Finite element discretization of state-constrained elliptic optimal control problems with semilinear state equation

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Abstract

We study a class of semilinear elliptic optimal control problems with pointwise state constraints. The purpose of this paper is twofold. First, we present convergence results for the finite element discretization of this problem class similarly to known results with finite-dimensional control space, thus extending results that are - for control functions - only available for linear-quadratic convex problems. We rely on a quadratic growth condition for the continuous problem that follows from second order sufficient conditions. Secondly, we show that the second order sufficient conditions for the continuous problem transfer to its discretized version. This is of interest for example when considering questions of local uniqueness of solutions or the convergence of solution algorithms such as the SQP method.

Keywords: optimal control, finite elements, semilinear elliptic PDE, state constraints, a priori error estimates

AMS subject classification: 49J20, 65K10, 65N15, 65N30

1 Introduction

In this paper, we are interested in the numerical analysis of nonconvex optimal control problems with pointwise state constraints governed by semi-linear elliptic equations. As a representative, we will consider the following model problem

$$\text{Minimize } J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1a}$$

subject to the semilinear elliptic PDE constraint

$$\begin{aligned} -\Delta y + d(\cdot, y) =& u \quad \text{in } \Omega, \\ y =& 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.1b}$$

as well as the pointwise state constraint

$$y(x) \geq y_c(x) \quad \text{for all } x \in \bar{\Omega}_0. \tag{1.1c}$$

We will refer to Problem (1.1) as $(\mathbb{P})$. In this setting, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded convex polygonal or polyhedral domain and $\Omega_0 \subset \Omega$ an interior subset. The precise conditions on the given quantities in $(\mathbb{P})$ will be summarized in Assumption 1 in the next section.

Finite element error estimates for state-constrained problems have been of considerable interest to the optimal control community in the recent past. Even though there are less published results than for purely control constrained problems some progress has recently been made. Plain convergence, i.e. convergence without any rates of convergence, for problems with control functions and only finitely many pointwise state constraints has been obtained by Casas in [7]. Convergence rates for situations with finitely many state constraints have later been proven by Casas and Mateos, see [11]. A broader class of perturbations for nonconvex problems like Problem $(\mathbb{P})$ including finite element discretization as well as regularization has been considered in [23]. There, convergence of the perturbed, for instance discretized, solutions has been proven based on the definition of local optimality. This ansatz, while not requiring second order sufficient optimality conditions, does not provide any rates of convergence. Most of the results on convergence rates deal with convex, linear-quadratic problems. Deckelnick and Hinze analyze a setting with variational control discretization, cf. [15]. They obtain error estimates of order $\mathcal{O}(h^{2-n/2-\varepsilon})$ in spatial dimensions $n = 2, 3$. The same order of convergence is proven by Meyer for a problem with pointwise state and control constraints and piecewise constant control discretization in [31]. Also for constant control discretization, Deckelnick and Hinze later obtained $\mathcal{O}(h|\ln h|)$ in two space dimensions and $\mathcal{O}(h^{\frac{1}{2}})$ in three space dimensions, cf. [16]. In [17, Corollary 3.3], the order $\mathcal{O}(h|\ln h|)$ is obtained for variational control discretization independently of the space dimension assuming uniform boundedness of the continuous and discrete optimal controls in

$L^\infty(\Omega)$. Just recently, Casas, Mateos, and Vexler were able to obtain the order $\mathcal{O}(h|\ln h|)$ for linear-quadratic optimal control problems in two and three space dimensions, see [12], by making use of new regularity results for the Lagrange multiplier which can be derived under additional regularity assumptions on the problem data. For state-constrained boundary control problems we refer to the results of Krumbiegel, Meyer, and Rösch in [26].

Another type of problem that has been investigated recently is a setting with finitely many control parameters rather than control functions that can vary arbitrarily in space. Merino, Tröltzsch, and Vexler considered such a nonconvex problem with only finitely many pointwise state constraints, cf. [30]. For this problem, the order $h^2|\ln h|$ could be obtained for the error in the controls. Under certain conditions, this higher convergence order is also obtained in a so called semi-infinite setting with finite dimensional control space, cf. [29, 28] for convex problems.

In this paper, we will provide error estimates for nonconvex problems with control functions. Here, we have to take care of handling function spaces rather than control parameters. In this context, we will make use of second order sufficient conditions. Second order sufficient optimality conditions play a role in many different aspects of optimal control. They are for instance used for convergence proofs of the SQP method, see e.g. [36] for control constrained problems or [20, 21] for problems with mixed pointwise control-state constraints. They also appear in the context of proving the so-called strong regularity of generalized equations, see e.g. [19] for elliptic state-constrained problems. This property can for instance be used to prove local uniqueness of local solutions, see [32] for a regularized parabolic problem, or [27] for a regularized elliptic problem. Motivated by this, the second main purpose of this paper is to provide a stability result of the second order sufficient optimality conditions with respect to the finite element discretization with sufficiently small mesh sizes. We point out similar results in [27], where stability of the SSC with respect to regularization has been proven.

