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A PRIORI ERROR ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF PARABOLIC OPTIMIZATION PROBLEMS WITH POINTWISE STATE CONSTRAINTS IN TIME*

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Abstract. In this paper, we consider an optimal control problem which is governed by a linear parabolic equation and is subject to state constraints pointwise in time. Optimal order error estimates are developed for a space-time finite element discretization of this problem. Numerical examples confirm the theoretical results. As a by-product of our analysis, we derive a new regularity result for the optimal control.

Key words. optimal control, heat equation, control constraints, state constraints, finite elements, a priori error estimates

AMS subject classifications. 49J20, 35K20, 49M05, 49M15, 49M25, 49M29, 65M12, 65M50, 65M60

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1. Introduction.

In this paper, we consider the following optimal control problem governed by the heat equation and subject to control and state constraints:

$$\text{Minimize } \frac{1}{2}\int_0^T\int_\Omega (u(t,x)-\hat{u}(t,x))^2\,dx\,dt + \frac{\alpha}{2}\int_0^T\int_\Omega q(t,x)^2\,dx\,dt \tag{1.1a}$$

subject to the equation constraints

$$\begin{aligned} \partial_t u - \Delta u &= f + q && \text{in } (0,T)\times\Omega,\\ u &= 0 && \text{on } (0,T)\times\Omega,\\ u &= u_0 && \text{in } \{\,0\,\}\times\Omega, \end{aligned} \tag{1.1b}$$

the control constraints

$$q_a \le q(t,x) \le q_b \quad \text{a.e. in } (0,T)\times\Omega \tag{1.1c}$$

for $q_a, q_b \in \mathbb{R}$, and the state constraint

$$\int_\Omega u(t,x)\omega(x)\,dx \le b \quad \text{in } [0,T] \tag{1.1d}$$

for given $\omega \in L^2(\Omega)$ and $b \in \mathbb{R}$. The precise functional analytic setting of (1.1) is formulated in section 2. Here, q denotes the (distributed) control and u the state variable. The cost functional (1.1a) is a quadratic functional of tracking type, and the control q enters the state equation (1.1b) via the right-hand side. Besides box

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constraints (1.1c) on the control variable, we consider state constraints (1.1d) which are integrated in space and are understood pointwise in time.

Parabolic optimal control problems with state constraints formulated pointwise in space and time, i.e.,

$$u(t,x) \leq b \quad \forall (t,x) \in [0,T] \times \bar{\Omega}, \tag{1.2}$$

are discussed in several publications; see, e.g., Casas [2] and Raymond and Zidani [27] for corresponding optimality conditions and Neitzel and Tröltzsch [22, 23] for regularization issues. The case of spatially integrated state constraints (1.1d) serves as an example for several applications, where some constraint which is formulated as a spatial functional (for instance, drag or lift coefficients in CFD) should hold continuously in time. Optimal control problems of this type are considered in Goldberg and Tröltzsch [14] and Bonnans and Jaisson [1]. In these publications necessary and sufficient optimality conditions as well as regularity results are discussed.

The main goal of this paper is to provide an a priori error analysis for a finite element discretization of the parabolic optimal control problem under consideration. To this end, we follow the strategy developed in Meidner and Vexler [19, 20] (see also Neitzel and Vexler [24]), where optimal control problems are analyzed in the absence of state constraints. We consider a discontinuous Galerkin scheme, the dG(0) method, for temporal discretization, conforming (bi-/tri-)linear finite elements for spatial discretization and cellwise constants for the discretization of the control variable; see section 3 for details.

The main difficulty in the numerical analysis of optimal control problems with state constraints is the lack of regularity caused by the fact that the Lagrange multiplier corresponding to the state constraint (1.1d) is a Borel measure $\mu \in C([0,T])^*$. This affects the regularity of the adjoint state and of the optimal control $\bar{q}$. Especially, the lack of temporal regularity complicates the derivation of a priori error estimates for the corresponding finite element discretization.

Error estimates for optimal control problems with state constraints governed by elliptic equations are derived in several publications. In Casas [3] error estimates are given for an optimal control problem with finitely many state constraints. In Deckelnick and Hinze [7, 8] error estimates of order $h^{1-\varepsilon}$ in two dimensions and $h^{\frac{1}{2}-\varepsilon}$ in three dimensions are derived for a problem with pointwise state constraints; see also a recent preprint by Rösch and Steinig [29] for an improvement to $h^{\frac{3}{4}}$ in three dimensions and [16] for an estimate of order $h|\log h|$ for a problem with control and state constraints. A similar result is obtained in Meyer [21] with a different technique avoiding the consideration of Lagrange multipliers on the discrete level. The latter technique is extended to problems governed by the Stokes equations in de los Reyes, Meyer, and Vexler [5]. The publications of Deckelnick, Günther, and Hinze [6] and Ortner and Wollner [25] are devoted to problems with pointwise state constraints on the gradient of the state.

We denote by k the maximum step size in the temporal discretization and by h the maximum cell size of the spatial mesh. The main result of this paper is the following estimate of the error between the optimal solution $\bar{q}$ of the continuous problem and the optimal solution $\bar{q}_\sigma$ of the discrete problem:

$$\|\bar{q} - \bar{q}_\sigma\|_{L^2(0,T;L^2(\Omega))} \leq \frac{C}{\sqrt{\alpha}}\Big(\ln\frac{T}{k}\Big)^{\frac{1}{2}}\big\{k^{\frac{1}{2}} + h\big\}. \tag{1.3}$$

This is to be compared to related results in Deckelnick and Hinze [9] for problem (1.1), but with state constraints pointwise in space and time ((1.2) instead of (1.1d)), which are of the lower order $\mathcal{O}(|\ln h|^{\frac{1}{4}}(h^{\frac{1}{2}} + k^{\frac{1}{4}}))$ in two dimensions and $\mathcal{O}(h^{\frac{1}{4}} + h^{-\frac{1}{4}}k^{\frac{1}{4}})$ in three dimensions.

One of the essential tools for the proof of the estimate (1.3) consists of error estimates with respect to the $L^\infty(0,T;L^2(\Omega))$-norm for the state equation with low regularity of the data. The derivation of these estimates (see section 5) is based on the techniques from Luskin and Rannacher [18] and Rannacher [26].

We emphasize that the proof of (1.3) does not rely on any additional regularity assumptions. We use only the natural regularity of the optimal control $\bar{q} \in L^2(0,T;H^1(\Omega)) \cap L^\infty((0,T)\times\Omega)$, which is obtained from the optimality system; cf. Remark 2.5. Based on the derived estimate (1.3), we are able to improve the regularity result on $\bar{q}$. Finally, we show

$$\bar{q} \in L^2(0,T;H^1(\Omega)) \cap L^\infty((0,T)\times\Omega) \cap H^s(0,T;L^2(\Omega)) \tag{1.4}$$

for all $0 \le s < \frac{1}{2}$; see Theorem 7.1.

The paper is organized as follows. In the next section the optimal control problem is precisely formulated on the continuous level and optimality conditions are discussed. In section 3 the three steps of discretization, i.e., temporal, spatial, and control discretization, are described. In section 4, we provide some stability estimates, which are needed in the following analysis. Section 5 is devoted to error estimates for the state equation with respect to the $L^\infty(0,T;L^2(\Omega))$ norm. The main result (1.3) is proved in section 6. As a by-product of our error analysis, we obtain a new regularity result (1.4) for optimal control $\bar{q}$ in section 7. In the last section, section 8, we present a numerical example for illustrating our theoretical results.

2. Continuous problem. To set up a weak formulation of the state equation (1.1b), we introduce the following notation. For a convex polygonal or polyhedral domain $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, we denote by V the Sobolev space $H_0^1(\Omega)$. Together with $H = L^2(\Omega)$, the Hilbert space V and its dual V^* form a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. Here and in what follows, we employ the usual notation for Lebesgue and Sobolev spaces.

For a time interval $I = (0,T)$, we introduce the "state space"

$$X := \left\{\, v\colon I\times\Omega \to \mathbb{R} \;\middle|\; v \in L^2(I,V) \text{ and } \partial_t v \in L^2(I,V^*) \,\right\}$$

and the "control space"

$$Q = L^2(I,H).$$

Remark 2.1. By obvious modifications, the error analysis derived below also applies to the case of finitely many (time-dependent) parameters instead of distributed control, i.e., for control spaces Q chosen as $Q = \mathbb{R}^l$ or $Q = L^2(I,\mathbb{R}^l)$ for $l \in \mathbb{N}$.

We use the following notation for the inner products and norms on $L^2(\Omega)$ and $L^2(I,L^2(\Omega))$:

$$\begin{aligned} (v,w) &:= (v,w)_{L^2(\Omega)}, & (v,w)_I &:= (v,w)_{L^2(I,L^2(\Omega))},\\ \|v\| &:= \|v\|_{L^2(\Omega)}, & \|v\|_I &:= \|v\|_{L^2(I,L^2(\Omega))}. \end{aligned}$$

Further, we write

$$\begin{aligned} \|v\|_{H^{-1}(\Omega)} &:= \|\nabla\Delta^{-1}v\|, & \|v\|_{H^{-2}(\Omega)} &:= \|\Delta^{-1}v\|,\\ \|v\|_{L^2(I,H^{-1}(\Omega))} &:= \|\nabla\Delta^{-1}v\|_I, & \|v\|_{L^2(I,H^{-2}(\Omega))} &:= \|\Delta^{-1}v\|_I \end{aligned}$$

for the norms of the dual spaces $H^{-1}(\Omega)$, $H^{-2}(\Omega)$, $L^2(I, H^{-1}(\Omega))$, and $L^2(I, H^{-2}(\Omega))$, respectively. Here, $(-\Delta)^{-1}\colon H^{-s}(\Omega) \to H^{2-s}(\Omega)$ denotes for $s \in \{ 1, 2 \}$ the solution operator of the Poisson equation in weak or very weak formulation, respectively.

In this setting, the weak formulation of the state equation (1.1b) for given $q \in Q$, $f \in L^2(I, H)$, and $u_0 \in H$ reads as follows: Find a state $u \in X$ satisfying

$$(2.1) \qquad \begin{aligned} (\partial_t u, \varphi)_I + (\nabla u, \nabla \varphi)_I &= (f + q, \varphi)_I \qquad \forall \varphi \in X, \\ u(0) &= u_0. \end{aligned}$$

A proof of existence and uniqueness of a solution $u \in X$ of (2.1) can be found, e.g., in [12].

Assumption 1. Throughout, we assume the data f and u_0 to exhibit the higher regularity $f \in L^\infty(I, L^2(\Omega))$ and $u_0 \in H^2(\Omega) \cap V$.

To formulate the optimal control problem, we observe the control constraint (1.1c) by introducing the admissible set Q_{ad} as

$$Q_{\mathrm{ad}} := \{ q \in Q \mid q_a \le q(t,x) \le q_b \text{ a.e. in } I \times \Omega \},$$

where the bounds $q_a, q_b \in \mathbb{R}$ fulfill $q_a < q_b$. Furthermore, for the given weight $\omega \in H$, we define the functional $G\colon H \to \mathbb{R}$ by

$$G(v) := (v, \omega).$$

The application of G to time-dependent functions $u\colon I \to H$ is defined by the setting $G(u)(t) := G(u(t))$. The state constraint (1.1d) is then formulated as

$$(2.2) \qquad G(u) \le b \quad \text{in } \bar{I}.$$

Remark 2.2. For $u \in X$, we have $G(u)(\cdot) \in C(\bar{I})$ by construction and due to the continuous embedding $X \hookrightarrow C(\bar{I}, H)$.

With the cost functional $J\colon Q \times L^2(I, H) \to \mathbb{R}$ defined as

$$J(q, u) := \tfrac{1}{2}\|u - \hat{u}\|_I^2 + \tfrac{1}{2}\alpha\|q\|_I^2,$$

the weak formulation of the optimal control problem (1.1) reads as follows:

$$(2.3) \qquad \text{Minimize } J(q,u) \text{ for } (q,u) \in Q_{\mathrm{ad}} \times X \text{ subject to (2.1) and (2.2)},$$

where $\hat{u} \in L^2(I, H)$ is the target state and $\alpha > 0$ the regularization parameter.

Assumption 2. Throughout, we assume the following Slater condition to be satisfied:

$$(2.4) \qquad \exists \tilde{q} \in Q_{\mathrm{ad}} : G(u(\tilde{q})) < b \quad \text{in } \bar{I},$$

where $u(\tilde{q})$ is the solution of (2.1) for the particular control $\tilde{q}$.

Remark 2.3. In view of the initial condition $u_0 \in H$, the relation $G(u_0) < b$ is necessary for the assumed Slater condition to be satisfied.

By standard arguments, the feasibility of the Slater point $\tilde{q}$ ensures the existence and uniqueness of optimal solutions to the considered problem (2.3). To formulate necessary optimality conditions, we employ the dual space of $C(\bar{I})$ denoted by $C(\bar{I})^*$ with its natural norm

$$\|\mu\|_{C(\bar{I})^*} = \sup \left\{ \langle v, \mu \rangle \;\middle|\; v \in C(\bar{I}),\ \|v\|_{C(\bar{I})} \le 1 \right\},$$

where the duality product $\langle \cdot, \cdot \rangle$ between $C(\bar{I})$ and $C(\bar{I})^*$ is given by

$$\langle v, \mu \rangle := \int_{\bar{I}} v \, d\mu.$$

THEOREM 2.4. *A control $\bar{q} \in Q_{ad}$ with associated state $\bar{u} = u(\bar{q})$ is an optimal solution of problem (2.3) if and only if $G(\bar{u}) \leq b$ and there exists an adjoint state $\bar{z} \in L^2(I, V)$ and a Lagrange multiplier $\mu \in C(\bar{I})^*$ with $\mu \geq 0$ such that*

$$(\partial_t \varphi, \bar{z})_I + (\nabla \varphi, \nabla \bar{z})_I = (\varphi, \bar{u} - \hat{u})_I + \langle G(\varphi), \mu \rangle \qquad \forall \varphi \in X, \ \varphi(0) = 0, \tag{2.5}$$

$$(\alpha \bar{q} + \bar{z}, q - \bar{q})_I \geq 0 \qquad \forall q \in Q_{ad}, \tag{2.6}$$

$$\langle b - G(\bar{u}), \mu \rangle = 0. \tag{2.7}$$

Proof. For given $u_0 \in H$ and $f \in L^2(I, H)$ the state equation (2.1) defines a continuous affine linear mapping $q \mapsto u$ from Q to X. Referring to [12] this mapping can be extended to a continuous affine linear mapping $S \colon L^2(I, V^*) \to X$. We denote the concatenation of S with the embedding $X \hookrightarrow L^2(I, H)$ by $\mathcal{S} \colon L^2(I, V^*) \to L^2(I, H)$. Since G is a continuous linear mapping from X to $C(\bar{I})$ (cf. Remark 2.2), we can define $\mathcal{G} \colon L^2(I, V^*) \to C(\bar{I})$ by $\mathcal{G} := G \circ S$. Furthermore, we define $\mathcal{K} \subset C(\bar{I})$ by $\mathcal{K} := \{ v \in C(\bar{I}) \mid v \leq b \text{ in } \bar{I} \}$. These definitions enable us to embed (2.3) into the following abstract setting of optimization problems (cf., e.g., [15]):

$$\text{Minimize } j(q) := J(q, \mathcal{S}(q)) \text{ for } q \in Q_{\text{ad}} \text{ subject to } \mathcal{G}(q) \in \mathcal{K}.$$

Then, by the generalized KKT theory (see, e.g., [17, 34]) the assumed Slater condition (2.4) (which postulates the existence of $\tilde{q} \in Q_{\text{ad}}$ such that $\mathcal{G}(\tilde{q}) \in \operatorname{int} \mathcal{K}$) implies that the optimality of $\bar{q}$ is equivalent to the existence of a Lagrange multiplier $\mu \in C(\bar{I})^*$ and an adjoint state $\bar{z} = \mathcal{S}'(\bar{q})^*(\mathcal{S}(\bar{q}) - \hat{u}) + \mathcal{G}'(\bar{q})^* \mu$ fulfilling

$$(\alpha \bar{q} + \bar{z}, q - \bar{q})_I \geq 0 \quad \forall q \in Q_{\text{ad}} \quad \text{and} \quad \langle v - \mathcal{G}(\bar{q}), \mu \rangle \leq 0 \quad \forall v \in \mathcal{K}.$$

Recalling the definitions of $\mathcal{S}$ and $\mathcal{G}$, we obtain that $\mathcal{S}'(\bar{q})^* \colon L^2(I, H) \to L^2(I, V)$ and $\mathcal{G}'(\bar{q})^* \colon C(\bar{I})^* \to L^2(I, V)$ which imply that $\bar{z} \in L^2(I, V)$. Furthermore, also by the definitions of $\mathcal{S}$ and $\mathcal{G}$ we conclude that the derived expression of $\bar{z}$ is equivalent to $\bar{z}$ being the solution of (2.5). We complete the proof by noting the equivalence

$$\langle v - \mathcal{G}(\bar{q}), \mu \rangle \leq 0 \quad \forall v \in \mathcal{K} \quad \iff \quad \mu \geq 0 \text{ and } \langle b - G(\bar{u}), \mu \rangle = 0.$$

This completes the proof. ∎

Remark 2.5. The variational inequality (2.6) can be equivalently rewritten using the pointwise projection $P_{Q_{\text{ad}}}$ onto the set of admissible controls Q_{ad} as follows:

$$\bar{q} = P_{Q_{\text{ad}}}(-\alpha^{-1} \bar{z}).$$

Therefore, the regularity $\bar{z} \in L^2(I, V)$ implies

$$\bar{q} \in L^2(I, H^1(\Omega)) \cap L^\infty(I \times \Omega).$$

In section 7 we will provide a stronger regularity result for the optimal control $\bar{q}$.

3. Discretization.

In this section we describe the space-time finite element discretization of the optimal control problem (2.3).

