Proof. In view of

$$|(e_u, 1)| \le |\Omega|^{1/2} \sup_{\varphi \in H^1(\Omega)} \frac{(e_u, \varphi)}{\|\varphi\|_{H^1(\Omega)}} = |\Omega|^{1/2}\|e_u\|_{H^{-1}(\Omega)},$$

the first part of the asserted estimate follows from Corollary 5.4. Next, we recall the Galerkin orthogonality relation (2.49) in the form

$$\alpha\langle e_q, \chi_h\rangle = -(\hat u - u_d, B\chi_h) + (\hat u_h - u_d, B_h\chi_h) \quad \forall \chi_h \in V_h^\partial.$$

Hence observing that the function $\chi_h \equiv 1$ satisfies $\chi_h \in V_h^\partial$ and $B\chi_h \equiv B_h\chi_h \equiv 1$, it follows that $\alpha\langle e_q, 1\rangle = -(e_u, 1)$. This implies the second part of the asserted estimate. ∎

Remark 5. The numerical experiments shown in the next section indicate that for the adjoint state there may hold the improved error estimate

$$\|e_z\| \le c_\alpha^2 h^{2-1/r}\Sigma_p, \quad 2 \le r < p_*^\Omega. \tag{5.23}$$

6. Numerical tests. In this section, we present some numerical results obtained for the optimization problem (1.1), (1.2) by the discretization described above. The purpose is to clarify the convergence rates to be expected for several configurations. For the computation the software libraries GASCOIGNE [3] and RoDoBo [16] have been used. Three different configurations have been considered in order to illustrate the sharpness of our theoretically derived error estimates:

1. regular domain (unit square) and known analytic solution,
2. general domain with $\omega_{\max} = \frac{5}{6}\pi$ and unknown solution,
3. regular domain (unit square) and "singular" data.

TABLE 6.1
Example with known analytic solution: convergence rates for a sequence of equidistant Cartesian meshes with $h_n = 2^{-n}\sqrt{2}$.

	$\lvert\hat{q}-\hat{q}_h\rvert$		$\lVert\hat{u}-\hat{u}_h\rVert$		$\lVert\hat{z}-\hat{z}_h\rVert$	
# cells	Error	Rate	Error	Rate	Error	Rate
64	4.81e-01	2.00	1.74e-02	2.00	3.04e-04	2.10
256	1.21e-01	2.00	4.25e-03	2.04	7.49e-05	2.02
1 024	3.02e-02	2.00	1.04e-03	2.03	1.87e-05	2.00
4 096	7.56e-03	2.00	2.58e-04	2.01	4.66e-06	2.00
16 384	1.89e-03	2.00	6.45e-05	2.00	1.17e-06	2.00
65 536	4.73e-04	2.00	1.61e-05	2.00	2.91e-07	2.00
Expected		1.00		1.50		2.00

TABLE 6.2
Example with known analytic solution: convergence rates for a sequence of tensor-product meshes with 10% *random shift of interior nodal points after each uniform refinement step.*

	$\lvert\hat{q}-\hat{q}_h\rvert$		$\lVert\hat{u}-\hat{u}_h\rVert$		$\lVert\hat{z}-\hat{z}_h\rVert$	
# cells	Error	Rate	Error	Rate	Error	Rate
64	4.91e-01	2.00	4.60e-02	2.00	3.71e-04	1.95
256	1.30e-01	1.92	1.30e-02	1.82	8.19e-05	2.18
1 024	4.53e-02	1.52	4.46e-03	1.54	2.00e-05	2.03
4 096	2.69e-02	0.75	1.98e-03	1.17	4.98e-06	2.01
16 384	1.74e-02	0.63	8.89e-04	1.16	1.24e-06	2.01
65 536	9.65e-03	0.85	3.48e-04	1.35	3.11e-07	1.99
Expected		1.00		1.50		2.00

TABLE 6.3
Example with known analytic solution: convergence rates for the mean values on a sequence of tensor-product meshes with 10% *random shift of nodal points after each uniform refinement step.*

	$\langle\hat{q}-\hat{q}_h, 1\rangle$		$(\hat{u}-\hat{u}_h, 1)$	
# cells	Error	Rate	Error	Rate
64	-2.66e+00	2.08	2.66e-02	2.08
256	-6.54e-01	2.02	6.54e-03	2.02
1 024	-1.65e-01	1.99	1.65e-03	1.99
4 096	-4.10e-02	2.01	4.11e-04	2.01
16 384	-1.02e-02	2.00	1.02e-04	2.01
65 536	-2.51e-03	2.03	2.46e-05	2.05
Expected		2.00		2.00

6.1. Example on a unit square with known analytic solution. The domain is the unit square $\Omega = (0,1)^2$, and the data is chosen as

$$f = -\tfrac{4}{\alpha}, \quad u_d = -\left(2+\tfrac{1}{\alpha}\right)\left(x(1-x)+y(1-y)\right), \quad \alpha = 0.01,$$

such that the optimal solution is given by

$$\hat{q} = -\tfrac{1}{\alpha}\left(x(1-x)+y(1-y)\right), \quad \hat{u} = -\tfrac{1}{\alpha}\left(x(1-x)+y(1-y)\right), \quad \hat{z} = xy(1-x)(1-y).$$

The obtained results are presented in Tables 6.1, 6.2, and 6.3. The test on uniform meshes shows second-order convergence for all quantities, which is probably due to superapproximation effects which are not captured by our analysis. These effects disappear on irregular meshes, and the observed orders of convergence agree well with the theoretically predicted rates.