2 Analysis of Problem ($\mathbb{P}$)

The purpose of this section is to summarize known analytical results for Problem ($\mathbb{P}$) that will be used in the numerical analysis. This includes in particular existence and regularity results for solutions of the state equation, as well as first and second order optimality conditions for local solutions of the optimal control problem.

2.1 General setting

Let us begin by stating assumptions on the setting of the optimal control problem, as well as laying out some general notation. Throughout, we employ the usual notation of Sobolev spaces. For convenience, we set

$$V := H_0^1(\Omega).$$

Let us note in passing that we denote by $C_0(\bar{\Omega})$ the space of functions that are continuous on $\bar{\Omega}$ and have compact support in Ω. The space $W_0^{1,\sigma}(\Omega)$ is defined analogously. By $W^{-1,\sigma'}(\Omega)$ with $1/\sigma + 1/\sigma' = 1$, we denote the dual space of $W_0^{1,\sigma}(\Omega)$. Moreover, we agree on the abbreviations

$$\|\cdot\| := \|\cdot\|_{L^2(\Omega)}, \quad (\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)},$$

$$\|\cdot\|_W := \|\cdot\|_{W^{1,\sigma}(\Omega)}, \quad \|\cdot\|_{W^*} := \|\cdot\|_{W_0^{1,\sigma}(\Omega)^*}$$

for $\sigma < n/(n-1)$, as well as

$$\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}, \quad \langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{C_0(\bar{\Omega}),C_0(\bar{\Omega})^*},$$

and point out that the space $C_0(\bar{\Omega})^*$ can be identified with the space $\mathcal{M}(\Omega)$ of regular Borel measures, see e.g. [2]. In the sequel, we will often consider L^∞-norms on interior subsets, and will denote this by an additional subscript, i.e. for $\Omega_0 \subset \Omega$, we abbreviate $\|\cdot\|_{\infty,\Omega_0} := \|\cdot\|_{L^\infty(\Omega_0)}$.

The assumptions formulated next shall be valid throughout the paper without explicit mentioning.

***Assumption* 1.**

- *Let $\Omega \subset \mathbb{R}^n, n = 2,3$, be a two- or threedimensional convex polygonal or polyhedral domain, respectively, with boundary $\Gamma := \partial\Omega$.*
- *The function $y_d \in L^2(\Omega)$, and the bound $y_c \in \mathbb{R}$ are given, fixed data.*
- *The set Ω_0 is an open inner subset of Ω, i.e.*

$$\bar{\Omega}_0 \subset \Omega, \quad dist(\Gamma, \Omega_0) > 0.$$

- *The nonlinearity $d = d(x,y)\colon \Omega \times \mathbb{R}$ is measurable with respect to $x \in \Omega$ for all fixed $y \in \mathbb{R}$, and twice continuously differentiable with respect to y, for almost all $x \in \Omega$.*
- *Moreover, for $y = 0$, d is bounded of order 2 with respect to x, i.e.*

$$\|d(\cdot,0)\|_\infty + \|d_y(\cdot,0)\|_\infty + \|d_{yy}(\cdot,0)\|_\infty \le c \tag{2.1}$$

  is satisfied.
- *Further, for almost all $x \in \Omega$, it holds that $d_y(x,y) \ge 0$.*
- *Last, the derivatives of d with respect to y up to order two are uniformly Lipschitz continuous on bounded sets, i.e. for all $M > 0$ there exists $L_M > 0$ such that d satisfies*

$$\|d_{yy}(\cdot,y_1) - d_{yy}(\cdot,y_2)\|_\infty \le L_M|y_1 - y_2| \tag{2.2}$$

  for all $y_i \in \mathbb{R}$ with $|y_i| \le M$, $i = 1,2$.

Here and throughout, we denote by d_y and d_{yy}, the partial derivatives $\partial_y d$ and $\partial_{yy} d$.

We point out that the theory presented in this paper can be extended to certain more general situations. We will comment on this in more detail in Section 6.