3.1. Semidiscretization in time. First, we define the semidiscretization in time of the state equation by "discontinuous Galerkin" methods; cf. [10, 19]. To this end, we consider a partitioning of the time interval $\bar{I} = [0, T]$ such as

$$\bar{I} = \{0\} \cup I_1 \cup I_2 \cup \cdots \cup I_M \tag{3.1}$$

with subintervals $I_m = (t_{m-1}, t_m]$ of size k_m and time points

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T.$$

The discretization parameter k is viewed as a piecewise constant function by setting $k\big|_{I_m} = k_m$ for $m = 1, 2, \ldots, M$. The maximum size of the time steps is also denoted by k, i.e., $k = \max_{m=1,2,\ldots,M} k_m$. We impose the following conditions on the time mesh:

(i) There are constants $c, \gamma > 0$ such that

$$\min_{m=1,2,\ldots,M} k_m \geq c k^\gamma.$$

(ii) There is a constant $\kappa > 0$ such that for all $m = 1, 2, \ldots, M-1$,

$$\kappa^{-1} \leq \frac{k_m}{k_{m+1}} \leq \kappa.$$

(iii) It holds that $k \leq \frac{1}{4} T$.

The semidiscrete trial and test spaces are defined as

$$X_k^r = \left\{ v_k \in L^2(I, V) \;\middle|\; v_k\big|_{I_m} \in \mathcal{P}_r(I_m, V),\ m = 1, 2, \ldots, M \right\}.$$

Here, $\mathcal{P}_r(I_m, V)$ is the space of polynomials of maximum degree r defined on I_m with values in V. On X_k^r we use the notation

$$(v, w)_{I_m} := (v, w)_{L^2(I_m, L^2(\Omega))}, \quad \|v\|_{I_m} := \|v\|_{L^2(I_m, L^2(\Omega))}.$$

To define the discontinuous Galerkin (abbreviated as dG(r)) approximation using the space X_k^r, we employ the following notation for functions $v_k \in X_k^r$:

$$v_{k,m}^+ := \lim_{t \to 0^+} v_k(t_m + t), \quad v_{k,m}^- := \lim_{t \to 0^+} v_k(t_m - t) = v_k(t_m), \quad [v_k]_m := v_{k,m}^+ - v_{k,m}^-$$

and define the bilinear form $B(\cdot, \cdot)$ for arguments $u_k, \varphi \in X_k^r$ by

$$B(u_k, \varphi) := \sum_{m=1}^{M} (\partial_t u_k, \varphi)_{I_m} + (\nabla u_k, \nabla \varphi)_I + \sum_{m=2}^{M} ([u_k]_{m-1}, \varphi_{m-1}^+) + (u_{k,0}^+, \varphi_0^+).$$

Then, the dG(r) semidiscretization of the state equation (2.1) for given control $q \in Q$ reads as follows: Find a state $u_k = u_k(q) \in X_k^r$ satisfying

$$B(u_k, \varphi) = (f + q, \varphi)_I + (u_0, \varphi_0^+) \quad \forall \varphi \in X_k^r. \tag{3.2}$$

The existence and uniqueness of solutions to (3.2) can be shown by using Fourier analysis; see [31] for details.

Remark 3.1. Using the density of X in $L^2(I,V) \supset X_k^0$, it is possible to show that the exact solution $u = u(q) \in X$ of the state equation (2.1) satisfies the identity

$$B(u,\varphi) = (f+q,\varphi)_I + (u_0,\varphi_0^+) \quad \forall \varphi \in X_k^r.$$

Thus, the dG(r) time discretization satisfies the "Galerkin orthogonality" equation

$$B(u-u_k,\varphi) = 0 \quad \forall \varphi \in X_k^r.$$

Throughout the paper, we restrict ourselves to the lowest-order case $r = 0$, i.e., piecewise constant approximation in time. The resulting dG(0) scheme is a variant of the implicit Euler method. Because of this, the notation for the discontinuous piecewise constant functions $v_k \in X_k^0$ can be simplified. Setting $v_{k,m} := v_{k,m}^-$, we have

$$v_{k,m} = v_k\big|_{I_m}, \quad v_{k,m}^+ = v_{k,m+1}, \quad \text{and} \quad [v_k]_m = v_{k,m+1} - v_{k,m}.$$

In this notation, the bilinear form $B(\cdot,\cdot)$ reads as

$$B(u_k,\varphi) := \sum_{m=1}^{M} (\partial_t u_k,\varphi)_{I_m} + (\nabla u_k,\nabla\varphi)_I + \sum_{m=2}^{M} ([u_k]_{m-1},\varphi_m) + (u_{k,1},\varphi_1). \tag{3.3}$$

Since $u_k \in X_k^0$ is piecewise constant in time, the state constraint $G(u_k) \le b$ can be written in the form of finitely many constraints,

$$G(u_k)\big|_{I_m} \le b \quad \text{for } m = 1,2,\dots,M. \tag{3.4}$$

Then, for the dG(0) time discretization, the semidiscrete optimization problem has the following form:

$$\text{Minimize } J(q_k,u_k) \text{ for } (q_k,u_k) \in Q_{\mathrm{ad}} \times X_k^0 \text{ subject to (3.2) and (3.4).} \tag{3.5}$$

In view of the Slater condition (2.4), by arguments as used later on in the proof of Lemma 6.2, we obtain the existence of a Slater point of the semidiscrete problem (3.5) for k small enough. As in the continuous case, this implies by standard arguments the existence of a unique optimal control $\bar{q}_k \in Q$.

Remark 3.2. We note that the optimal control $\bar{q}_k$ is searched for in the subset Q_{ad} of the continuous control space Q, and the subscript k indicates the usage of the semidiscretized state equation.

Similarly to the continuous setting, we can formulate the following result on existence and optimality conditions.

THEOREM 3.3. *A control $\bar{q}_k \in Q_{ad}$ with associated state $\bar{u}_k = u_k(\bar{q}_k)$ is an optimal solution of problem* (3.5) *if and only if $G(\bar{u}_k)\big|_{I_m} \le b$ for $m = 1,2,\dots,M$ and there exists an adjoint state $\bar{z}_k \in X_k^0$ and a Lagrange multiplier $\mu_k \in C(\bar{I})^*$ given for any $v \in C(\bar{I})$ by*

$$\langle v,\mu_k\rangle = \sum_{l=1}^{M} \frac{\mu_{k,l}}{k_l} \int_{I_l} v(t)\,dt \quad \textit{with} \quad \mu_{k,l} \in \mathbb{R},\ \mu_{k,l} \ge 0\ (l = 1,2,\dots,M) \tag{3.6}$$

such that

$$B(\varphi,\bar{z}_k) = (\varphi,\bar{u}_k - \hat{u})_I + \langle G(\varphi),\mu_k\rangle \qquad \forall \varphi \in X_k^0, \tag{3.7}$$

$$(\alpha\bar{q}_k + \bar{z}_k, q - \bar{q}_k)_I \ge 0 \qquad \forall q \in Q_{ad}, \tag{3.8}$$

$$\langle b - G(\bar{u}_k),\mu_k\rangle = 0. \tag{3.9}$$

Proof. Following the argument used in the proof of Theorem 2.4, we extend the mapping $q \mapsto u_k \in X_k^0$ to a linear mapping $S_k\colon (X_k^0)^* \to X_k^0$ and denote the concatenation of S_k with the embedding $X_k^0 \hookrightarrow L^2(I,H)$ by $\mathcal{S}_k$. We directly obtain the continuity of S_k and consequently also that of $\mathcal{S}_k$. The finitely many state constraints are described with the help of the continuous linear operator $\mathcal{G}_k\colon X_k^{0^*} \to \mathbb{R}^M$ with $(\mathcal{G}_k)_m := (G \circ S_k)\big|_{I_m}$ for $m = 1, 2, \ldots, M$. By means of the set $\mathcal{K}_k := \{\, v \in \mathbb{R}^M \mid v_m \le b,\ m = 1, 2, \ldots, M \,\}$ we can rewrite problem (3.5) as follows:

$$\text{Minimize } j_k(q) := J(q, \mathcal{S}_k(q)) \text{ for } q \in Q_{\text{ad}} \text{ subject to } \mathcal{G}_k(q) \in \mathcal{K}_k.$$

In view of the Slater condition (2.4), by arguments as used later on in the proof of Lemma 6.2, we obtain that $\mathcal{G}_k(\tilde{q}) \in \operatorname{int} \mathcal{K}_k$ is fulfilled for k small enough. Hence, as in the proof of Theorem 2.4, we obtain that the optimality of $\bar{q}_k$ is equivalent to the existence of a Lagrange multiplier $(\mu_{k,l})_{l=1}^M \in \mathbb{R}_+^M$ and an adjoint state $\bar{z}_k \in X_k^0$ fulfilling (3.7), (3.8), and (3.9). Via the construction given in (3.6), μ_k is then defined as an element of $C(\bar{I})^*$. ∎

Remark 3.4. As on the continuous level (see Remark 2.5), the variational inequality (3.8) can be equivalently rewritten using the pointwise projection $P_{Q_{\text{ad}}}$ as

$$\bar{q}_k = P_{Q_{\text{ad}}}(-\alpha^{-1} \bar{z}_k).$$

Although the control has not yet explicitly been discretized, from this projection formula, we obtain that $\bar{q}_k\big|_{I_m} \in \mathcal{P}_0(I_m, H^1(\Omega))$ for $m = 1, 2, \ldots, M$.

Remark 3.5. We note that using integration by parts in time, the bilinear form $B(\varphi, z_k)$ in (3.7) defined by (3.3) can equivalently be expressed as follows:

$$B(\varphi, z_k) = -\sum_{m=1}^{M} (\varphi, \partial_t z_k)_{I_m} + (\nabla\varphi, \nabla z_k)_I - \sum_{m=1}^{M-1} (\varphi_m, [z_k]_m) + (\varphi_M, z_{k,M}). \tag{3.10}$$

3.2. Discretization in space. To define the Galerkin finite element discretization in space, we consider families of two- or three-dimensional meshes covering the computational domain $\bar{\Omega}$, which satisfy the usual regularity conditions such as conformity and shape regularity (see, e.g., [4]). The meshes consist of quadrilateral or hexahedral cells K and are denoted by $\mathcal{T}_h = \{K\}$, where we define the discretization parameter h as a cellwise constant function by setting $h\big|_K = h_K$ with the diameter h_K of the cell K. We use the symbol h also for the maximum cell size, i.e., $h = \max h_K$.

On the mesh $\mathcal{T}_h$, we construct a conforming finite element space $V_h \subset V$ in a standard way:

$$V_h^s = \left\{\, v \in V \;\middle|\; v\big|_K \in \mathcal{Q}_s(K) \text{ for } K \in \mathcal{T}_h \,\right\}.$$

Here, $\mathcal{Q}_s(K)$ consists of shape functions obtained via (bi-/tri-)linear transformations of polynomials in $\widehat{\mathcal{Q}_s}(\widehat{K})$ defined on the reference cell $\widehat{K} = (0,1)^n$, where

$$\widehat{\mathcal{Q}_s}(\widehat{K}) = \operatorname{span}\left\{ \prod_{j=1}^{n} x_j^{\alpha_j} \;\middle|\; \alpha_j \in \mathbb{N}_0,\ \alpha_j \le s \right\}.$$

To obtain the fully discretized versions of the time discretized state equation (3.2), we introduce the space-time finite element space

$$X_{k,h}^{r,s} = \left\{ v_{kh} \in L^2(I, V_h^s) \;\middle|\; v_{kh}\big|_{I_m} \in \mathcal{P}_r(I_m, V_h^s),\ m = 1, 2, \ldots, M \right\} \subset X_k^r.$$

Then, the so-called cG(s)dG(r) discretization (continuous in space and discontinuous in time) of the state equation for given control $q \in Q$ has the following form: Find a state $u_{kh} = u_{kh}(q) \in X_{k,h}^{r,s}$ satisfying

$$B(u_{kh}, \varphi) = (f + q, \varphi)_I + (u_0, \varphi_0^+) \quad \forall \varphi \in X_{k,h}^{r,s}. \tag{3.11}$$

Throughout this paper we will restrict our analysis to the lowest-order case of (bi-/tri-)linear elements; i.e., we set $s = 1$ and consider the cG(1)dG(0) scheme. The state constraint on this level of discretization is given as in subsection 3.1 by

$$G(u_{kh})\big|_{I_m} \le b \quad \text{for } m = 1, 2, \dots, M. \tag{3.12}$$

Then, the corresponding fully discrete optimal control problem reads as follows:

$$\text{Minimize } J(q_{kh}, u_{kh}) \text{ for } (q_{kh}, u_{kh}) \in Q_{\text{ad}} \times X_{k,h}^{0,1} \text{ subject to (3.11) and (3.12),} \tag{3.13}$$

and the optimality conditions are given by the following theorem.

THEOREM 3.6. *A control $\bar{q}_{kh} \in Q_{ad}$ with associated state $\bar{u}_{kh} = u_{kh}(\bar{q}_{kh})$ is an optimal solution of problem* (3.13) *if and only if $G(\bar{u}_{hk})\big|_{I_m} \le b$ for $m = 1, 2, \dots, M$ and there exists an adjoint state $\bar{z}_{kh} \in X_{k,h}^{0,1}$ and a Lagrange multiplier $\mu_{kh} \in C(\bar{I})^*$ given for any $v \in C(\bar{I})$ by*

$$\langle v, \mu_{kh} \rangle = \sum_{l=1}^{M} \frac{\mu_{kh,l}}{k_l} \int_{I_l} v(t)\, dt \quad \textit{with} \quad \mu_{kh,l} \in \mathbb{R},\ \mu_{kh,l} \ge 0\ (l = 1, 2, \dots, M) \tag{3.14}$$

such that

$$B(\varphi, \bar{z}_{kh}) = (\varphi, \bar{u}_{kh} - \hat{u})_I + \langle G(\varphi), \mu_{kh} \rangle \qquad \forall \varphi \in X_{k,h}^{0,1}, \tag{3.15}$$

$$(\alpha \bar{q}_{kh} + \bar{z}_{kh}, q - \bar{q}_{kh})_I \ge 0 \qquad \forall q \in Q_{ad}, \tag{3.16}$$

$$\langle b - G(\bar{u}_{kh}), \mu_{kh} \rangle = 0. \tag{3.17}$$

Proof. The theorem can be proved by repeating the steps of the proof of Theorem 3.3. ∎

Remark 3.7. As for $\bar{q}$ and $\bar{q}_k$ (see Remarks 2.5 and 3.4), we obtain

$$\bar{q}_{kh} = P_{Q_{\text{ad}}}(-\alpha^{-1} \bar{z}_{kh}) \tag{3.18}$$

and therefore $\bar{q}_{kh}\big|_{I_m} \in \mathcal{P}_0(I_m, H^1(\Omega))$ for $m = 1, 2, \dots, M$. We note that since $\bar{z}_{kh}$ is cellwise (bi-/tri-)linear, $\bar{q}_{kh}\big|_{I_m}$ may have kinks in the interior of a cell and therefore is in general not in $\mathcal{P}_0(I_m, V_h)$.

3.3. Discretization of the controls. In this subsection, we describe the discretization of the control variable by lowest-order finite elements, i.e., cellwise constant functions. We employ the same time partitioning and the same spatial mesh as for the discretization of the state variable and set

$$Q_d = \left\{ q \in Q \;\middle|\; q\big|_{I_m \times K} \in \mathcal{P}_0(I_m \times K),\ m = 1, 2, \dots, M,\ K \in \mathcal{T}_h \right\}.$$

For this choice of the subspace $Q_d \subset Q$, we introduce the corresponding admissible set $Q_{d,\text{ad}}$ by

$$Q_{d,\text{ad}} := Q_d \cap Q_{\text{ad}}.$$

The state constraint can be expressed as in the previous sections by the conditions

$$G(u_\sigma)\big|_{I_m} \le b \quad \text{for } m = 1, 2, \dots, M. \tag{3.19}$$

Then, the optimal control problem on this level of discretization reads as follows:

$$\text{Minimize } J(q_\sigma, u_\sigma) \text{ for } (q_\sigma, u_\sigma) \in Q_{d,\text{ad}} \times X_{k,h}^{0,1} \text{ subject to (3.11) and (3.19).} \tag{3.20}$$

The unique optimal solution of (3.20) is denoted by $(\bar{q}_\sigma, \bar{u}_\sigma)$, where the subscript σ represents all three discretization parameters k, h, and d. The corresponding first-order necessary optimality conditions are stated in the following theorem.

THEOREM 3.8. *A control $\bar{q}_\sigma \in Q_{d,ad}$ with associated state $\bar{u}_\sigma = u_{kh}(\bar{q}_\sigma)$ is an optimal solution of problem* (3.20) *if and only if $G(\bar{u}_\sigma)\big|_{I_m} \le b$ for $m = 1, 2, \dots, M$ and there exists an adjoint state $\bar{z}_\sigma \in X_{k,h}^{0,1}$ and a Lagrange multiplier $\mu_\sigma \in C(\bar{I})^*$ given for any $v \in C(\bar{I})$ by*

$$\langle v, \mu_\sigma \rangle = \sum_{l=1}^{M} \frac{\mu_{\sigma,l}}{k_l} \int_{I_l} v(t)\, dt \quad \text{with} \quad \mu_{\sigma,l} \in \mathbb{R},\ \mu_{\sigma,l} \ge 0\ (l = 1, 2, \dots, M) \tag{3.21}$$

such that

$$B(\varphi, \bar{z}_\sigma) = (\varphi, \bar{u}_\sigma - \hat{u})_I + \langle G(\varphi), \mu_\sigma \rangle \qquad \forall \varphi \in X_{k,h}^{0,1}, \tag{3.22}$$

$$(\alpha \bar{q}_\sigma + \bar{z}_\sigma, q - \bar{q}_\sigma)_I \ge 0 \qquad \forall q \in Q_{d,ad}, \tag{3.23}$$

$$\langle b - G(\bar{u}_\sigma), \mu_\sigma \rangle = 0. \tag{3.24}$$

Proof. The theorem can be proved by repeating the steps of the proof of Theorem 3.3. □

4. Stability estimates. In this section, we provide several stability estimates for adjoint solutions arising from the optimality conditions of the optimization problem and for additional auxiliary solutions defined below in subsection 4.2.

4.1. Semidiscrete and discrete adjoint solutions. At first, we consider the solution of the discrete adjoint equation (3.15).