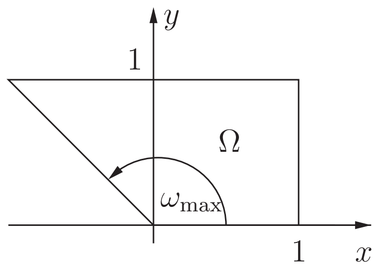

FIG. 6.1. *General domain with* $\omega_{\max} = \frac{5}{6}\pi$.

TABLE 6.4
Results for a domain with maximum interior angle $\omega_{\max} = \frac{5}{6}\pi$*: convergence rates on a sequence of irregular meshes.*

$\omega_{\max} = \frac{5}{6}\pi$	$\lvert\hat{q} - \hat{q}_h\rvert$		$\lVert\hat{u} - \hat{u}_h\rVert$		$\lVert\hat{z} - \hat{z}_h\rVert$	
# cells	Error	Rate	Error	Rate	Error	Rate
64	4.40e-02	0.99	9.98e-03	1.50	2.34e-03	1.70
256	2.42e-02	0.87	4.04e-03	1.31	6.95e-04	1.75
1 024	1.47e-02	0.72	1.72e-03	1.23	2.43e-04	1.52
4 096	8.43e-03	0.80	7.25e-04	1.25	7.21e-05	1.75
16 384	5.35e-03	0.65	3.26e-04	1.15	2.21e-05	1.71
65 536	3.40e-03	0.65	1.47e-04	1.15	6.76e-06	1.71
Expected		0.60		1.10		1.20

6.2. Example on a general polygonal domain with unknown solution. Next, we test the convergence rates for a domain with maximum interior angle $\omega_{\max} = \frac{5}{6}\pi$ (see Figure 6.1), where the optimal solution has only reduced regularity. The data is taken as $\alpha = 1$ and

$$f = 1, \quad u_d = \begin{cases} -1 & \text{for } 0 \le y < 0.5, \\ 1 & \text{for } 0.5 \le y \le 1. \end{cases}$$

Since $\hat{u} - u_d \in L^\infty(\Omega)$, the adjoint state satisfies $\hat{z} \in W^{2,p}(\Omega)$ for $2 \le p < p_*^\Omega$, where in this case $p_*^\Omega = 2\omega_{\max}(2\omega_{\max} - \pi)^{-1} = \frac{5}{2}$. The results obtained for this configuration are shown in Table 6.4. The "reference solution" has been calculated on a very fine mesh with more than 10^6 cells.

6.3. Example on a unit square with irregular data. The following example has been adopted from Casas and Raymond [7]. The domain again is the unit square $\Omega = (0,1)^2$, $f = 0$, and $u_d = (x^2 + y^2)^{-\frac{1}{3}}$. Hence $p_*^\Omega = \infty$. Since u_d has a singularity at the boundary Γ such that $u_d \in L^p(\Omega)$ for $2 \le p < 3$ but $u_d \notin L^3(\Omega)$, the optimal solution has only reduced regularity $\{\hat{u}, \hat{q}, \hat{z}\} \in H^{\frac{3}{2}-\frac{1}{p}}(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma) \times W^{2,p}(\Omega)$, with $2 \le p < p_*^d = 3$. Hence, according to our theory, we expect the errors for $\hat{q}$, $\hat{u}$, and $\hat{z}$ to converge with the rates $\approx 1 - \frac{1}{p_*^d} = \frac{2}{3}$, $\approx \frac{3}{2} - \frac{1}{p_*^d} = \frac{7}{6}$, and $\approx 2 - \frac{1}{p_*^d} - \frac{1}{p_*^\Omega} = \frac{5}{3}$, respectively. The results are shown in Table 6.5. The orders of convergence observed for e_q and e_u are in reasonable agreement with the theoretically predicted ones, while that for e_z seems to be too high.

A similar observation was made in section 6.2 above. Together with section 6.1 this suggests that the presented estimates for the control $\hat{q}$ and the primal state $\hat{u}$

TABLE 6.5
Results for irregular data: convergence rates on a sequence of irregular meshes.

	$\lvert\hat{q}-\hat{q}_h\rvert$		$\|\hat{u}-\hat{u}_h\|$		$\|\hat{z}-\hat{z}_h\|$	
# cells	Error	Rate	Error	Rate	Error	Rate
16	3.18e-02	-	1.28e-02	-	4.70e-03	-
64	1.95e-02	0.70	5.14e-03	1.32	1.28e-03	1.88
256	1.16e-02	0.75	2.07e-03	1.31	3.40e-04	1.91
1 024	6.21e-03	0.90	8.45e-04	1.29	8.94e-05	1.93
4 096	3.66e-03	0.76	3.31e-04	1.35	2.27e-05	1.98
16 384	2.21e-03	0.73	1.39e-04	1.26	5.79e-06	1.97
65 536	1.30e-03	0.77	5.69e-05	1.29	1.42e-06	2.03
Expected		0.67		1.17		1.67

are optimal-order. The estimate for the adjoint state $\hat{z}$ (while being a considerable improvement over the order of $\mathcal{O}(h^{1-1/p})$ given in Casas and Raymond [7]) seems to be slightly suboptimal.

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