2.2 The control-to-state operator and its derivatives

We begin the detailed discussion of the optimal control problem by collecting solvability and regularity results for the uncontrolled state equation with given right-hand-side $f \in L^2(\Omega)$, whose weak formulation is given by

$$\text{Find } y \in V \text{ s.t. } (\nabla y, \nabla \varphi) + (d(\cdot, y), \varphi) = (f, \varphi) \quad \forall \varphi \in V. \tag{2.3}$$

The following standard result is readily available:

Theorem 1. *For every right-hand-side $f \in L^2(\Omega)$ there exists a unique solution $y \in V \cap C_0(\bar{\Omega})$ of the semilinear elliptic boundary value problem (2.3). Moreover, the solution satisfies the additional regularity $y \in H^2(\Omega)$. The estimates*

$$\|y\|_{H^2(\Omega)} \leq c\|f - d(\cdot, 0)\|$$

and

$$\|y\|_\infty \leq c\|f - d(\cdot, 0)\|$$

are satisfied for a constant $c > 0$.

Proof. Existence of $y \in V \cap C_0(\bar{\Omega})$ follows as in [6]. Noting that $d(\cdot, y) \in L^2(\Omega)$ by Assumption 1, the H^2-regularity follows after considering

$$(\nabla y, \nabla \varphi) = (f - d(\cdot, y), \varphi) \quad \forall \varphi \in V \tag{2.4}$$

and applying regularity results from [22]. □

Now, in a standard way, we use Theorem 1 to introduce the control-to-state operator

$$G : L^2(\Omega) \to V \cap C_0(\bar{\Omega}),$$

which maps a given control $u \in L^2(\Omega)$ to the weak solution of (2.3) with right-hand-side $f = u$. It is well known that G is of class C^2 with its first derivative $y_v = G'(u)v \in V$ for all $u, v \in L^2(\Omega)$ defined by

$$\text{Find } y_v \in V \text{ s.t. } (\nabla y_v, \nabla \varphi) + (d_y(\cdot, y)y_v, \varphi) = (v, \varphi) \quad \forall \varphi \in V \tag{2.5}$$

with $y = G(u)$, and second derivative $y_{v_{1,2}} = G''(u)(v_1, v_2)$ for every $u, v_1, v_2 \in L^2(\Omega)$ defined as the solution of

$$\text{Find } y_{v_{1,2}} \in V \text{ s.t. } (\nabla y_{v_{1,2}}, \nabla \varphi) + (d_y(\cdot, y)y_{v_{1,2}}, \varphi) = -(d_{yy}(\cdot, y)y_{v_1}y_{v_2}, \varphi) \quad \forall \varphi \in V, \tag{2.6}$$

where $y = G(u)$ and $y_{v_i} = G'(u)v_i$, $i = 1, 2$. For details, we refer for instance to [35].

***Remark* 1.** *We note in passing that the regularity result from Theorem 1 holds for* (2.5) *and* (2.6)*, accordingly, with the obvious modifications due to linearization. In particular, we obtain*

$$\|\nabla y_v\| + \|y_v\|_\infty \leq c\|v\|$$

as well as

$$\|\nabla y_{v_{1,2}}\| + \|y_{v_{1,2}}\|_\infty \leq c\|v_1\|\|v_2\|.$$

Here, the constant c may depend on u.

Analogously to [31], we make use of the following properties of the linearized control-to-state mapping, which hold in general Lipschitz domains.

***Theorem* 2.** *Let $\Omega \subset \mathbb{R}^{2,3}$ be a (not necessarily convex) Lipschitz domain. There exists $\underline{\sigma} < 4/3$ if $n = 2$ and $\underline{\sigma} < 3/2$ if $n = 3$, such that the linearized control-to-state operator $G'(u)$ is continuous from $W^{-1,\sigma'}(\Omega)$ to $W_0^{1,\sigma'}(\Omega)$ for all $\underline{\sigma} < \sigma < n/(n-1)$, $1/\sigma + 1/\sigma' = 1$ and for all $u \in L^2(\Omega)$.*

This follows from [25, Theorem 0.5]. We will later use this theorem to estimate the L^∞-norm of certain (auxiliary) states, making use of appropriate regularity for the optimal control that can be obtained by means of the optimality conditions for $\sigma < n/(n-1)$.