THEOREM 4.1. *For the solution $\bar{z}_{kh} \in X_{k,h}^{0,1}$ of* (3.15) *there holds*

$$\|\nabla \bar{z}_{kh}\|_I \le C\Big\{\|\bar{u}_{kh} - \hat{u}\|_I + \|\omega\| \|\mu_{kh}\|_{C(\bar{I})^*}\Big\}. \tag{4.1}$$

Proof. By means of the definition of μ_{kh}, the definition of G, and the setting $\varphi_l = \varphi\big|_{I_l}$, (3.15) can be rewritten in the form

$$B(\varphi, \bar{z}_{kh}) = (\varphi, \bar{u}_{kh} - \hat{u})_I + \sum_{l=1}^{M} \mu_{kh,l}(\varphi_l, \omega) \quad \forall \varphi \in X_{k,h}^{0,1}.$$

Defining the solutions z_k^l for $l = 0, 1, \dots, M$ by

$$B(\varphi, z_{kh}^0) = (\varphi, \bar{u}_{kh} - \hat{u})_I \qquad \forall \varphi \in X_{k,h}^{0,1},$$
$$B(\varphi, z_{kh}^l) = (\varphi_l, \omega) \qquad \forall \varphi \in X_{k,h}^{0,1},\ l = 1, 2, \dots, M,$$

we have the representation

$$\bar{z}_{kh} = z_{kh}^0 + \sum_{l=1}^{M} \mu_{kh,l} z_{kh}^l.$$

Hence, we get

$$\|\nabla \bar{z}_{kh}\|_I \leq \|\nabla z^0_{kh}\|_I + \sum_{l=1}^{M} \mu_{kh,l} \|\nabla z^l_{kh}\|_I \leq \|\nabla z^0_{kh}\|_I + \max_{l=1,2,\dots,M} \|\nabla z^l_{kh}\|_I \sum_{l=1}^{M} \mu_{kh,l}.$$

To estimate $\|\nabla z^l_{kh}\|_I$ for $l = 0, 1, \dots, M$, we consider the solution $\tilde{z}_{kh} \in X^{0,1}_{k,h}$ of

$$B(\varphi, \tilde{z}_{kh}) = (\varphi, g)_I + (\varphi_M, \tilde{z}_T) \quad \forall \varphi \in X^{0,1}_{k,h},$$

with $g \in L^2(I, H)$ and $\tilde{z}_T \in H$. By means of (3.10) and the setting $z_{kh,M+1} \coloneqq \tilde{z}_T$ this can be rewritten as the following system of equations:

$$(\nabla \varphi, \nabla \tilde{z}_{kh})_{I_m} - (\varphi_m, [\tilde{z}_{kh}]_m) = (\varphi, g)_I \quad \forall \varphi \in \mathcal{P}_0(I_m, V_h),\ m = 1, 2, \dots, M.$$

Choosing $\varphi = \tilde{z}_{kh}$ and using the algebraic identity

$$(y_m, [y]_m) = \tfrac{1}{2}\|y_{m+1}\|^2 - \tfrac{1}{2}\|[y]_m\|^2 - \tfrac{1}{2}\|y_m\|^2 \tag{4.2}$$

implies

$$\|\tilde{z}_{kh,m}\|^2 + 2\|\nabla \tilde{z}_{kh}\|^2_{I_m} \leq \|\tilde{z}_{kh,m+1}\|^2 + 2\|g\|_{I_m}\|\tilde{z}_{kh}\|_{I_m}.$$

Hence, by the inequalities of Poincaré and Young and summing up for $m = 1, 2, \dots, M$, we end up with

$$\|\nabla \tilde{z}_{kh}\|^2_I \leq C\big\{\|g\|^2_I + \|\tilde{z}_T\|^2\big\}.$$

Application of this estimate to the solutions z^l_{kh} with $l = 0, 1, \dots, M$ yields

$$\|\nabla z^0_{kh}\|_I \leq C\|\bar{u}_{kh} - \hat{u}\|_I \qquad \text{and} \qquad \|\nabla z^l_{kh}\|_I \leq C\|\omega\|,\ l = 1, 2, \dots, M.$$

Then, these estimates together with

$$\sum_{l=1}^{M} \mu_{kh,l} = \langle \mu_{kh}, 1 \rangle \leq \|\mu_{kh}\|_{C(\bar{I})^*}$$

imply the assertion. □

A similar result holds for the solution z_σ of the discrete adjoint equation (3.22).

COROLLARY 4.2. *For the solution $\bar{z}_\sigma \in X^{0,1}_{k,h}$ of (3.22) there holds*

$$\|\nabla \bar{z}_\sigma\|_I \leq C\Big\{\|\bar{u}_\sigma - \hat{u}\|_I + \|\omega\|\|\mu_\sigma\|_{C(\bar{I})^*}\Big\}. \tag{4.3}$$

Proof. The assertion follows immediately by repeating the steps of the proof of Theorem 4.1. □

4.2. Continuous and semidiscrete auxiliary solutions. We consider the following forward and backward auxiliary problems: Find $v \in X$ fulfilling

$$\begin{aligned} (\partial_t v, \varphi)_I + (\nabla v, \nabla \varphi)_I &= 0 \qquad \forall \varphi \in X, \\ v(0) &= v_0, \end{aligned} \tag{4.4}$$

with initial value $v_0 \in H$, and find $y \in X$ fulfilling

$$\begin{aligned} -(\varphi, \partial_t y)_I + (\nabla \varphi, \nabla y)_I &= 0 \qquad \forall \varphi \in X, \\ y(T) &= y_T, \end{aligned} \tag{4.5}$$

with terminal value $y_T \in H$. Existence and uniqueness of the solutions $v \in X$ and $y \in X$ of (4.4) and (4.5) are proved, e.g., in [12].

The corresponding semidiscrete analogues are given as follows: Find $v_k \in X_k^0$ fulfilling

$$B(v_k, \varphi) = (v_0, \varphi_1) \quad \forall \varphi \in X_k^0, \tag{4.6}$$

and find $y_k \in X_k^0$ fulfilling

$$B(\varphi, y_k) = (\varphi_M, y_T) \quad \forall \varphi \in X_k^0. \tag{4.7}$$

Furthermore the discrete variants are given by the following formulation: Find $v_{kh} \in X_{k,h}^{0,1}$ fulfilling

$$B(v_{kh}, \varphi) = (v_0, \varphi_1) \quad \forall \varphi \in X_{k,h}^{0,1}, \tag{4.8}$$

and find $y_{kh} \in X_{k,h}^{0,1}$ fulfilling

$$B(\varphi, y_{kh}) = (\varphi_M, y_T) \quad \forall \varphi \in X_{k,h}^{0,1}. \tag{4.9}$$

For the solution of (4.5), we have the following stability result.

THEOREM 4.3. *For the solution $y \in X$ of* (4.5) *there holds*

$$\|\nabla y\|_I + \max_{t \in \bar{I}} \|y(t)\| \le \|y_T\|. \tag{4.10}$$

Proof. See, for instance, [12]. ∎

Next, we prove an a priori estimate for the solution of (4.5) with respect to "time-weighted norms."

THEOREM 4.4. *For the solution $y \in X$ of* (4.5) *there hold the a priori estimates*

$$\int_I (T-t)\|\partial_t y(t)\|^2 \, dt \le C\|y_T\|^2 \tag{4.11}$$

$$\int_{I \setminus I_M} \|\partial_t y(t)\| \, dt \le C\Big(\ln \frac{T}{k}\Big)^{\frac{1}{2}} \|y_T\|. \tag{4.12}$$

Proof. To estimate $\int_I (T-t)\|\partial_t y(t)\|^2 \, dt$, we choose $\varphi = -(T-t)\partial_t y$ in (4.5) obtaining

$$\begin{aligned} 0 &= \int_I (T-t)\|\partial_t y(t)\|^2 \, dt - ((T-t)\nabla y, \nabla \partial_t y)_I \\ &= \int_I (T-t)\|\partial_t y(t)\|^2 \, dt - \frac{1}{2}\int_I \frac{d}{dt}\big((T-t)\|\nabla y(t)\|^2\big) \, dt - \frac{1}{2}\|\nabla y\|_I^2. \end{aligned}$$

From this, we conclude

$$2\int_I (T-t)\|\partial_t y\|^2 \, dt + T\|\nabla y(0)\|^2 \le \|\nabla y\|_I^2.$$

Then, the application of the a priori estimate from Theorem 4.3 yields the first of the asserted estimates. The second then follows immediately from

$$\begin{aligned} \int_{I \setminus I_M} \|\partial_t y(t)\| \, dt &\le \left(\int_{I \setminus I_M} (T-t)^{-1} \, dt\right)^{\frac{1}{2}} \left(\int_{I \setminus I_M} (T-t)\|\partial_t y(t)\|^2 \, dt\right)^{\frac{1}{2}} \\ &\le C\Big(\ln \frac{T}{k}\Big)^{\frac{1}{2}} \left(\int_I (T-t)\|\partial_t y(t)\|^2 \, dt\right)^{\frac{1}{2}}. \end{aligned}$$

The proof is complete. ∎

In the following theorem, we derive a stability estimate for the semidiscrete solutions of (4.6) and (4.7).

THEOREM 4.5. *For the solutions $v_k \in X_k^0$ of* (4.6) *and $y_k \in X_k^0$ of* (4.7) *there hold the a priori estimates*

$$T^2\|\Delta v_{k,M}\|^2 + T\|\nabla v_{k,M}\|^2 + \|v_{k,M}\|^2 + \sum_{m=1}^{M} t_m \|\Delta v_k\|_{I_m}^2 + \|\nabla v_k\|_I^2 + \sum_{m=2}^{M} \frac{t_{m-1}^2}{k_m} \|[\nabla v_k]_{m-1}\|^2 \le C\|v_0\|^2 \tag{4.13}$$

and

$$T^2\|\Delta y_{k,1}\|^2 + T\|\nabla y_{k,1}\|^2 + \|y_{k,1}\|^2 + \sum_{m=1}^{M} \tau_{k,m} \|\Delta y_k\|_{I_m}^2 + \|\nabla y_k\|_I^2 + \sum_{m=1}^{M-1} \frac{\tau_{k,m+1}^2}{k_m} \|[\nabla y_k]_m\|^2 \le C\|y_T\|^2, \tag{4.14}$$

with $\tau_{k,m} = \tau_k\big|_{I_m} \coloneqq T - t_{m-1}$.

Proof. For proving the assertion for v_k, we recall (4.6), which by means of the setting $v_{k,0} \coloneqq v_0$ can be rewritten as the following system of equations:

$$(\nabla v_k, \nabla\varphi)_{I_m} + ([v_k]_{m-1}, \varphi_m) = 0 \quad \forall \varphi \in \mathcal{P}_0(I_m, V),\ m = 1, 2, \dots, M. \tag{4.15}$$

(i) Choosing $\varphi = v_k$ in (4.15) leads us to

$$\|\nabla v_k\|_{I_m}^2 + ([v_k]_{m-1}, v_{k,m}) = 0.$$

Then, the algebraic identity

$$([v]_{m-1}, v_m) = \tfrac{1}{2}\|v_m\|^2 + \tfrac{1}{2}\|[v]_{m-1}\|^2 - \tfrac{1}{2}\|v_{m-1}\|^2 \tag{4.16}$$

is used to obtain

$$\|v_{k,m}\|^2 + 2\|\nabla v_k\|_{I_m}^2 \le \|v_{k,m-1}\|^2.$$

By adding these inequalities for $m = 1, 2, \dots, M$, we arrive at

$$\|v_{k,M}\|^2 + 2\|\nabla v_k\|_I^2 \le \|v_0\|^2. \tag{4.17}$$

(ii) Integrating by parts in (4.15) and choosing $\varphi\big|_{I_m} = -\frac{t_{m-1}^2}{k_m}[\Delta v_k]_{m-1}$ gives us

$$\frac{t_{m-1}^2}{k_m}(\Delta v_k, [\Delta v_k]_{m-1})_{I_m} + \frac{t_{m-1}^2}{k_m}\|[\nabla v_k]_{m-1}\|^2 = 0,$$

and thus

$$t_{m-1}^2(\Delta v_{k,m}, [\Delta v_k]_{m-1}) + \frac{t_{m-1}^2}{k_m}\|[\nabla v_k]_{m-1}\|^2 = 0.$$

Then, the algebraic identity (4.16) and the relation

$$t_{m-1}^2 \ge t_m^2 - 2k_m t_m$$

are used to obtain

$$t_m^2\|\Delta v_{k,m}\|^2 + 2\frac{t_{m-1}^2}{k_m}\|[\nabla v_k]_{m-1}\|^2 \le t_{m-1}^2\|\Delta v_{k,m-1}\|^2 + 2k_m t_m\|\Delta v_{k,m}\|^2.$$

By adding these inequalities for $m = 2, 3, \dots, M$ and using $t_1 = k_1$, we arrive at

$$\begin{aligned} T^2\|\Delta v_{k,M}\|^2 + 2\sum_{m=2}^{M}\frac{t_{m-1}^2}{k_m}\|[\nabla v_k]_{m-1}\|^2 &\le t_1 k_1\|\Delta v_{k,1}\|^2 + 2\sum_{m=2}^{M} t_m\|\Delta v_k\|_{I_m}^2 \\ &\le 2\sum_{m=1}^{M} t_m\|\Delta v_k\|_{I_m}^2. \end{aligned} \tag{4.18}$$

(iii) Integrating by parts in (4.15) and choosing $\varphi\big|_{I_m} = -t_m\Delta v_k$ gives us

$$t_m\|\Delta v_k\|_{I_m}^2 + t_m([\nabla v_k]_{m-1}, \nabla v_{k,m}) = 0.$$

Then, the algebraic identity (4.16) and the relation $k_m \le \kappa^{-1}k_{m-1}$ imply that

$$t_m\|\nabla v_{k,m}\|^2 + 2t_m\|\Delta v_k\|_{I_m}^2 \le t_{m-1}\|\nabla v_{k,m-1}\|^2 + \kappa^{-1}k_{m-1}\|\nabla v_{k,m-1}\|^2.$$

By adding these inequalities for $m = 2, 3, \dots, M$ and using $t_1 = k_1$, we arrive at

$$\begin{aligned} T\|\nabla v_{k,M}\|^2 + 2\sum_{m=2}^{M} t_m\|\Delta v_k\|_{I_m}^2 &\le k_1\|\nabla v_{k,1}\|^2 + \kappa^{-1}\sum_{m=1}^{M-1}\|\nabla v_k\|_{I_m}^2 \\ &\le (1+\kappa^{-1})\|\nabla v_k\|_I^2. \end{aligned} \tag{4.19}$$

Hence, it remains to estimate $t_1\|\Delta v_k\|_{I_1}^2$.

(iv) Integrating by parts in (4.15) and choosing $\varphi = -\Delta v_k$ leads us for $m = 1$ to

$$\|\Delta v_k\|_{I_1}^2 = (v_{k,1} - v_0, \Delta v_{k,1}) \le \|v_{k,1} - v_0\|\|\Delta v_{k,1}\| = k_1^{-\frac{1}{2}}\|v_{k,1} - v_0\|\|\Delta v_k\|_{I_1}$$

and consequently to

$$\|\Delta v_k\|_{I_1} \le k_1^{-\frac{1}{2}}\|v_{k,1} - v_0\|.$$

This implies

$$k_1\|\Delta v_{k,1}\| = k_1^{\frac{1}{2}}\|\Delta v_k\|_{I_1} \le \|v_{k,1} - v_0\|,$$

and using $t_1 = k_1$,

$$t_1\|\Delta v_k\|_{I_1}^2 = k_1^2\|\Delta v_{k,1}\|^2 \le 2\big\{\|v_0\|^2 + \|v_{k,1}\|^2\big\}. \tag{4.20}$$

Combining the estimates (4.17), (4.18), (4.19), and (4.20) yields the first of the asserted estimates. The second on y_k follows by inspection of (4.7), which by means of the setting $y_{k,M+1} \coloneqq y_T$ can be rewritten as the following system of equations:

$$(\nabla\varphi, \nabla y_k)_{I_m} - (\varphi_m, [y_k]_m) = 0 \quad \forall\varphi \in \mathcal{P}_0(I_m, V),\ m = 1, 2, \dots, M. \tag{4.21}$$

We repeat the above steps (i)–(iv) and employ the algebraic identity (4.2) and the relation

$$\tau_{k,m+1}^2 \ge \tau_{k,m}^2 - 2k_m\tau_{k,m}$$

to derive the desired result. $\square$

For the case of more regular initial and terminal values for the solutions v_0 and y_T of (4.6) and (4.7), respectively, we have the following results.

THEOREM 4.6. *If $v_0, y_T \in H^2(\Omega) \cap V$, for the solutions $v_k \in X_k^0$ of (4.6) and $y_k \in X_k^0$ of (4.7) there hold the a priori estimates*

$$T\|\nabla\Delta v_{k,M}\|^2 + \|\Delta v_{k,M}\|^2 + \|\nabla\Delta v_k\|_I^2 + \sum_{m=2}^{M} \frac{t_{m-1}}{k_m}\|[\Delta v_k]_{m-1}\|^2 \le C\|\Delta v_0\|^2 \tag{4.22}$$

and

$$T\|\nabla\Delta y_{k,1}\|^2 + \|\Delta y_{k,1}\|^2 + \|\nabla\Delta y_k\|_I^2 + \sum_{m=1}^{M-1} \frac{\tau_{k,m+1}}{k_m}\|[\Delta y_k]_m\|^2 \le C\|\Delta y_T\|^2, \tag{4.23}$$

with $\tau_{k,m} = \tau_k\big|_{I_m} := T - t_{m-1}$.

Proof. The proof of the assertion for v_k is based on (4.15).

(i) Integrating by parts in (4.15) and choosing $\varphi = \Delta^2 v_k$ gives us

$$\|\nabla\Delta v_k\|_{I_m}^2 + ([\Delta v_k]_{m-1}, \Delta v_{k,m}) = 0,$$

and further, applying (4.16),

$$\|\Delta v_{k,m}\|^2 + 2\|\nabla\Delta v_k\|_{I_m}^2 \le \|\Delta v_{k,m-1}\|^2.$$

By summing up for $m = 1, 2, \ldots, M$, we obtain

$$\|\Delta v_{k,M}\|^2 + 2\|\nabla\Delta v_k\|_I^2 \le \|\Delta v_0\|^2. \tag{4.24}$$

(ii) Integrating by parts in (4.15) and choosing $\varphi\big|_{I_m} = \frac{t_{m-1}}{k_m}[\Delta^2 v_k]_{m-1}$ gives us

$$\frac{t_{m-1}}{k_m}(\nabla\Delta v_k, [\nabla\Delta v_k]_{m-1})_{I_m} + \frac{t_{m-1}}{k_m}\|[\Delta v_k]_{m-1}\|^2 = 0$$

and, consequently,

$$t_{m-1}(\nabla\Delta v_{k,m}, [\nabla\Delta v_k]_{m-1}) + \frac{t_{m-1}}{k_m}\|[\Delta v_k]_{m-1}\|^2 = 0.$$

Then, the algebraic identity (4.16) implies

$$t_m\|\nabla\Delta v_{k,m}\|^2 + 2\frac{t_{m-1}}{k_m}\|[\Delta v_k]_{m-1}\|^2 \le t_{m-1}\|\nabla\Delta v_{k,m-1}\|^2 + k_m\|\nabla\Delta v_{k,m}\|^2.$$

By adding these inequalities for $m = 2, 3, \ldots, M$ and using $t_1 = k_1$, we arrive at

$$\begin{aligned} T\|\nabla\Delta v_{k,M}\|^2 &+ 2\sum_{m=2}^{M} \frac{t_{m-1}}{k_m}\|[\Delta v_k]_{m-1}\|^2 \\ &\le k_1\|\nabla\Delta v_{k,1}\|^2 + \sum_{m=2}^{M}\|\nabla\Delta v_k\|_{I_m}^2 = \|\nabla\Delta v_k\|_I^2. \end{aligned} \tag{4.25}$$

Finally, the estimates (4.24) and (4.25) imply the assertion.

The assertion for y_k follows by repeating steps (i) and (ii) for (4.21) employing identity (4.2). ∎

5. Analysis of the discretization error for the state equation. The aim of this section is to prove a priori error estimates for the (uncontrolled) state equation (2.1) in the norm of $L^\infty(I, L^2(\Omega))$. These estimates form the basis of the error analysis for the whole optimization problem (2.3), which will be developed in section 6. In contrast to the $L^\infty(I, L^2(\Omega))$ estimates available in the literature (cf. [10, 11]) the estimates we derive here require only that the right-hand side f be in $L^\infty(I, L^2(\Omega))$. Later this requirement carries over to the boundedness of the control q in $L^\infty(I, L^2(\Omega))$, which is fulfilled due to the prescribed control constraints.

The parabolic duality technique for proving such "rough forcing" error estimates has been developed in [18] for the spatial semidiscretization by finite element methods in combination with time discretization by the backward Euler scheme. Similar arguments have been used in [26] and [31]. In the following analysis spatial and temporal semidiscretization appears in reversed order compared to [18], and the dG(0) method is considered in time. Further, the logarithmic factor occurring in the estimates depends only on the time step k and not on the spatial discretization parameter h as in [18].

Let $u \in X$ be the solution of the state equation (2.1) for $q = 0$, let $u_k \in X_k^r$ be the solution of the corresponding semidiscretized equation (3.2), and let $u_{kh} \in X_{k,h}^{r,s}$ be the solution of the fully discretized state equation (3.11). In order to separate the influences of the space and time discretizations, we split the total discretization error $e := u - u_{kh}$ into its temporal part $e_k := u - u_k$ and its spatial part $e_h := u_k - u_{kh}$. The temporal discretization error will be estimated in the following subsection; the spatial discretization error is treated in subsection 5.2.

5.1. Analysis of the temporal discretization error. In this subsection, we will prove an error estimate for the temporal discretization error e_k. For this, we need, in addition to the solution $y \in X$ of (4.5), the solution $y^k \in X$ of the auxiliary equation

$$\begin{aligned} -(\varphi, \partial_t y^k)_{I^*} + (\nabla\varphi, \nabla y^k)_{I^*} &= 0 \qquad \forall \varphi \in X, \\ y^k(t^*) &= y_T, \end{aligned} \tag{5.1}$$

where $I^* = (0, t^*)$ with some $t^* \in I_M = (T - k_M, T]$. Please note that $y^k \in X$ is the continuous solution of (5.1) and the superscript k symbolizes the dependence on k through the choice of t^*.

The following lemma provides an estimate for the error between y and y^k.

LEMMA 5.1. *For the solutions $y \in X$ of (4.5) and $y^k \in X$ of (5.1) there holds*

$$\|y - y^k\|_{L^1(I^*, L^2(\Omega))} + \|(y - y^k)(0)\|_{H^{-2}(\Omega)} \le Ck\Big(\ln \frac{T}{k}\Big)^{\frac{1}{2}} \|y_T\|. \tag{5.2}$$

Proof. Using the notation $\xi := y^k - y$, we have to estimate the two quantities $\|\xi\|_{L^1(I^*, L^2(\Omega))}$ and $\|\xi(0)\|_{H^{-2}(\Omega)}$. Since y satisfies $y(T) = y_T$ and

$$-(\varphi, \partial_t y)_{I^*} + (\nabla\varphi, \nabla y)_{I^*} = 0 \quad \forall \varphi \in X,$$

the difference ξ solves

$$-(\varphi, \partial_t \xi)_{I^*} + (\nabla\varphi, \nabla\xi)_{I^*} + (\varphi(t^*), \xi(t^*)) = (\varphi(t^*), y(T) - y(t^*)) \quad \forall \varphi \in X. \tag{5.3}$$

(i) Integrating by parts in (5.3) and choosing $\varphi = \Delta^{-2}\xi$, we obtain

$$-(\Delta^{-2}\xi, \partial_t\xi)_{I^*} - (\Delta^{-1}\xi, \xi)_{I^*} + (\Delta^{-2}\xi(t^*), \xi(t^*)) = (\Delta^{-2}\xi(t^*), y(T) - y(t^*)).$$

This implies

$$\|\Delta^{-1}\xi(t^*)\|^2 + \|\Delta^{-1}\xi(0)\|^2 + 2\|\nabla\Delta^{-1}\xi\|_{I^*}^2 \le \|\Delta^{-1}\xi(t^*)\|^2 + \|\Delta^{-1}(y(T) - y(t^*))\|^2$$

and, consequently,

$$\|\Delta^{-1}\xi(0)\|^2 + 2\|\nabla\Delta^{-1}\xi\|_{I^*}^2 \le \|\Delta^{-1}(y(T) - y(t^*))\|^2.$$

By virtue of $\partial_t y = -\Delta y$ and the stability estimate from Theorem 4.3, the right-hand side can be estimated as

$$\begin{aligned}\|\Delta^{-1}(y(T) - y(t^*))\|^2 &= \int_\Omega \left(\int_{t^*}^T \Delta^{-1}\partial_t y(t)\,dt \right)^2 dx \\ &\le k_M \int_{t^*}^T \|\Delta^{-1}\partial_t y(t)\|^2\,dt \\ &= k_M \int_{t^*}^T \|y(t)\|^2\,dt \le Ck^2\|y_T\|^2.\end{aligned}$$

By the definition of the norms of H^{-1} and H^{-2}, this leads us to

$$\|\xi(0)\|_{H^{-2}(\Omega)}^2 + \|\xi\|_{L^2(I^*,H^{-1}(\Omega))}^2 \le Ck^2\|y_T\|^2. \tag{5.4}$$

(ii) Integrating by parts in (5.3) and choosing $\varphi = -\tau\Delta^{-1}\xi$ with

$$\tau(t) := \max\{t^* - t, k\}$$

implies

$$(\tau\Delta^{-1}\xi, \partial_t\xi)_{I^*} + (\tau\xi, \xi)_{I^*} - k(\Delta^{-1}\xi(t^*), \xi(t^*)) = -k(\Delta^{-1}\xi(t^*), y(T) - y(t^*)).$$

By the relation

$$(\tau\Delta^{-1}\xi, \partial_t\xi)_{I^*} = \frac{1}{2}\int_{I^*} \frac{d}{dt}\Big(\tau(\Delta^{-1}\xi(t), \xi(t))\Big)\,dt - \frac{1}{2}\int_{I^*} \tau'(\Delta^{-1}\xi(t), \xi(t))\,dt$$

and observing $-\tau' \le 1$, we conclude that

$$\begin{aligned}&k\|\nabla\Delta^{-1}\xi(t^*)\|^2 + t^*\|\nabla\Delta^{-1}\xi(0)\|^2 + 2\|\sqrt{\tau}\xi\|_{I^*}^2 \\ &\qquad \le \|\nabla\Delta^{-1}\xi\|_{I^*}^2 + k\|\nabla\Delta^{-1}\xi(t^*)\|^2 + k\|\nabla\Delta^{-1}(y(T) - y(t^*))\|^2,\end{aligned}$$

and, consequently,

$$2\|\sqrt{\tau}\xi\|_{I^*}^2 + t^*\|\nabla\Delta^{-1}\xi(0)\|^2 \le \|\nabla\Delta^{-1}\xi\|_{I^*}^2 + k\|\nabla\Delta^{-1}(y(T) - y(t^*))\|^2.$$

For estimating the second term on the right-hand side, we use $\partial_t y = -\Delta y$ to obtain

$$\begin{aligned}\|\nabla\Delta^{-1}(y(T) - y(t^*))\|^2 &= \int_\Omega \left(\int_{t^*}^T \nabla\Delta^{-1}\partial_t y(t)\,dt \right)^2 dx \\ &\le k_M \int_{t^*}^T \|\nabla\Delta^{-1}\partial_t y(t)\|^2\,dt \\ &= k_M \int_{t^*}^T \|\nabla y(t)\|^2\,dt \le Ck\|y_T\|^2.\end{aligned}$$

From this, using (5.4), we obtain by the definition of the H^{-1} norm that

$$\|\sqrt{\tau}\xi\|_{I^*}^2 \le Ck^2\|y_T\|^2.$$

Then, the estimate of $\|\xi\|_{L^1(I^*,L^2(\Omega))}$ follows from

$$\|\xi\|^2_{L^1(I^*,L^2(\Omega))} \le \|\sqrt{\tau^{-1}}\|^2_{I^*}\|\sqrt{\tau}\xi\|^2_{I^*} \le \int_I \tau(t)^{-1}\,dt\,\|\sqrt{\tau}\xi\|^2_{I^*}$$
$$\le Ck^2\Big(\ln\frac{T}{k}+1\Big)\|y_T\|^2 \le Ck^2\ln\frac{T}{k}\|y_T\|^2,$$

where in the last inequality the assumption $k \le \frac{1}{4}T$ is used. ∎

Next, we provide an estimate for the error between y and its semidiscrete analogue y_k.

LEMMA 5.2. *For the solutions $y \in X$ of (4.5) and $y_k \in X_k^0$ of (4.7) there holds*

$$\|y-y_k\|_{L^1(I,L^2(\Omega))} + \|y(0)-y_{k,1}\|_{H^{-2}(\Omega)} \le Ck\Big(\ln\frac{T}{k}\Big)^{\frac{1}{2}}\|y_T\|. \tag{5.5}$$

Proof. We define a semidiscrete projection $\pi_k^*\colon C(\bar I\setminus I_M,V)\to X_k^0$, for $m = 1,2,\dots,M$, by

$$\pi_k^* y\big|_{I_m} = y(t_{m-1}). \tag{5.6}$$

By inserting $\pi_k^* y$, we obtain, due to the definition of π_k^*, that

$$\|y-y_k\|_{L^1(I,L^2(\Omega))} + \|y(0)-y_{k,1}\|_{H^{-2}(\Omega)}$$
$$\le \|y-\pi_k^*y\|_{L^1(I,L^2(\Omega))} + \|\pi_k^*y-y_k\|_{L^1(I,L^2(\Omega))} + \|\pi_k^*y(0)-y_{k,1}\|_{H^{-2}(\Omega)}.$$

For the first term, we have

$$\|y-\pi_k^*y\|_{L^1(I,L^2(\Omega))} = \int_{I\setminus I_M}\|y(t)-\pi_k^*y(t)\|\,dt + \int_{I_M}\|y(t)-\pi_k^*y(t)\|\,dt$$
$$\le Ck\left\{\int_{I\setminus I_M}\|\partial_t y(t)\|\,dt + \max_{t\in\bar I}\|y(t)\|\right\}.$$

Then, Theorem 4.4 and the a priori estimate from Theorem 4.3 imply

$$\|y-\pi_k^*y\|_{L^1(I,L^2(\Omega))} \le Ck\Big(\ln\frac{T}{k}\Big)^{\frac{1}{2}}\|y_T\|.$$

Using the notation $\xi_k := \pi_k^*y - y_k$, we have to estimate the two quantities $\|\xi_k\|_{L^1(I,L^2(\Omega))}$ and $\|\xi_{k,1}\|_{H^{-2}(\Omega)}$. Employing Galerkin orthogonality and the definition of π_k^*, for $\varphi \in X_k^0 \cap L^2(I,H^2(\Omega))$, we have

$$B(\varphi,\xi_k) = -B(\varphi, y-\pi_k^*y) = (\Delta\varphi, y-\pi_k^*y)_I$$
$$= \sum_{m=1}^M\left\{\int_{I_m}(\Delta\varphi_m, y(t))\,dt - k(\Delta\varphi_m, y(t_{m-1}))\right\}$$
$$= \sum_{m=1}^M\int_{I_m}(t_m-t)(\Delta\varphi_m,\partial_t y(t))\,dt.$$

By means of (3.10) and the definition $\xi_{k,M+1} := 0$, this equality can be rewritten as the following system of equations for $m = 1,2,\dots,M$:

$$(\nabla\varphi,\nabla\xi_k)_{I_m} - (\varphi_m,[\xi_k]_m) = \int_{I_m}(t_m-t)(\Delta\varphi,\partial_t y(t))\,dt \quad \forall\varphi\in\mathcal{P}_0(I_m,H^2(\Omega)\cap V). \tag{5.7}$$

(i) Setting $\varphi = \Delta^{-2}\xi_k$ in (5.7), after integration by parts and observing $\partial_t y = -\Delta y$, we obtain

$$-(\Delta^{-1}\xi_k, \xi_k)_{I_m} - (\Delta^{-1}\xi_{k,m}, [\Delta^{-1}\xi_k]_m) = -\int_{I_m} (t_m - t)(\xi_k, y(t))\,dt.$$

Estimating the right-hand side by

$$-\int_{I_m} (t_m - t)(\xi_k, y(t))\,dt \le \frac{1}{2}\|\nabla\Delta^{-1}\xi_k\|_{I_m}^2 + \frac{1}{2}\int_{I_m} (t_m - t)^2\|\nabla y(t)\|^2\,dt$$

and applying the identity (4.2) to the left-hand side leads us to

$$\|\Delta^{-1}\xi_{k,m}\|^2 + \|\nabla\Delta^{-1}\xi_k\|_{I_m}^2 \le \|\Delta^{-1}\xi_{k,m+1}\|^2 + k_m^2\|\nabla y\|_{I_m}^2.$$

Summing this for $m = 1, 2, \ldots, M$ yields

$$\|\Delta^{-1}\xi_{k,1}\|^2 + \|\nabla\Delta^{-1}\xi_k\|_I^2 \le k^2\|\nabla y\|_I^2.$$

Consequently, by the a priori estimate from Theorem 4.3 and the definition of the norms of $H^{-1}(\Omega)$ and $H^{-2}(\Omega)$, we get

$$\|\xi_{k,1}\|_{H^{-2}(\Omega)}^2 + \|\xi_k\|_{L^2(I,H^{-1}(\Omega))}^2 \le Ck^2\|y_T\|^2. \tag{5.8}$$

(ii) Setting $\varphi\big|_{I_m} = -\tau_{k,m}\Delta^{-1}\xi_k$ in (5.7), after integration by parts, we obtain

$$\tau_{k,m}\|\xi_k\|_{I_m}^2 - \tau_{k,m}(\nabla\Delta^{-1}\xi_{k,m}, [\nabla\Delta^{-1}\xi_k]_m) = \tau_{k,m}\int_{I_m} (t_m - t)(\xi_k, \partial_t y(t))\,dt.$$

Estimating the right-hand side by

$$\tau_{k,m}\int_{I_m} (t_m - t)(\xi_k, \partial_t y(t))\,dt \le \frac{\tau_{k,m}}{2}\|\xi_k\|_{I_m}^2 + \frac{\tau_{k,m}}{2}\int_{I_m} (t_m - t)^2\|\partial_t y\|^2\,dt$$

and using the identities (4.2) and $\tau_{k,m} = \tau_{k,m+1} + k_m$ leads us to

$$\begin{aligned}\tau_{k,m}\|\xi_k\|_{I_m}^2 + \tau_{k,m}\|\nabla\Delta^{-1}\xi_{k,m}\|^2 &\le \tau_{k,m+1}\|\nabla\Delta^{-1}\xi_{k,m+1}\|^2 \\ &\quad + k_m\|\nabla\Delta^{-1}\xi_{k,m+1}\|^2 + \tau_{k,m}\int_{I_m} (t_m - t)^2\|\partial_t y\|^2\,dt.\end{aligned}$$

Observing $k_m \le \kappa k_{m+1}$, summing these equations for $m = 1, 2, \ldots, M$, we obtain

$$\begin{aligned}T\|\nabla\Delta^{-1}\xi_{k,1}\|^2 + \sum_{m=1}^{M}\tau_{k,m}\|\xi_k\|_{I_m}^2 \\ \le \kappa\|\nabla\Delta^{-1}\xi_k\|_I^2 + \sum_{m=1}^{M}\tau_{k,m}\int_{I_m} (t_m - t)^2\|\partial_t y\|^2\,dt.\end{aligned} \tag{5.9}$$

Since for $t \in I_m$, with $m \le M - 1$, we have

$$\tau_{k,m} \le T - t_m + \kappa k_{m+1} \le (1 + \kappa)(T - t_m) \le (1 + \kappa)(T - t)$$

and for $m = M$, we have $\tau_{k,M} = k_M$, the second term on the right-hand side of (5.9) can be estimated as follows:

$$\begin{aligned}\sum_{m=1}^{M} \tau_{k,m} \int_{I_m} (t_m - t)^2 \|\partial_t y\|^2 \, dt \\ &\le \sum_{m=1}^{M-1} k_m^2 \int_{I_m} \tau_{k,m} \|\partial_t y\|^2 \, dt + k_M^2 \int_{I_M} (T-t) \|\partial_t y\|^2 \, dt \\ &\le (1+\kappa) k^2 \int_I (T-t) \|\partial_t y\|^2 \, dt.\end{aligned}$$

Using this, (5.8), and Theorem 4.4, we conclude from (5.9) that

$$\sum_{m=1}^{M} \tau_{k,m} \|\xi_k\|_{I_m}^2 \le C k^2 \|y_T\|^2.$$

Then, the desired estimate for $\|\xi_k\|_{L^1(I,L^2(\Omega))}$ follows from

$$\begin{aligned}\|\xi_k\|_{L^1(I,L^2(\Omega))}^2 &\le \sum_{m=1}^{M} k_m \tau_{k,m}^{-1} \sum_{m=1}^{M} \tau_{k,m} \|\xi_k\|_{I_m}^2 \\ &\le C k^2 \Big(\ln \frac{T}{k} + 1 \Big) \|y_T\|^2 \le C k^2 \ln \frac{T}{k} \|y_T\|^2,\end{aligned}$$

where in the last inequality the assumption $k \le \frac{1}{4} T$ is used. ∎

After these preparations, we can prove the following two theorems leading to the main result of this subsection. We begin with an estimate for the interpolation error $u(\cdot) - u(t_m)$.