***Assumption* 2.** *In all what follows, suppose $\underline{\sigma} < \sigma < n/(n-1)$, with $\underline{\sigma}$ chosen as in Theorem 2.*

By $\sigma' > n$ and hence $W_0^{1,\sigma'}(\Omega) \hookrightarrow C_0(\bar{\Omega})$, cf. [1], we obtain:

***Corollary* 1.** *The linearized control-to-state mapping $G'(u)$ is continuous from $W^{-1,\sigma'}(\Omega)$ to $C_0(\bar{\Omega})$ for all $u \in L^2(\Omega)$.*

***Corollary* 2.** *Let $u_1, u_2 \in L^2(\Omega)$ with associated states $y_1 = G(u_1)$ and $y_2 = G(u_2)$ be given. Then there exists a constant $c > 0$ such that the control-to-state operator satisfies the following Lipschitz property:*

$$\|y_1 - y_2\|_\infty \leq c\|u_1 - u_2\|_{W^*}.$$

Proof. This is a consequence of Corollary 1, noting that $y_1 - y_2 \in V$ fulfills the linearized equation

$$(\nabla(y_1 - y_2), \nabla\varphi) + (d_y(\cdot, y_\xi)(y_1 - y_2), \varphi) = (u_1 - u_2, \varphi) \quad \forall \varphi \in V$$

with some $y_\xi = y_1 + \xi(y_2 - y_1)$, $0 < \xi < 1$. □

***Corollary* 3.** *Let $u_1, u_2 \in L^2(\Omega)$ and $v \in W^{-1,\sigma'}(\Omega)$. Furthermore, let $y_1 = G'(u_1)v$ and $y_2 = G'(u_2)v$. Then there is a constant $c > 0$ such that the linearized states fulfill*

$$\|y_1 - y_2\|_\infty \leq c\|u_1 - u_2\|_{W^*}\|v\|_{W^*}.$$

Proof. First, we observe that $w := y_1 - y_2 \in V$ satisfies

$$(\nabla w, \nabla\varphi) + (d_y(\cdot, G(u_1))w, \varphi) = ((d_y(\cdot, G(u_2)) - d_y(\cdot, G(u_1)))y_2, \varphi) \quad \forall \varphi \in V.$$

Thus, Corollary 1, the Lipschitz continuity of d_y and Corollary 2 yield the assertion. □

2.3 The optimal control problem

The control-to-state mapping G defined in the last section can now be used to obtain a reduced formulation of Problem ($\mathbb{P}$) in the usual way, i.e.

$$\text{Minimize} \quad f(u) := J(G(u), u) \text{ subject to } G(u)(x) \geq y_c \quad \forall\, x \in \bar{\Omega}_0. \qquad (\mathbb{P})$$

For ease of discussion, we introduce the set of feasible controls U_{feas} by

$$U_{\text{feas}} := \{u \in L^2(\Omega) \colon\ G(u)(x) \geq y_c \quad \forall\, x \in \bar{\Omega}_0\}.$$

The following result on existence of an optimal solution to Problem ($\mathbb{P}$) is then a simple consequence of the fact that in our setting U_{feas} is nonempty on the one hand, and the convexity of the objective function with respect to the control on the other hand.

Theorem 3. *Problem ($\mathbb{P}$) admits at least one global solution $\bar{u} \in U_{feas}$.*

Proof. Note that $\operatorname{dist}(\Gamma, \Omega_0) > 0$ holds by assumption and hence there exists a function $y_{\text{feas}} \in C_0(\bar{\Omega}) \cap H^2(\Omega)$ with $y_{\text{feas}}(x) \geq y_c$ for all x in $\bar{\Omega}_0$. Any such function defines a feasible control

$$u_{\text{feas}} := -\Delta y_{\text{feas}} + d(\cdot, y_{\text{feas}}) \in L^2(\Omega)$$

for Problem ($\mathbb{P}$). The remainder of the proof can be carried out along the lines of e.g. [35]. □

In the sequel, we will deal with local solutions due to the nonconvexity of Problem ($\mathbb{P}$). These will be considered in the sense of $L^2(\Omega)$.

Definition 1. *We call a feasible control $\bar{u} \in U_{feas}$ a local solution of Problem ($\mathbb{P}$) if there exists a constant $\rho > 0$ such that*

$$f(u) \geq f(\bar{u})$$

for all $u \in U_{feas}$ with $\|u - \bar{u}\| \leq \rho$.

2.4 Optimality condition for ($\mathbb{P}$)

In this section we summarize the first order necessary and second order sufficient optimality conditions for Problem ($\mathbb{P}$) for later use. For that purpose, let us first point out that the reduced objective function is of class C^2 due to the differentiability properties of G as well as the chain rule. For $u, v, v_1, v_2 \in L^2(\Omega)$, consider the associated states $y = G(u)$, $y_{v_i} = G'(u)v_i$, $i = 1, 2$, and $y_{v_{1,2}} = G''(u)(v_1, v_2)$. Then, the first and second order derivatives of the objective function can be expressed by

$$f'(u)v = (y - y_d, y_v) + \nu(u, v)$$

as well as

$$f''(u)(v_1, v_2) = (y - y_d, y_{v_{1,2}}) + (y_{v_1}, y_{v_2}) + \nu(v_1, v_2),$$

respectively.