THEOREM 5.3. *On each time interval I_m with $m = 1, 2, \ldots, M$, for the solution $u \in X$ of* (2.1)*, there holds*

$$\|u(\cdot) - u(t_m)\|_{L^\infty(I_m, L^2(\Omega))} \le C k \Big(\ln \frac{T}{k} \Big)^{\frac{1}{2}} \big\{ \|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \big\}. \qquad (5.10)$$

Proof. For simplicity, we consider only the last time interval I_M and a fixed time point $t^* \in I_M$. Let y and y^k be the solutions of (4.5) and (5.1) with $y_T = u(t^*) - u(T)$. Using (4.5), (5.1), and (2.1), we obtain by integration by parts in time and the condition $u(0) = u_0$ that

$$\begin{aligned}(u(T), u(t^*) - u(T)) &= (f, y)_I + (u_0, y(0)), \\ (u(t^*), u(t^*) - u(T)) &= (f, y^k)_{I^*} + (u_0, y^k(0)).\end{aligned}$$

This implies the relation

$$\begin{aligned}\|u(t^*) - u(T)\|^2 &= (f, y^k)_{I^*} - (f, y)_I + (u_0, (y^k - y)(0)) \\ &= (f, y^k - y)_{I^*} - \int_{t^*}^{T} (f(t), y(t)) \, dt + (u_0, (y^k - y)(0)) \\ &\le \big\{ \|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \big\} \\ &\quad \times \big\{ \|y^k - y\|_{L^1(I^*, L^2(\Omega))} + k \|y\|_{L^\infty(I,L^2(\Omega))} + \|(y^k - y)(0)\|_{H^{-2}(\Omega)} \big\}.\end{aligned}$$

By the a priori estimate from Theorem 4.3, we directly obtain

$$\|y\|_{L^\infty(I,L^2(\Omega))} \le C\|u(t^*) - u(T)\|,$$

and the assertion of Lemma 5.1 completes the proof. ∎

Furthermore, we estimate the error $u(t_m) - u_k(t_m)$.

THEOREM 5.4. *For the solution $u \in X$ of (2.1) and the dG(0) semidiscretized solution $u_k \in X_k^0$ of (3.2), there holds on each time interval I_m with $m = 1, 2, \dots, M$ the error estimate*

$$\|u(t_m) - u_{k,m}\| \le Ck\Big(\ln \frac{T}{k}\Big)^{\frac{1}{2}} \big\{\|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\big\}. \tag{5.11}$$

Proof. For simplicity, we consider only the last time point $t_M = T$. The proof employs a duality argument. Let $y \in X$ and $y_k \in X_k^0$ be the solutions of (4.5) and (4.7) with $y_T = e_{k,M} = u(t_M) - u_{k,M}$. By Galerkin orthogonality, we have

$$\begin{aligned}\|e_{k,M}\|^2 &= B(e_k, y) = B(e_k, y - y_k) = B(u, y - y_k)\\ &= (f, y - y_k)_I + (u_0, y(0) - y_{k,1})\\ &\le \big\{\|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\big\}\big\{\|y - y_k\|_{L^1(I,L^2(\Omega))} + \|y(0) - y_{k,1}\|_{H^{-2}(\Omega)}\big\}.\end{aligned}$$

Then, the assertion of Lemma 5.2 completes the proof. ∎

Based on the previous theorems, we can now state the main result of this subsection.

COROLLARY 5.5. *For the error $e_k := u - u_k$ between the solution $u \in X$ of (2.1) and the dG(0) semidiscretized solution $u_k \in X_k^0$ of (3.2), there holds the estimate*

$$\|e_k\|_{L^\infty(I,L^2(\Omega))} \le Ck\Big(\ln \frac{T}{k}\Big)^{\frac{1}{2}} \big\{\|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\big\}. \tag{5.12}$$

Proof. We decompose the error on I_m for $m = 1, 2, \dots, M$ as follows:

$$\begin{aligned}\|e_k\|_{L^\infty(I_m,L^2(\Omega))} &\le \|u(\cdot) - u(t_m)\|_{L^\infty(I_m,L^2(\Omega))} + \|u(t_m) - u_k(\cdot)\|_{L^\infty(I_m,L^2(\Omega))}\\ &= \|u(\cdot) - u(t_m)\|_{L^\infty(I_m,L^2(\Omega))} + \|u(t_m) - u_{k,m}\|.\end{aligned}$$

Then, the assertion of the corollary follows from Theorems 5.3 and 5.4. ∎

5.2. Analysis of the spatial discretization error. In this subsection, we analyze the spatial discretization error e_h. However, to derive the main result, we need to prove a sequence of auxiliary lemmas.

LEMMA 5.6. *For the solutions $v_k \in X_k^0$ of (4.6), $v_{kh} \in X_{k,h}^{0,1}$ of (4.8), $y_k \in X_k^0$ of (4.7), and $y_{kh} \in X_{k,h}^{0,1}$ of (4.9), the following hold:*

(a) *If $v_0,\ y_T \in H$,*

$$\|v_k - v_{kh}\|_I \le Ch\|v_0\|, \tag{5.13}$$

$$\|y_k - y_{kh}\|_I \le Ch\|y_T\|. \tag{5.14}$$

(b) *If $v_0,\ y_T \in V \cap H^2(\Omega)$,*

$$\|v_k - v_{kh}\|_I \le C\sqrt{T}h^2\|\Delta v_0\|, \tag{5.15}$$

$$\|y_k - y_{kh}\|_I \le C\sqrt{T}h^2\|\Delta y_T\|. \tag{5.16}$$

Proof. We prove only the assertions for $v_k - v_{kh}$. The assertions for $y_k - y_{kh}$ can be obtained by similar arguments. We use the splitting

$$\eta_h := v_k - v_{kh} = v_k - P_h v_k + P_h \eta_h,$$

where $P_h \colon V \to V_h$ denotes the L^2 projection in space. The application of P_h to time-dependent arguments has to be understood pointwise in time. By Galerkin orthogonality, it holds that

$$B(P_h\eta_h, \varphi) = B(P_h v_k - v_k, \varphi) \quad \forall \varphi \in X^{0,1}_{k,h},$$

which can be rewritten, by means of the definitions $P_h\eta_{h,0} := 0$ and $P_h v_{k,0} - v_{k,0} := 0$, as the following system of equations for $m = 1, 2, \ldots, M$:

$$\begin{aligned}(5.17)\quad &(\nabla P_h\eta_h, \nabla\varphi)_{I_m} + ([P_h\eta_h]_{m-1}, \varphi_m)\\ &\quad = (\nabla(P_h v_k - v_k), \nabla\varphi)_{I_m} + ([P_h v_k - v_k]_{m-1}, \varphi_m) \quad \forall \varphi \in \mathcal{P}_0(I_m, V_h).\end{aligned}$$

For $u \in V$ and $u_h \in V_h$, we define the Ritz projection $R_h \colon V \to V_h$ and the discrete Laplacian $\Delta_h \colon V_h \to V_h$ by the relations

$$(\nabla R_h u, \nabla\varphi) = (\nabla u, \nabla\varphi) \quad \forall \varphi \in V_h \quad \text{and} \quad (-\Delta_h u_h, \varphi) = (\nabla u_h, \nabla\varphi) \quad \forall \varphi \in V_h.$$

As for P_h, the application of Δ_h and R_h to time-dependent arguments has to be understood pointwise in time. Taking $\varphi = -\Delta_h^{-1} P_h\eta_h$ in (5.17) and observing the definitions of the projectors P_h and R_h, we conclude that

$$\begin{aligned}\|P_h\eta_h\|^2_{I_m} - ([P_h\eta_h]_{m-1}, \Delta_h^{-1} P_h\eta_{h,m}) &= - (\nabla(P_h v_k - v_k), \nabla\Delta_h^{-1} P_h\eta_h)_{I_m}\\ &= - (\nabla(P_h v_k - R_h v_k), \nabla\Delta_h^{-1} P_h\eta_h)_{I_m}\\ &\quad - (\nabla(R_h v_k - v_k), \nabla\Delta_h^{-1} P_h\eta_h)_{I_m}\\ &= (P_h v_k - R_h v_k, P_h\eta_h)_{I_m}.\end{aligned}$$

By the definition of Δ_h^{-1} and R_h, this implies

$$\|P_h\eta_h\|^2_{I_m} + ([\nabla\Delta_h^{-1} P_h\eta_h]_{m-1}, \nabla\Delta_h^{-1} P_h\eta_{h,m}) = (P_h v_k - R_h v_k, P_h\eta_h)_{I_m}.$$

We remark that the trick of comparing v_{kh} to $P_h v_k$ and $R_h v_k$ rather than directly to v_k is crucial and has been introduced into the error analysis of parabolic problems in [33]. Then, the algebraic identity (4.16) and Young's inequality lead us to

$$\|P_h\eta_h\|^2_{I_m} + \|\nabla\Delta_h^{-1} P_h\eta_{h,m}\|^2 \le \|\nabla\Delta_h^{-1} P_h\eta_{h,m-1}\| + \|P_h v_k - R_h v_k\|^2_{I_m}.$$

By adding these identities for $m = 1, 2, \ldots, M$, we arrive at

$$\|P_h\eta_h\|^2_I \le \|P_h v_k - R_h v_k\|^2_I,$$

and observing $\eta_h = v_k - P_h v_k + P_h\eta_h$ gives us

$$\|\eta_h\|^2_I \le \|v_k - P_h v_k\|^2_I + \|P_h v_k - R_h v_k\|^2_I.$$

(a) For $v_0 \in H$, we have

$$\|\eta_h\|^2_I \le Ch^2 \|\nabla v_k\|^2_I$$

and Theorem 4.5 implies the asserted estimate.

(b) For $v_0 \in V \cap H^2(\Omega)$, we have

$$\|\eta_h\|_I^2 \le Ch^4 \|\Delta v_k\|_I^2 \le CTh^4 \max_{m=1,2,\dots,M} \|\Delta v_{k,m}\|^2,$$

and Theorem 4.6 implies the asserted estimate.

The proof is completed. □

LEMMA 5.7. *For the solutions $v_k \in X_k^0$ of* (4.6), $v_{kh} \in X_{k,h}^{0,1}$ *of* (4.8), $y_k \in X_k^0$ *of* (4.7), *and* $y_{kh} \in X_{k,h}^{0,1}$ *of* (4.9), *the following hold:*

(a) *If* $v_0,\ y_T \in H$,

$$\sqrt{T}\|v_{k,M} - v_{kh,M}\| \le Ch\|v_0\|, \tag{5.18}$$

$$\sqrt{T}\|y_{k,1} - y_{kh,1}\| \le Ch\|y_T\|. \tag{5.19}$$

(b) *If* $v_0,\ y_T \in V \cap H^2(\Omega)$,

$$\|v_{k,M} - v_{kh,M}\| \le Ch^2\|\Delta v_0\|, \tag{5.20}$$

$$\|y_{k,1} - y_{kh,1}\| \le Ch^2\|\Delta y_T\|. \tag{5.21}$$

Proof. We prove only the assertion for $v_k - v_{kh}$. The assertion for $y_k - y_{kh}$ can be proved similarly. We use the splitting

$$\eta_h \coloneqq v_k - v_{kh} = v_k - R_h v_k + R_h \eta_h,$$

where R_h denotes the Ritz projection onto V_h. By Galerkin orthogonality, there holds

$$B(R_h\eta_h, \varphi) = B(R_h v_k - v_k, \varphi) \quad \forall \varphi \in X_{k,h}^{0,1}.$$

By means of the definitions $R_h\eta_{h,0} \coloneqq 0$ and $R_h v_{k,0} - v_{k,0} \coloneqq 0$, this can be rewritten as the following system of equations for $m = 1, 2, \dots, M$:

$$(\nabla R_h\eta_h, \nabla\varphi)_{I_m} + ([R_h\eta_h]_{m-1}, \varphi_m) = (\nabla(R_h v_k - v_k), \nabla\varphi)_{I_m} + ([R_h v_k - v_k]_{m-1}, \varphi_m)$$

for all $\varphi \in \mathcal{P}_0(I_m, V_h)$. Taking $\varphi\big|_{I_m} = t_{m-1}R_h\eta_h$ and using the definition of the projector R_h yields

$$t_{m-1}\|\nabla R_h\eta_h\|_{I_m}^2 + t_{m-1}([R_h\eta_h]_{m-1}, R_h\eta_{h,m}) = t_{m-1}([R_h v_k - v_k]_{m-1}, R_h\eta_{h,m}).$$

The identity (4.16) implies

$$\begin{aligned} t_m\|R_h\eta_{h,m}\|^2 + 2t_{m-1}\|\nabla R_h\eta_h\|_{I_m}^2 &\le t_{m-1}\|R_h\eta_{h,m-1}\|^2 \\ &\quad + k_m\|R_h\eta_{h,m}\|^2 + \frac{t_{m-1}^2}{k_m}\|[R_h v_k - v_k]_{m-1}\|^2 + k_m\|R_h\eta_{h,m}\|^2. \end{aligned}$$

Summing this for $m = 2, 3, \dots, M$ and using $t_1 = k_1$, we obtain

$$T\|R_h\eta_{h,M}\|^2 \le k_1\|R_h\eta_{h,1}\|^2 + \sum_{m=2}^{M} \frac{t_{m-1}^2}{k_m}\|[R_h v_k - v_k]_{m-1}\|^2 + 2\sum_{m=2}^{M} k_m\|R_h\eta_{h,m}\|^2.$$

Consequently,

$$T\|R_h\eta_{h,M}\|^2 \le \sum_{m=2}^{M} \frac{t_{m-1}^2}{k_m}\|[R_h v_k - v_k]_{m-1}\|^2 + 2\|R_h\eta_h\|_I^2.$$

Since $\eta_h = v_k - R_h v_k + R_h \eta_h$, we conclude that

$$T\|\eta_{h,M}\|^2 \le 2T\|R_h v_{k,M} - v_{k,M}\|^2 + 2\sum_{m=2}^{M} \frac{t_{m-1}^2}{k_m}\|[R_h v_k - v_k]_{m-1}\|^2 + 4\|R_h v_k - v_k\|_I^2 + 4\|\eta_h\|_I^2.$$

(a) For $v_0 \in H$, by using the approximation properties of R_h we obtain that

$$T\|\eta_{h,M}\|^2 \le Ch^2\Big\{T\|\nabla v_{k,M}\|^2 + \|\nabla v_k\|_I^2 + \sum_{m=2}^{M} \frac{t_{m-1}^2}{k_m}\|[\nabla v_k]_{m-1}\|^2\Big\} + C\|\eta_h\|_I^2.$$

Then, Theorem 4.5 and Lemma 5.6(a) imply the asserted estimate

(b) For $v_0 \in V \cap H^2(\Omega)$, by using the error estimates for R_h we obtain that

$$T\|\eta_{h,M}\|^2 \le Ch^4 T\left\{\max_{m=1,2,\dots,M}\|\Delta v_{k,m}\|^2 + \sum_{m=2}^{M} \frac{t_{m-1}}{k_m}\|[\Delta v_k]_{m-1}\|^2\right\} + C\|\eta_h\|_I^2.$$

Then, Theorem 4.6 and Lemma 5.6(b) imply the asserted estimate.

The proof is complete. □

LEMMA 5.8. *For the solutions $v_k \in X_k^0$ of (4.6), $v_{kh} \in X_{k,h}^{0,1}$ of (4.8), $y_k \in X_k^0$ of (4.7), and $y_{kh} \in X_{k,h}^{0,1}$ of (4.9), there holds that*

$$\|v_{k,M} - v_{kh,M}\|_{H^{-2}(\Omega)} \le Ch^2\|v_0\|, \tag{5.22}$$

$$\|y_{k,1} - y_{kh,1}\|_{H^{-2}(\Omega)} \le Ch^2\|y_T\|. \tag{5.23}$$

Proof. We prove only the assertion for $v_k - v_{kh}$. The assertion for $y_k - y_{kh}$ can be proved similarly. For the proof, we employ another duality argument. For $\eta_h := v_k - v_{kh}$, we recall that

$$\|\eta_{h,M}\|_{H^{-2}(\Omega)} = \sup_{\psi \in H^2(\Omega)\cap V} \frac{(\eta_{h,M}, \psi)}{\|\Delta\psi\|}.$$

For any fixed $\psi \in H^2(\Omega) \cap V$, we consider the solutions $y_k \in X_k^0$ of (4.7) and $y_{kh} \in X_{k,h}^{0,1}$ of (4.9) with $y_T = \psi$. Using (4.7) with $\varphi = v_k$, (4.6) with $\varphi = y_k$, (4.9) with $\varphi = v_{kh}$, and (4.8) with $\varphi = y_{kh}$, we obtain

$$(v_{k,M}, \psi) = B(v_k, y_k) = (v_0, y_{k,1}) \quad \text{and} \quad (v_{kh,M}, \psi) = B(v_{kh}, y_{kh}) = (v_0, y_{kh,1}).$$

Consequently,

$$(\eta_{h,M}, \psi) = (v_0, y_{k,1} - y_{kh,1}) \le \|v_0\|\|y_{k,1} - y_{kh,1}\|.$$

Then, Lemma 5.7(b) implies

$$(\eta_{h,M}, \psi) \le Ch^2\|v_0\|\|\Delta\psi\|,$$

which yields the asserted estimate. □

LEMMA 5.9. *For the solutions $y_k \in X_k^0$ of (4.7) and $y_{kh} \in X_{k,h}^{0,1}$ of (4.9), there holds*

$$T\|y_{k,1} - y_{kh,1}\| \le Ch^2\|y_T\|. \tag{5.24}$$

Proof. The proof employs a "bootstrap" argument based on the suboptimal error estimate of Lemma 5.7. We use the solutions $v_k \in X_k^0$ of (4.6) and $v_{kh} \in X_{k,h}^{0,1}$ of (4.8) with $v_0 = y_{k,1} - y_{kh,1}$. Considering the equations (4.6) with $\varphi = y_k$, (4.7) with $\varphi = v_k$, (4.8) with $\varphi = y_{kh}$, and (4.9) with $\varphi = v_{kh}$ on $\{0\} \cup I_1 \cup I_2 \cup \cdots \cup I_{\widetilde{m}}$ with some $\widetilde{m} \le M$ and $\varphi = 0$ on the remaining subintervals yields

$$(y_{k,1} - y_{kh,1}, y_{k,1}) = B(v_k, y_k) = (v_{k,\widetilde{m}}, y_{k,\widetilde{m}+1})$$

and

$$(y_{k,1} - y_{kh,1}, y_{kh,1}) = B(v_{kh}, y_{kh}) = (v_{kh,\widetilde{m}}, y_{kh,\widetilde{m}+1}).$$

Hence, with $\xi_h := y_k - y_{kh}$ and $\eta_h := v_k - v_{kh}$, we obtain

$$\begin{aligned} \|\xi_{h,1}\|^2 &= (\eta_{h,\widetilde{m}}, y_{k,\widetilde{m}+1}) - (\eta_{h,\widetilde{m}}, \xi_{h,\widetilde{m}+1}) + (v_{k,\widetilde{m}}, \xi_{h,\widetilde{m}+1}) \\ &\le \|\eta_{h,\widetilde{m}}\|_{H^{-2}(\Omega)} \|\Delta y_{k,\widetilde{m}+1}\| + \|\eta_{h,\widetilde{m}}\| \|\xi_{h,\widetilde{m}+1}\| \\ &\quad + \|\Delta v_{k,\widetilde{m}}\| \|\xi_{h,\widetilde{m}+1}\|_{H^{-2}(\Omega)}. \end{aligned} \tag{5.25}$$

From Theorem 4.5, we have

$$t_{\widetilde{m}} \|\Delta v_{k,\widetilde{m}}\| \le C \|\xi_{h,1}\| \quad \text{and} \quad \tau_{k,\widetilde{m}+1} \|\Delta y_{k,\widetilde{m}+1}\| \le C \|y_T\|,$$

and further, by Lemma 5.8,

$$\|\eta_{h,\widetilde{m}}\|_{H^{-2}(\Omega)} \le Ch^2 \|\xi_{h,1}\| \quad \text{and} \quad \|\xi_{h,\widetilde{m}+1}\|_{H^{-2}(\Omega)} \le Ch^2 \|y_T\|,$$

and by Lemma 5.7,

$$\sqrt{t_{\widetilde{m}}} \|\eta_{h,\widetilde{m}}\| \le Ch \|\xi_{h,1}\| \quad \text{and} \quad \sqrt{\tau_{k,\widetilde{m}+1}} \|\xi_{h,\widetilde{m}+1}\| \le Ch \|y_T\|.$$

For the three terms on the right-hand side of (5.25), we obtain

$$\begin{aligned} \|\eta_{h,\widetilde{m}}\|_{H^{-2}(\Omega)} \|\Delta y_{k,\widetilde{m}+1}\| &\le Ch^2 \tau_{k,\widetilde{m}+1}^{-1} \|\xi_{h,1}\| \|y_T\|, \\ \|\eta_{h,\widetilde{m}}\| \|\xi_{h,\widetilde{m}+1}\| &\le Ch^2 \sqrt{t_{\widetilde{m}}^{-1} \tau_{k,\widetilde{m}+1}^{-1}} \|\xi_{h,1}\| \|y_T\|, \\ \|\Delta v_{k,\widetilde{m}}\| \|\xi_{h,\widetilde{m}+1}\|_{H^{-2}(\Omega)} &\le Ch^2 t_{\widetilde{m}}^{-1} \|\xi_{h,1}\| \|y_T\|. \end{aligned}$$

We choose $\widetilde{m}$ such that $\frac{1}{2}T \in I_{\widetilde{m}}$. Using the assumption $k \le \frac{1}{4}T$, this implies

$$t_{\widetilde{m}} \ge \tfrac{1}{2}T \quad \text{and} \quad \tau_{k,\widetilde{m}+1} = T - t_{\widetilde{m}} \ge \tfrac{1}{2}T - k_{\widetilde{m}} \ge \tfrac{1}{4}T.$$

Hence, we conclude

$$\|\xi_{h,1}\|^2 \le \frac{Ch^2}{T} \|\xi_{h,1}\| \|y_T\|,$$

which implies the asserted estimate. ∎

After these preparations, we can now prove the main result of this subsection.

THEOREM 5.10. *For the* dG(0) *semidiscretized solution* $u_k \in X_k^0$ *of* (3.2) *and the fully discretized solution* $u_{kh} \in X_{k,h}^{0,1}$ *of* (3.11), *we have the error estimate*

$$\max_{m=1,2,\ldots,M} \|u_{k,m} - u_{kh,m}\| \le Ch^2 \ln \frac{T}{k} \left\{ \|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\| \right\}. \tag{5.26}$$

Proof. For simplicity, we consider only the last time point $t_M = T$. The proof again employs a duality argument. Let $y_k \in X_k^0$ and $y_{kh} \in X_{k,h}^{0,1}$ be the solutions of (4.7) and (4.9), respectively, with $y_T = e_{h,M} = u_{k,M} - u_{kh,M}$. Using Galerkin orthogonality, we obtain

$$\begin{aligned}\|e_{h,M}\|^2 &= B(e_h, y_k) = B(e_h, y_k - y_{kh}) = B(u_k, y_k - y_{kh}) \\ &= (f, y_k - y_{kh})_I + (u_0, y_{k,1} - y_{kh,1}) \\ &\le \big\{\|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\big\}\big\{\|y_k - y_{kh}\|_{L^1(I,L^2(\Omega))} + \|y_{k,1} - y_{kh,1}\|_{H^{-2}(\Omega)}\big\}.\end{aligned}$$

Then, in view of the assumption $k \le \frac{1}{4}T$, Lemma 5.9 implies

$$\begin{aligned}\|y_k - y_{kh}\|_{L^1(I,L^2(\Omega))} &\le \sum_{m=1}^{M} k_m \tau_{k,m}^{-1} \max_{m=1,2,\dots,M}\big(\tau_{k,m}\|y_{k,m} - y_{kh,m}\|\big) \\ &\le Ch^2\Big(\ln\frac{T}{k} + 1\Big)\|e_{h,M}\| \le Ch^2 \ln\frac{T}{k}\|e_{h,M}\|,\end{aligned}$$

and the assertion of Lemma 5.8 completes the proof. ☐

The following corollary, which is a direct consequence of the previous theorem, states the main result of this subsection.

COROLLARY 5.11. *For the error* $e_h := u_k - u_{kh}$ *between the* dG(0) *semidiscretized solution* $u_k \in X_k^0$ *of* (3.2) *and the fully discretized solution* $u_{kh} \in X_{k,h}^{0,1}$ *of* (3.11), *there holds the estimate*

$$\|e_h\|_{L^\infty(I,L^2(\Omega))} \le Ch^2 \ln\frac{T}{k}\big\{\|f\|_{L^\infty(I,L^2(\Omega))} + \|\Delta u_0\|\big\}. \tag{5.27}$$

Proof. Since u_k and u_{kh} are constant on the time intervals I_m, $m = 1, 2, \dots, M$, the assertion is directly implied by Theorem 5.10. ☐

6. Error analysis for the optimal control problem.

In this section, we will prove the main result of this article.

THEOREM 6.1. *Let* $\bar q \in Q_{ad}$ *be the solution of the optimal control problem* (2.3) *with optimal state* $\bar u \in X$, *and let* $\bar q_\sigma \in Q_{d,ad}$ *be the solution of the fully discrete optimal control problem* (3.20) *with discrete optimal state* $\bar u_\sigma \in X_{k,h}^{0,1}$. *Then, the following error estimate holds:*

$$\sqrt{\alpha}\|\bar q - \bar q_\sigma\|_I + \|\bar u - \bar u_\sigma\|_I \le C\left\{k^{\frac{1}{2}}\Big(\ln\frac{T}{k}\Big)^{\frac{1}{4}} + h\Big(\ln\frac{T}{k}\Big)^{\frac{1}{2}} + \frac{1}{\sqrt{\alpha}}h\right\}. \tag{6.1}$$

The proof of this result is divided into three steps reflecting the three steps of discretization introduced in section 3. In each step the important tools will be the estimates for the state equation from the previous section and the (uniform) boundedness of the discrete Lagrange multipliers; cf. [8, 6].

6.1. Estimates for the error due to time discretization of the state.

LEMMA 6.2. *Let* $\bar q_k \in Q_{ad}$ *be the solution of* (3.5) *with state* $\bar u_k \in X_k^0$ *and corresponding Lagrange multiplier* $\mu_k \in C(\bar I)^*$. *Then, there exists* $k_0 > 0$ *such that*

$$\|\bar q_k\|_I + \|\bar u_k\|_I + \|\mu_k\|_{C(\bar I)^*} \le C \quad \forall k \le k_0. \tag{6.2}$$

Proof. Since $G(u(\tilde{q}))(\cdot) \in C(\bar{I})$, the Slater condition (2.4) ensures the existence of $\delta > 0$ such that $G(u(\tilde{q})) \leq b - \delta$ in $\bar{I}$. Since $Q_{\mathrm{ad}} \subset L^\infty(I, L^2(\Omega))$, we have from Corollary 5.5 for $q \in Q_{\mathrm{ad}}$ that

$$\|u(q) - u_k(q)\|_{L^\infty(I,L^2(\Omega))} \to 0 \quad (k \to 0),$$

we also have

$$G(u_k(\tilde{q})) = G(u(\tilde{q})) + G(u_k(\tilde{q}) - u(\tilde{q})) \leq b - \delta + \|\omega\| \|u(\tilde{q}) - u_k(\tilde{q})\|_{L^\infty(I,L^2(\Omega))} < b$$

for $k \leq k_0$. This implies

$$\begin{aligned} J(\bar{q}_k, \bar{u}_k) &\leq J(\tilde{q}, u_k(\tilde{q})) = \tfrac{1}{2}\|u_k(\tilde{q}) - \hat{u}\|_I^2 + \tfrac{1}{2}\alpha\|\tilde{q}\|_I^2 \\ &\leq \|u_k(\tilde{q}) - u(\tilde{q})\|_I^2 + \|u(\tilde{q}) - \hat{u}\|_I^2 + \tfrac{1}{2}\alpha\|\tilde{q}\|_I^2 \leq C \end{aligned}$$

for $k \leq k_0$ and, consequently, the bound

$$\|\bar{q}_k\|_I + \|\bar{u}_k\|_I \leq C. \tag{6.3}$$

For $p := \frac{1}{2}\bar{q} + \frac{1}{2}\tilde{q}$, we have for $k \leq k_0$ that

$$\begin{aligned} G(u_k(p)) &= \tfrac{1}{2}G(\bar{u}) + \tfrac{1}{2}G(u(\tilde{q})) + G(u_k(p) - u(p)) \\ &\leq \tfrac{1}{2}G(\bar{u}) + \tfrac{1}{2}G(u(\tilde{q})) + \|\omega\| \|u_k(p) - u(p)\|_{L^\infty(I,L^2(\Omega))} \\ &\leq \tfrac{1}{2}b + \tfrac{1}{2}b - \tfrac{1}{2}\delta + \tfrac{1}{4}\delta = b - \tfrac{1}{4}\delta. \end{aligned} \tag{6.4}$$

Since $p \in Q_{\mathrm{ad}}$, by means of (3.8), (3.2), and (3.7) and using (6.3), (6.4), and (3.9), we conclude

$$\begin{aligned} 0 &\leq (\bar{z}_k + \alpha\bar{q}_k, p - \bar{q}_k)_I = (\bar{z}_k, p - \bar{q}_k)_I + \alpha(\bar{q}_k, p - \bar{q}_k)_I \\ &= B(u_k(p) - \bar{u}_k, \bar{z}_k) + \alpha(\bar{q}_k, p - \bar{q}_k)_I \\ &= (u_k(p) - \bar{u}_k, \bar{u}_k - \hat{u})_I + \langle G(u_k(p)) - G(\bar{u}_k), \mu_k\rangle + \alpha(\bar{q}_k, p - \bar{q}_k)_I \\ &\leq C + \langle G(u_k(p)) - G(\bar{u}_k), \mu_k\rangle \leq C + \langle b - G(\bar{u}_k), \mu_k\rangle - \tfrac{1}{4}\delta\langle 1, \mu_k\rangle \\ &= C - \tfrac{1}{4}\delta\langle 1, \mu_k\rangle. \end{aligned}$$

Then, $\mu_k \geq 0$ implies the asserted bound for the third term,

$$\|\mu_k\|_{C(\bar{I})^*} = \langle 1, \mu_k\rangle \leq C.$$

This completes the proof. ∎

THEOREM 6.3. *Let $\bar{q} \in Q_{ad}$ be the solution of the optimal control problem* (2.3) *with optimal state $\bar{u} \in X$, and let $\bar{q}_k \in Q_{ad}$ be the solution of the semidiscrete optimal control problem* (3.5) *with semidiscrete optimal state $\bar{u}_k \in X_k^0$. Then, the following error estimate holds:*

$$\begin{aligned} \alpha\|\bar{q} - \bar{q}_k\|_I^2 + \|\bar{u} - \bar{u}_k\|_I^2 \leq C\big\{&\|u(\bar{q}) - u_k(\bar{q})\|_I + \|u(\bar{q}_k) - u_k(\bar{q}_k)\|_I \\ &+ \|u(\bar{q}) - u_k(\bar{q})\|_{L^\infty(I,L^2(\Omega))} + \|u(\bar{q}_k) - u_k(\bar{q}_k)\|_{L^\infty(I,L^2(\Omega))}\big\}. \end{aligned} \tag{6.5}$$

Proof. Choosing $q = \bar{q}_k$ in (2.6) and $q = \bar{q}$ in (3.8) gives us

$$(-\alpha\bar{q} - \bar{z}, \bar{q} - \bar{q}_k)_I \geq 0 \quad \text{and} \quad (\alpha\bar{q}_k + \bar{z}_k, \bar{q} - \bar{q}_k)_I \geq 0.$$

By adding these inequalities, we obtain

$$\alpha\|\bar{q}-\bar{q}_k\|_I^2 \leq \underbrace{(\bar{z},\bar{q}_k-\bar{q})_I}_{\text{(I)}} + \underbrace{(\bar{z}_k,\bar{q}-\bar{q}_k)_I}_{\text{(II)}}. \tag{6.6}$$

The terms (I) and (II) will be treated separately.

(i) For (I), we have, due to the state and adjoint equations (2.1) and (2.5), respectively, and since $(u(\bar{q}_k)-\bar{u})(0)=0$, that

$$\begin{aligned}(\bar{z},\bar{q}_k-\bar{q})_I &= (\partial_t(u(\bar{q}_k)-\bar{u}),\bar{z})_I + (\nabla(u(\bar{q}_k)-\nabla\bar{u}),\nabla\bar{z})_I\\ &= (u(\bar{q}_k)-\bar{u},\bar{u}-\hat{u})_I + \langle G(u(\bar{q}_k)-\bar{u}),\mu\rangle.\end{aligned} \tag{6.7}$$

By means of the pointwise projection P_b onto $(-\infty,b]$, the last term in (6.7) can be estimated by means of (2.7) as

$$\begin{aligned}\langle G(u(\bar{q}_k)-\bar{u}),\mu\rangle &= \langle G(u(\bar{q}_k))-P_bG(u(\bar{q}_k)),\mu\rangle + \langle P_bG(u(\bar{q}_k))-G(\bar{u}),\mu\rangle\\ &\leq \langle G(u(\bar{q}_k))-P_bG(u(\bar{q}_k)),\mu\rangle + \langle b-G(\bar{u}),\mu\rangle\\ &= \langle G(u(\bar{q}_k))-P_bG(u(\bar{q}_k)),\mu\rangle.\end{aligned}$$

Employing the relation $|P_b(t)-P_b(s)|\leq|t-s|$, we obtain

$$\begin{aligned}\langle G(u(\bar{q}_k))-P_bG(u(\bar{q}_k)),\mu\rangle &\leq \|G(u(\bar{q}_k))-P_bG(u(\bar{q}_k))\|_{L^\infty(I)}\|\mu\|_{C(\bar{I})^*}\\ &\leq \|G(u(\bar{q}_k))-G(\bar{u}_k)\|_{L^\infty(I)}\|\mu\|_{C(\bar{I})^*}\\ &\quad + \|P_bG(\bar{u}_k)-P_bG(u(\bar{q}_k))\|_{L^\infty(I)}\|\mu\|_{C(\bar{I})^*}\\ &\leq 2\|\omega\|\|u(\bar{q}_k)-u_k(\bar{q}_k)\|_{L^\infty(I,L^2(\Omega))}\|\mu\|_{C(\bar{I})^*}.\end{aligned}$$

Collecting all the preceding estimates, we obtain the desired bound for (I):

$$\begin{aligned}(\bar{z},\bar{q}_k-\bar{q})_I \leq{}& (u(\bar{q}_k)-\bar{u},\bar{u}-\hat{u})_I\\ &+ 2\|\omega\|\|u(\bar{q}_k)-u_k(\bar{q}_k)\|_{L^\infty(I,L^2(\Omega))}\|\mu\|_{C(\bar{I})^*}.\end{aligned} \tag{6.8}$$

(ii) The term (II) is estimated similarly to (I). In view of the semidiscrete state and adjoint equations (3.2) and (3.7), we have

$$\begin{aligned}(\bar{z}_k,\bar{q}-\bar{q}_k)_I &= B(u_k(\bar{q})-\bar{u}_k,\bar{z}_k)\\ &= (u_k(\bar{q})-\bar{u}_k,\bar{u}_k-\hat{u})_I + \langle G(u_k(\bar{q})-\bar{u}_k),\mu_k\rangle.\end{aligned} \tag{6.9}$$

Using the projection P_b defined above, the last term in (6.9) can be estimated as before by

$$\begin{aligned}\langle G(u_k(\bar{q})-\bar{u}_k),\mu_k\rangle &\leq \langle G(u_k(\bar{q}))-P_bG(u_k(\bar{q})),\mu_k\rangle\\ &\leq 2\|\omega\|\|u_k(\bar{q})-u(\bar{q})\|_{L^\infty(I,L^2(\Omega))}\|\mu_k\|_{C(\bar{I})^*}.\end{aligned}$$

This yields the desired bound for (II):

$$\begin{aligned}(\bar{z}_k,\bar{q}-\bar{q}_k)_I \leq{}& (u_k(\bar{q})-\bar{u}_k,\bar{u}_k-\hat{u})_I\\ &+ 2\|\omega\|\|u_k(\bar{q})-u(\bar{q})\|_{L^\infty(I,L^2(\Omega))}\|\mu_k\|_{C(\bar{I})^*}.\end{aligned} \tag{6.10}$$

Using the estimates (6.8) and (6.10) in (6.6) yields

$$\begin{aligned}\alpha\|\bar q-\bar q_k\|_I^2 &\le (u(\bar q_k)-\bar u,\bar u-\hat u)_I+(u_k(\bar q)-\bar u_k,\bar u_k-\hat u)_I\\ &\quad+2\|\omega\|\|\mu\|_{C(\bar I)^*}\|u(\bar q_k)-u_k(\bar q_k)\|_{L^\infty(I,L^2(\Omega))}\\ &\quad+2\|\omega\|\|\mu_k\|_{C(\bar I)^*}\|u(\bar q)-u_k(\bar q)\|_{L^\infty(I,L^2(\Omega))}.\end{aligned}\tag{6.11}$$

The first two terms on the right-hand side of (6.11) can be transformed as follows:

$$\begin{aligned}&(u(\bar q_k)-\bar u,\bar u-\hat u)_I+(u_k(\bar q)-\bar u_k,\bar u_k-\hat u)_I\\ &\quad=-\|\bar u-\bar u_k\|_I^2+(\bar u-u_k(\bar q),\hat u-\bar u_k)_I+(\bar u_k-u(\bar q_k),\hat u-\bar u)_I\\ &\quad\le-\|\bar u-\bar u_k\|_I^2+\big\{\|\hat u\|_I+\|\bar u_k\|_I\big\}\|u(\bar q)-u_k(\bar q)\|_I\\ &\qquad+\big\{\|\hat u\|_I+\|\bar u\|_I\big\}\|u_k(\bar q_k)-u(\bar q_k)\|_I.\end{aligned}$$

Then, the boundedness of $\|\bar u\|_I$ and $\|\mu\|_{C(\bar I)^*}$, $\|\bar u_k\|_I$ and $\|\mu_k\|_{C(\bar I)^*}$ (cf. Lemma 6.2), as well as that of $\|\hat u\|_I$ and $\|\omega\|$, implies the desired estimate. ∎

COROLLARY 6.4. *Under the conditions of Theorem* 6.3 *there holds*

$$\alpha\|\bar q-\bar q_k\|_I^2+\|u(\bar q)-u_k(\bar q_k)\|_I^2\le Ck\Big(\ln\frac{T}{k}\Big)^{\frac12}.\tag{6.12}$$

Proof. The assertion follows directly from the estimates in Theorem 6.3 and Corollary 5.5 and the $L^2(I,L^2(\Omega))$ error estimates in Theorem 5.1 in [19]. ∎

6.2. Estimates for the error due to space discretization of the state.

LEMMA 6.5. *Let $\bar q_{kh}\in Q_{ad}$ be the solution of* (3.13) *with state $\bar u_{kh}\in X^{0,1}_{k,h}$ and corresponding Lagrange multiplier $\mu_{kh}\in C(\bar I)^*$. Then, there exist $k_0>0$ and $h_0>0$ such that*

$$\|\bar q_{kh}\|_I+\|\bar u_{kh}\|_I+\|\mu_{kh}\|_{C(\bar I)^*}\le C\quad\forall k\le k_0,\ h\le h_0.\tag{6.13}$$

Proof. Similarly to the proof of Lemma 6.2, by the Slater condition (2.4), we have the existence of $\delta>0$ such that $G(u(\tilde q))\le b-\delta$ in $\bar I$. Since $Q_{\mathrm{ad}}\subset L^\infty(I,L^2(\Omega))$, we have from Corollaries 5.5 and 5.11 for $q\in Q_{\mathrm{ad}}$ that

$$\begin{aligned}\|u(q)-u_k(q)\|_{L^\infty(I,L^2(\Omega))}&\to0\qquad(k\to0),\\ \|u_k(q)-u_{kh}(q)\|_{L^\infty(I,L^2(\Omega))}&\to0\qquad(h\to0),\end{aligned}$$

and we obtain, as before,

$$G(u_{kh}(\tilde q))<b$$

for $k\le k_0$, $h\le h_0$. This implies

$$\begin{aligned}J(\bar q_{kh},\bar u_{kh})&\le J(\tilde q,u_{kh}(\tilde q))=\tfrac12\|u_{kh}(\tilde q)-\hat u\|_I^2+\tfrac12\alpha\|\tilde q\|_I^2\\ &\le\|u_{kh}(\tilde q)-u(\tilde q)\|_I^2+\|u(\tilde q)-\hat u\|_I^2+\tfrac12\alpha\|\tilde q\|_I^2\le C\end{aligned}$$

for $k\le k_0$, $h\le h_0$, and consequently,

$$\|\bar q_{kh}\|_I+\|\bar u_{kh}\|_I\le C.\tag{6.14}$$

For $p:=\frac12\bar q+\frac12\tilde q$, as in the proof of Lemma 6.2, we have for $k\le k_0$, $h\le h_0$ that

$$G(u_{kh}(p))\le b-\tfrac14\delta.\tag{6.15}$$

Since $p \in Q_{\text{ad}}$, by means of (3.16), (3.11), (3.15) and using (6.14), (6.15), (3.17), we obtain as in the proof of Lemma 6.2 that

$$0 \le (\bar z_{kh} + \alpha \bar q_{kh}, p - \bar q_{kh})_I \le C - \tfrac{1}{4}\delta\langle 1, \mu_{kh}\rangle.$$

Then, $\mu_{kh} \ge 0$ implies the asserted bound for the third term,

$$\|\mu_{kh}\|_{C(\bar I)^*} = \langle 1, \mu_{kh}\rangle \le C.$$

This completes the proof. □

THEOREM 6.6. *Let $\bar q_k \in Q_{ad}$ be the solution of the semidiscrete optimal control problem* (3.5) *with semidiscrete optimal state $\bar u_k \in X_k^0$, and let $\bar q_{kh} \in Q_{ad}$ be the solution of the semidiscrete optimal control problem* (3.13) *with discrete optimal state $\bar u_{kh} \in X_{k,h}^{0,1}$. Then, the following estimate holds:*

$$\begin{aligned}\alpha\|\bar q_k - \bar q_{kh}\|_I^2 + \|\bar u_k - \bar u_{kh}\|_I^2 \le C\big\{&\|u_k(\bar q_k) - u_{kh}(\bar q_k)\|_I + \|u_k(\bar q_{kh}) - u_{kh}(\bar q_{kh})\|_I \\ &+ \|u_k(\bar q_k) - u_{kh}(\bar q_k)\|_{L^\infty(I,L^2(\Omega))} \\ &+ \|u_k(\bar q_{kh}) - u_{kh}(\bar q_{kh})\|_{L^\infty(I,L^2(\Omega))}\big\}.\end{aligned} \tag{6.16}$$

Proof. Choosing $q = \bar q_{kh}$ in (3.8) and $q = \bar q_k$ in (3.16) gives us

$$(-\alpha \bar q_k - \bar z_k, \bar q_k - \bar q_{kh})_I \ge 0 \quad \text{and} \quad (\alpha \bar q_{kh} + \bar z_{kh}, \bar q_k - \bar q_{kh})_I \ge 0.$$

By adding these inequalities, we obtain

$$\alpha\|\bar q_k - \bar q_{kh}\|_I^2 \le \underbrace{(\bar z_k, \bar q_{kh} - \bar q_k)_I}_{\text{(I)}} + \underbrace{(\bar z_{kh}, \bar q_k - \bar q_{kh})_I}_{\text{(II)}}. \tag{6.17}$$

The terms (I) and (II) will be treated separately.

(i) For (I), in view of the semidiscrete state and adjoint equations (3.2) and (3.7), respectively, we obtain as in the proof of Theorem 6.3 that

$$\begin{aligned}(\bar z_k, \bar q_{kh} - \bar q_k)_I \le{}& (u_k(\bar q_{kh}) - \bar u_k, \bar u_k - \hat u)_I \\ &+ 2\|\omega\|\|u_k(\bar q_{kh}) - u_{kh}(\bar q_{kh})\|_{L^\infty(I,L^2(\Omega))}\|\mu_k\|_{C(\bar I)^*}.\end{aligned} \tag{6.18}$$

(ii) For the term (II), we proceed as in the proof of Theorem 6.3 using now the semidiscrete state and adjoint equations (3.11) and (3.15). We get

$$\begin{aligned}(\bar z_{kh}, \bar q_k - \bar q_{kh})_I \le{}& (u_{kh}(\bar q_k) - \bar u_{kh}, \bar u_{kh} - \hat u)_I \\ &+ 2\|\omega\|\|u_{kh}(\bar q_k) - u_k(\bar q_k)\|_{L^\infty(I,L^2(\Omega))}\|\mu_{kh}\|_{C(\bar I)^*}.\end{aligned} \tag{6.19}$$

Using the estimates (6.18) and (6.19) in (6.17) and performing the same transformations as in the proof of Theorem 6.3, we conclude that

$$\begin{aligned}(u_k(\bar q_{kh}) - \bar u_k, \bar u_k - \hat u)_I &+ (u_{kh}(\bar q_k) - \bar u_{kh}, \bar u_{kh} - \hat u)_I \\ &\le -\|\bar u_k - \bar u_{kh}\|_I^2 + \big\{\|\hat u\|_I + \|\bar u_{kh}\|_I\big\}\|u_k(\bar q_k) - u_{kh}(\bar q_k)\|_I \\ &\qquad + \big\{\|\hat u\|_I + \|\bar u_k\|_I\big\}\|u_{kh}(\bar q_{kh}) - u_k(\bar q_{kh})\|_I.\end{aligned}$$

Then, the boundedness of $\|\bar u_k\|_I$ and $\|\mu_k\|_{C(\bar I)^*}$ (cf. Lemma 6.2), $\|\bar u_{kh}\|_I$ and $\|\mu_{kh}\|_{C(\bar I)^*}$ (cf. Lemma 6.5), as well as that of $\|\hat u\|_I$ and $\|\omega\|$, implies the desired estimate. □

COROLLARY 6.7. *Under the conditions of Theorem* 6.6 *there holds*

$$\alpha\|\bar q_k - \bar q_{kh}\|_I^2 + \|u_k(\bar q_k) - u_{kh}(\bar q_{kh})\|_I^2 \le Ch^2 \ln\frac{T}{k}. \tag{6.20}$$

Proof. The assertion follows directly from the estimates of Theorem 6.6 and Corollary 5.11 and the $L^2(I, L^2(\Omega))$ error estimates from Theorem 5.5 in [19]. □

6.3. Estimates for the error due to discretization of the control. To estimate the error due to the discretization of the control, we introduce the L^2 projection $\pi_d\colon Q \to Q_d$. Because of the cellwise constant discretization in time and space, we have $\pi_d Q_{\mathrm{ad}} \subset Q_{d,\mathrm{ad}}$.

LEMMA 6.8. *Let $\bar{q}_\sigma \in Q_{d,ad}$ be the solution of* (3.20) *with associated state $\bar{u}_\sigma \in X^{0,1}_{k,h}$ and corresponding Lagrange multiplier $\mu_\sigma \in C(\bar{I})^*$. Then there exist $k_0 > 0$ and $h_0 > 0$ such that*

$$\|\bar{q}_\sigma\|_I + \|\bar{u}_\sigma\|_I + \|\mu_\sigma\|_{C(\bar{I})^*} \le C \quad \forall k \le k_0,\ h \le h_0. \tag{6.21}$$

Proof. Similarly to the proofs of Lemmas 6.2 and 6.5, by the Slater condition (2.4), we have the existence of $\delta > 0$ such that $G(u(\tilde{q})) \le b - \delta$ in $\bar{I}$. As before, we have for $q \in Q_{\mathrm{ad}}$ from Corollaries 5.5 and 5.11 that

$$\|u(q) - u_k(q)\|_{L^\infty(I,L^2(\Omega))} \to 0 \qquad (k \to 0),$$
$$\|u_k(q) - u_{kh}(q)\|_{L^\infty(I,L^2(\Omega))} \to 0 \qquad (h \to 0).$$

To prove

$$\|u(\tilde{q}) - u_{kh}(\pi_d\tilde{q})\|_{L^\infty(I,L^2(\Omega))} \to 0 \quad (k,h \to 0)$$

and, consequently,

$$G(u_{kh}(\pi_d\tilde{q})) < b$$

for $k \le k_0$, $h \le h_0$, it remains to show the convergence

$$\|u_{kh}(q) - u_{kh}(\pi_d q)\|_{L^\infty(I,L^2(\Omega))} \to 0 \quad (k,h \to 0).$$

Using the stability of the continuous solution $u \in X$ of (2.1), we have

$$\begin{aligned}\|u_{kh}(q) - u_{kh}(\pi_d q)\|_{L^\infty(I,L^2(\Omega))} \le{}& \|u_{kh}(q) - u(q)\|_{L^\infty(I,L^2(\Omega))} \\ &+ \|u(q) - u(\pi_d q)\|_{L^\infty(I,L^2(\Omega))} \\ &+ \|u(\pi_d q) - u_{kh}(\pi_d q)\|_{L^\infty(I,L^2(\Omega))} \\ \le{}& \|u_{kh}(q) - u(q)\|_{L^\infty(I,L^2(\Omega))} + C\|q - \pi_d q\|_I \\ &+ \|u(\pi_d q) - u_{kh}(\pi_d q)\|_{L^\infty(I,L^2(\Omega))}.\end{aligned}$$

Then, by the properties of the L^2 projection π_d, we obtain

$$\|q - \pi_d q\|_I \to 0 \quad (k,h \to 0) \quad \text{and} \quad \|\pi_d q\|_{L^\infty(I,L^2(\Omega))} \le C\|q\|_{L^\infty(I,L^2(\Omega))},$$

and further, by Corollaries 5.5 and 5.11, we conclude the desired convergence behavior of $\|u_{kh}(q) - u_{kh}(\pi_d q)\|_{L^\infty(I,L^2(\Omega))}$. This implies

$$\begin{aligned}J(\bar{q}_\sigma, \bar{u}_\sigma) &\le J(\pi_d\tilde{q}, u_{kh}(\pi_d\tilde{q})) \\ &= \tfrac{1}{2}\|u_{kh}(\pi_d\tilde{q}) - \hat{u}\|_I^2 + \tfrac{1}{2}\alpha\|\pi_d\tilde{q}\|_I^2 \\ &\le \|u_{kh}(\pi_d\tilde{q}) - u(\tilde{q})\|_I^2 + \|u(\tilde{q}) - \hat{u}\|_I^2 + \tfrac{1}{2}\alpha\|\pi_d\tilde{q}\|_I^2 \le C\end{aligned}$$

for $k \le k_0$, $h \le h_0$, and consequently,

$$\|\bar{q}_\sigma\|_I + \|\bar{u}_\sigma\|_I \le C. \tag{6.22}$$

For $p := \frac{1}{2}\bar{q} + \frac{1}{2}\tilde{q}$, as in the proofs of Lemmas 6.2 and 6.5, we have for $k \le k_0$, $h \le h_0$ that

$$G(u_{kh}(\pi_d p)) \le b - \tfrac{1}{4}\delta. \tag{6.23}$$

Further, since $\pi_d p \in Q_{d,\mathrm{ad}}$, by means of (3.11) and (3.22) using (6.22), (6.23), and (3.24) as in the proof of Lemma 6.2, we find

$$0 \le (\bar{z}_\sigma + \alpha\bar{q}_\sigma, \pi_d p - \bar{q}_\sigma)_I \le C - \tfrac{1}{4}\delta\langle 1, \mu_\sigma\rangle.$$

Then, $\mu_\sigma \ge 0$ implies the asserted bound for the third term,

$$\|\mu_\sigma\|_{C(\bar{I})^*} = \langle 1, \mu_\sigma\rangle \le C.$$

This completes the proof. ∎

THEOREM 6.9. *Let $\bar{q}_{kh} \in Q_{ad}$ be the solution of the semidiscrete optimal control problem* (3.13) *with discrete optimal state $\bar{u}_{kh} \in X_{k,h}^{0,1}$, and let $\bar{q}_\sigma \in Q_{d,ad}$ be the solution of the discrete optimal control problem* (3.20) *with discrete optimal state $\bar{u}_\sigma \in X_{k,h}^{0,1}$. Then the following estimate holds:*

$$\alpha\|\bar{q}_{kh} - \bar{q}_\sigma\|_I^2 + \|\bar{u}_{kh} - \bar{u}_\sigma\|_I^2 \le C\|\bar{z}_\sigma - \pi_d\bar{z}_\sigma\|_I\|\bar{q}_{kh} - \pi_d\bar{q}_{kh}\|_I. \tag{6.24}$$

Proof. Choosing $q = \bar{q}_\sigma$ in (3.16) and $q = \pi_d\bar{q}_{kh}$ in (3.23) gives us

$$(\alpha\bar{q}_{kh} + \bar{z}_{kh}, \bar{q}_\sigma - \bar{q}_{kh})_I \ge 0 \quad \text{and} \quad (-\alpha\bar{q}_\sigma - \bar{z}_\sigma, \bar{q}_\sigma - \pi_d\bar{q}_{kh})_I \ge 0.$$

By adding these inequalities and using the properties of π_d, we obtain

$$\begin{aligned}
\alpha\|\bar{q}_{kh} - \bar{q}_\sigma\|_I^2 &\le (\bar{z}_{kh}, \bar{q}_\sigma - \bar{q}_{kh})_I - (\bar{z}_\sigma, \bar{q}_\sigma - \bar{q}_{kh})_I - (\alpha\bar{q}_\sigma + \bar{z}_\sigma, \bar{q}_{kh} - \pi_d\bar{q}_{kh})_I \\
&= B(\bar{u}_\sigma - \bar{u}_{kh}, \bar{z}_{kh}) - B(\bar{u}_\sigma - \bar{u}_{kh}, \bar{z}_\sigma) - (\bar{z}_\sigma - \pi_d z_{kh}(\bar{q}_\sigma), \bar{q}_{kh} - \pi_d\bar{q}_{kh})_I \\
&= (\bar{u}_\sigma - \bar{u}, \bar{u}_{kh} - \hat{u})_I + \langle G(\bar{u}_\sigma) - G(\bar{u}_{kh}), \mu_{kh}\rangle - (\bar{u}_\sigma - \bar{u}_{kh}, \bar{u}_\sigma - \hat{u})_I \\
&\quad - \langle G(\bar{u}_\sigma) - G(\bar{u}_{kh}), \mu_\sigma\rangle - (\bar{z}_\sigma - \pi_d\bar{z}_\sigma, \bar{q}_{kh} - \pi_d\bar{q}_{kh})_I \\
&\le -\|\bar{u}_{kh} - \bar{u}_\sigma\|_I^2 + \langle b - G(\bar{u}_{kh}), \mu_{kh}\rangle + \langle b - G(\bar{u}_\sigma), \mu_\sigma\rangle \\
&\quad + \|\bar{z}_\sigma - \pi_d\bar{z}_\sigma\|_I\|\bar{q}_{kh} - \pi_d\bar{q}_{kh}\|_I.
\end{aligned}$$

Then, the complementarity relations (3.17) and (3.24) imply the asserted estimate. ∎

COROLLARY 6.10. *Under the conditions of Theorem* 6.6 *there holds*

$$\alpha\|\bar{q}_{kh} - \bar{q}_\sigma\|_I^2 + \|\bar{u}_{kh} - \bar{u}_\sigma\|_I^2 \le C\alpha^{-1}h^2. \tag{6.25}$$

Proof. From Theorem 6.9, we have

$$\alpha\|\bar{q}_{kh} - \bar{q}_\sigma\|_I^2 + \|\bar{u}_{kh} - \bar{u}_\sigma\|_I^2 \le C\|\bar{z}_\sigma - \pi_d\bar{z}_\sigma\|_I\|\bar{q}_{kh} - \pi_d\bar{q}_{kh}\|_I$$

Further, for $v_k \in X_k^0$, there holds

$$\|v_k - \pi_d v_k\|_I \le Ch\|\nabla v_k\|_I.$$

We use the fact that $\bar{q}_{kh}\big|_{I_m} \in \mathcal{P}_0(I_m, H^1(\Omega))$ for $m = 1, 2, \dots, M$ (see Remark 3.7) and obtain

$$\alpha\|\bar{q}_{kh} - \bar{q}_\sigma\|_I^2 + \|\bar{u}_{kh} - \bar{u}_\sigma\|_I^2 \le Ch^2\|\nabla\bar{z}_\sigma\|_I\|\nabla\bar{q}_{kh}\|_I.$$

Employing the projection formula (3.18), the H^1 stability of the pointwise projection $P_{Q_{\rm ad}}$ (cf. [32]),

$$\|\nabla\bar{q}_{kh}\|_I \le \alpha^{-1}\|\nabla\bar{z}_{kh}\|_I,$$

and the estimates from Theorem 4.1 and Corollary 4.2 implies

$$\begin{aligned}\alpha\|\bar{q}_{kh} - \bar{q}_\sigma\|_I^2 &+ \|\bar{u}_{kh} - \bar{u}_\sigma\|_I^2 \\ &\le C\alpha^{-1}h^2\big\{\|\bar{u}_{kh} - \hat{u}\|_I^2 + \|\bar{u}_\sigma - \hat{u}\|_I^2 + \|\omega\|^2\|\mu_{kh}\|_{C(\bar{I})^*}^2 + \|\omega\|^2\|\mu_\sigma\|_{C(\bar{I})^*}^2\big\}.\end{aligned}$$

The boundedness of $\|\bar{u}_{kh}\|_I$, $\|\mu_{kh}\|_{C(\bar{I})^*}$ (cf. Lemma 6.5) and $\|\bar{u}_\sigma\|_I$, $\|\mu_\sigma\|_{C(\bar{I})^*}$ (cf. Lemma 6.8) eventually yields the asserted estimate for the third term. ∎

Finally, the assertion of Theorem 6.1 is obtained by combining the results of Corollaries 6.4, 6.7, and 6.10.

7. Additional regularity.

In this section, we provide a regularity result for the optimal control $\bar{q}$. The proof is based on the error estimate from Corollary 6.4. Usually, regularity properties are used for deriving error estimates. Here, the error estimate in Corollary 6.4 is derived without assuming temporal regularity of $\bar{q}$, and the regularity result can be inferred from the error estimate.

THEOREM 7.1. *Let $\bar{q} \in Q_{ad}$ be the solution of the optimal control problem* (2.3). *Then, there holds*

$$\bar{q} \in L^2(I, H^1(\Omega)) \cap L^\infty(I \times \Omega) \cap H^s(I, H) \tag{7.1}$$

for all $0 \le s < \frac{1}{2}$.

Proof. The regularity $\bar{q} \in L^2(I, H^1(\Omega)) \cap L^\infty(I \times \Omega)$ follows directly from the optimality system in Theorem 2.4 (see Remark 2.5). In order to show $\bar{q} \in H^s(I, H)$, we consider a sequence of uniformly refined temporal meshes with step size $k_n = 2^{-n}T$ and the corresponding sequence of solutions $\bar{q}_n := \bar{q}_{k_n}$ to the semidiscrete problem (3.5). From the optimality system in Theorem 3.3, we obtain

$$\bar{q}_n = P_{Q_{\rm ad}}(-\alpha^{-1}\bar{z}_n)$$

with the corresponding adjoint state $\bar{z}_n := \bar{z}_{k_n} \in X_{k_n}^0$. This implies that $\bar{q}_n\big|_{I_m} \in \mathcal{P}_0(I_m, H^1(\Omega))$, and therefore $\bar{q}_n \in H^s(I, H)$, for $0 \le s < \frac{1}{2}$, which can be verified directly, since functions in $X_{k_n}^0$ are piecewise constant in time.

Due to the fact that $\bar{q}_n \to \bar{q}$ in $L^2(I, H)$, we obtain the identity

$$\bar{q} = \bar{q}_1 + \sum_{n=1}^{\infty}(\bar{q}_{n+1} - \bar{q}_n), \tag{7.2}$$

where the convergence of the series is understood in $L^2(I, H)$. It remains to show that the above series converges in $H^s(I, H)$. We employ the inverse inequality for $v_k \in X_k^0$, to obtain

$$\|v_k\|_{H^s(I,H)} \le Ck^{-s}\,\|v_k\|_I.$$

Then, using the error estimate from Corollary 6.4, it follows for large n such that k_n is sufficiently small that

$$\begin{aligned}\|\bar{q}_{n+1}-\bar{q}_n\|_{H^s(I,H)} &\le Ck_n^{-s}\,\|\bar{q}_{n+1}-\bar{q}_n\|_I \le Ck_n^{-s}\big\{\|\bar{q}_{n+1}-\bar{q}\|_I+\|\bar{q}-\bar{q}_n\|_I\big\}\\ &\le Ck_n^{-s}\left\{k_n^{\frac12}\Big(\ln\frac{T}{k_n}\Big)^{\frac14}+k_{n+1}^{\frac12}\Big(\ln\frac{T}{k_{n+1}}\Big)^{\frac14}\right\}\\ &\le Ck_n^{\frac12-s}\Big(\ln\frac{T}{k_{n+1}}\Big)^{\frac14}=C2^{-n(\frac12-s)}\big((n+1)\ln 2\big)^{\frac14}.\end{aligned}$$

For $s<\frac{1}{2}$, we obtain absolute convergence of (7.2) in $H^s(I,H)$, which completes the proof. □

8. Numerical results. In this section, we are going to numerically validate the a priori error estimates for the error in the control, state, and adjoint state. To this end, we consider the following concretion of the optimal control problem (2.3) with known exact solution on $\Omega\times I=(0,1)^2\times(0,1)$ and homogeneous Dirichlet boundary conditions. For $\gamma\in(0,1)$ and $\lambda\coloneqq 2\pi^2$, the right-hand side f, the desired state $\hat{u}$, and the initial condition u_0 are given in terms of the functions

$$\varepsilon(t,x_1,x_2)\coloneqq\big(e^{-\frac{\lambda}{2}}-e^{-\lambda t}\big)\sin(\pi x_1)\sin(\pi x_2)$$

and

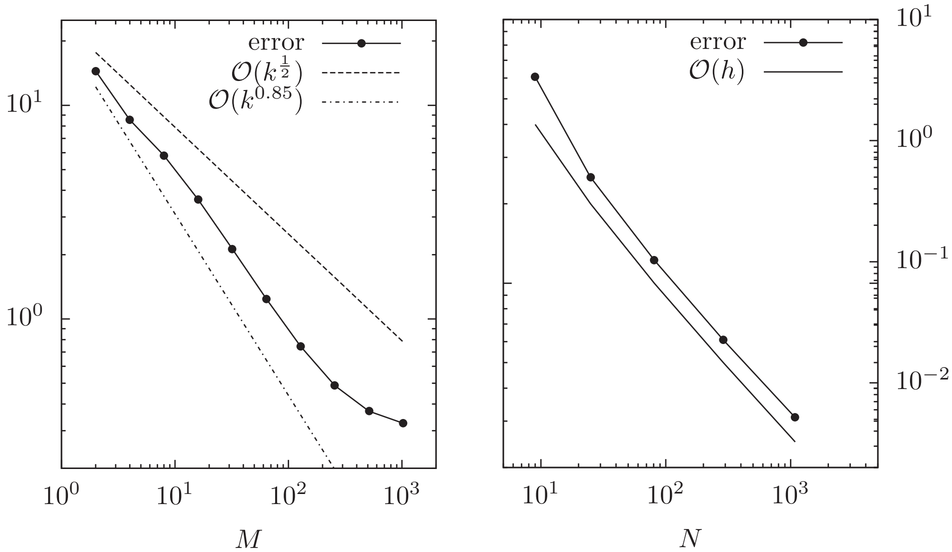
$$a(t,x_1,x_2)\coloneqq\frac{e^{\frac{\lambda}{2}-\lambda t}}{2}\varepsilon(1,x_1,x_2)^{1-\gamma},\quad b(t,x_1,x_2)\coloneqq\frac{e^{\frac{\lambda}{2}-\lambda t}}{2\lambda}\varepsilon(1,x_1,x_2)^{1-\gamma}$$

as

$$\begin{aligned}f(t,x_1,x_2)&\coloneqq\frac{1}{\lambda}\big(e^{\lambda(t-1)}-1\big)\sin(\pi x_1)\sin(\pi x_2)\\ &\quad+\frac{e^{\lambda t}}{\lambda(1-\gamma)}\begin{cases}\lambda(a(t)t+b(t)-\frac{a(t)}{2})+a(t), & t\le\frac12,\\ \varepsilon(1)^{1-\gamma}-\varepsilon(t)^{1-\gamma}, & t>\frac12,\end{cases}\\ \hat{u}(t,x_1,x_2)&\coloneqq\sin(\pi x_1)\sin(\pi x_2)\\ &\quad-\frac{e^{\lambda t}}{\lambda(1-\gamma)}\begin{cases}\frac{1}{2\lambda}\varepsilon(1)^{1-\gamma}-a(t)t-b(t)+\frac{a(t)}{2}, & t\le\frac12,\\ 0, & t>\frac12,\end{cases}\\ u_0(x_1,x_2)&\coloneqq-\frac{1}{\lambda(1-\gamma)}\left(\frac{1}{2\lambda}\varepsilon(1)^{1-\gamma}-b(0)+\frac{a(0)}{2}\right).\end{aligned}$$

For this choice of data, with the regularization parameter α chosen as $\alpha=1$, the weight $\omega(x_1,x_2)\coloneqq\sin(\pi x_1)\sin(\pi x_2)$ in the definition of the constraint G, and the upper bound $b=0$, the optimal solution triple $(\bar{q},\bar{u},\bar{z})$ of control problem (2.3) is given by

$$\begin{aligned}\bar{q}(t,x_1,x_2)&\coloneqq-\frac{1}{\lambda}\big(e^{\lambda(t-1)}-1\big)\sin(\pi x_1)\sin(\pi x_2)\\ &\quad-\frac{e^{\lambda t}}{\lambda(1-\gamma)}\begin{cases}\varepsilon(1)^{1-\gamma}, & t\le\frac12,\\ \varepsilon(1)^{1-\gamma}-\varepsilon(t)^{1-\gamma}, & t>\frac12,\end{cases}\\ \bar{u}(t,x_1,x_2)&\coloneqq\hat{u}(t,x_1,x_2)-\sin(\pi x_1)\sin(\pi x_2),\\ \bar{z}(t,x_1,x_2)&\coloneqq-\bar{q}(t,x_1,x_2),\end{aligned}$$



(a) Refinement of the time steps for $N = 1089$ spatial nodes

(b) Refinement of the spatial triangulation for $M = 2048$ time steps

FIG. 8.1. *Discretization error* $\|\bar{q} - \bar{q}_\sigma\|_I$.

and the Lagrange multiplier μ associated with the state constraint is

$$\mu(t) := \begin{cases} 0, & t \leq \frac{1}{2}, \\ \varepsilon(t)^{-\gamma}, & t > \frac{1}{2}. \end{cases}$$

We are going to validate the estimates developed in the previous section by separating the discretization errors. That is, first we consider the behavior of the error for a sequence of discretizations with decreasing size of the time steps and a fixed spatial triangulation with $N = 1089$ nodes. Second, we examine the behavior of the error under refinement of the spatial triangulation for $M = 2048$ time steps.

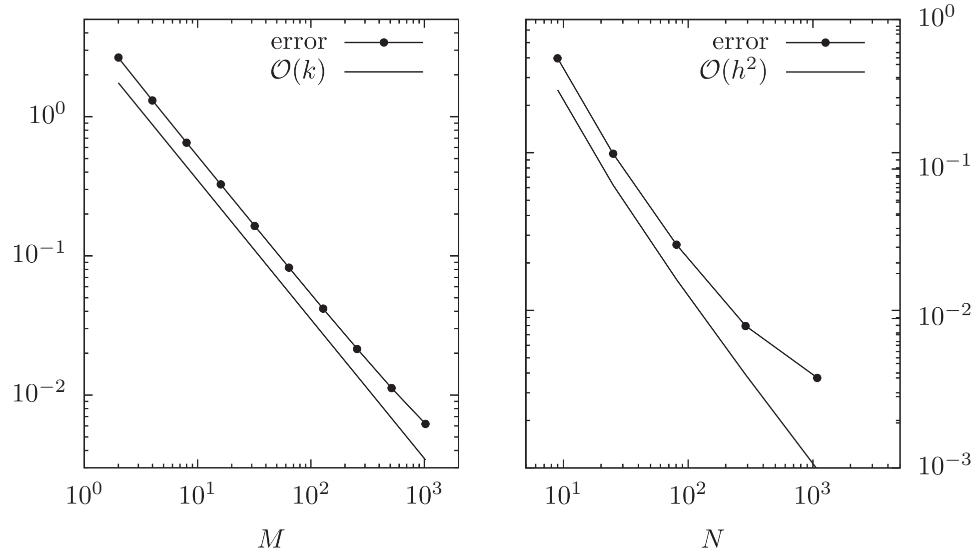
As in the theoretical part of this article, the state discretization is chosen as cG(1)dG(0). The control variable is discretized by piecewise constants on the same temporal and spatial meshes as those used for the state variable. For the following computations, we choose the free parameter γ to be 0.6.

The optimal control problems are solved by the optimization library RoDoBo [28] and the finite element toolkit Gascoigne [13] using a regularization by interior point methods (cf. [30]) for the state constrained reduced problem (3.20) and by Newton's method combined with an inner conjugate gradient method to solve the regularized problem.

Figure 8.1(a) depicts the development of the error in the control variable under refinement of the temporal step size k. Up to the spatial discretization error it exhibits at least the proven convergence order $\mathcal{O}(k^{\frac{1}{2}})$. The observed better convergence behavior of approximately $\mathcal{O}(k^{0.85})$ may be due to the constructed problem data, which do not exhibit the full irregularity covered by our analysis derived in section 6.

In Figure 8.1(b) the development of the error in the control variable under spatial refinement is shown. The expected order $\mathcal{O}(h)$ is observed.

Figure 8.2 shows the errors in the state variable for separate refinement of the time and space discretizations. Thereby, we observe convergence of order $\mathcal{O}(k + h^2)$.



(a) Refinement of the time steps for $N = 1089$ spatial nodes

(b) Refinement of the spatial triangulation for $M = 2048$ time steps

FIG. 8.2. *Discretization error* $\|\bar{u} - \bar{u}_\sigma\|_I$.

Thus, the estimate $\|\bar{u} - \bar{u}_\sigma\|_I = \mathcal{O}(k^{\frac{1}{2}} + h)$ for the error in the state variable proved in Theorem 6.1 as a by-product of the analysis for the error in the control variable seems not to be optimal.

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