For state-constrained problems, it is a standard procedure to obtain first order optimality conditions in form of a KKT system by means of a (linearized) Slater condition. Note that in contrast to problems with additional control constraints the existence of a Slater point need not be assumed but can be proven similarly to the existence of feasible controls.

***Corollary* 1.** *Let $\bar{u}$ be a local solution of Problem $(\mathbb{P})$ in the sense of Definition 1. Then $\bar{u}$ satisfies a linearized Slater condition, i.e., there exists $\gamma > 0$ and $u_\gamma \in L^2(\Omega)$, such that*

$$G(\bar{u})(x) + G'(\bar{u})(u_\gamma - \bar{u})(x) \geq y_c + \gamma \quad \forall x \in \bar{\Omega}_0$$

is satisfied.

Proof. Let $\bar{y} = G(\bar{u})$ denote the optimal state associated with $\bar{u}$, and $\tilde{y}_{\bar{u}} := G'(\bar{u})\bar{u}$ a linearized state. We choose a constant $\gamma \in \mathbb{R}^+$ and a function $\tilde{y}_{\text{slater}} \in C_0(\bar{\Omega}) \cap H^2(\Omega)$ with

$$\tilde{y}_{\text{slater}} \geq y_c + \gamma - \bar{y} + \tilde{y}_{\bar{u}} \quad \text{in } \bar{\Omega}_0.$$

Then

$$u_\gamma := -\Delta \tilde{y}_{\text{slater}} + d_y(\cdot, \bar{y})\tilde{y}_{\text{slater}} \in L^2(\Omega)$$

fulfills the required Slater point property, since it satisfies $\tilde{y}_{\text{slater}} = G'(\bar{u})u_\gamma$. We refer also to [12, Theorem 2.1]. □

Based on the linearized Slater condition, first order necessary optimality conditions for Problem $(\mathbb{P})$ can be established, which include the existence of a regular Borel measure as a Lagrange multiplier with respect to the state constraints. From the theory of Casas in [6] we obtain:

***Theorem* 4.** *Suppose that $\bar{u}$ with associated state $\bar{y}$ is a local solution of Problem $(\mathbb{P})$. Then, there exist a regular Borel measure $\bar{\mu} \in \mathcal{M}(\Omega)$ and an adjoint state $\bar{p} \in W_0^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$, such that the following optimality system is satisfied:*

$$(\nabla \bar{y}, \nabla \varphi) + (d(\cdot, \bar{y}), \varphi) = (\bar{u}, \varphi) \quad \forall \varphi \in V, \tag{2.7}$$

$$-(\bar{p}, \Delta \varphi) + (d_y(\cdot, \bar{y})\bar{p}, \varphi) = (\bar{y} - y_d, \varphi) - \langle \varphi, \bar{\mu} \rangle \quad \forall \varphi \in H^2(\Omega) \cap V, \tag{2.8}$$

$$\nu \bar{u} + \bar{p} = 0, \tag{2.9}$$

$$\langle y_c - \bar{y}, \bar{\mu} \rangle = 0, \quad \bar{y}(x) \geq y_c \quad \forall\, x \in \bar{\Omega}_0, \quad \bar{\mu} \geq 0. \tag{2.10}$$

Note that the support of the Borel measure $\bar{\mu}$ is contained in $\bar{\Omega}_0$, since the state constraints are only prescribed in this subdomain of Ω. However, extending $\bar{\mu}$ to an element of $\mathcal{M}(\Omega)$ will be convenient for notational purposes.

Before continuing with second order sufficient conditions, let us collect some observations from the optimality system that will be used in our further analysis. First, we emphasize that the adjoint state fulfills the stability estimate

$$\|p\|_W \leq c(\|\bar{y} - y_d\| + \|\bar{\mu}\|_{\mathcal{M}(\Omega)}) \leq c(\|\bar{u}\| + \|y_d\| + \|\bar{\mu}\|_{\mathcal{M}(\Omega)}), \tag{2.11}$$

cf. Theorem 1 of [4]. Next, note that the gradient equation (2.9) combined with the adjoint equation (2.8) implies uniqueness of the dual variables.

***Corollary* 4.** *The adjoint state $\bar{p}$ and the Lagrange multiplier $\bar{\mu}$ from Theorem 4 are uniquely determined.*

From the optimality system, more precisely the gradient equation (2.9) and the regularity result for the adjoint equation (2.8), we directly deduce the following higher regularity result for the optimal control:

***Corollary* 5.** *Let $\bar{u}$ be a locally optimal control of $(\mathbb{P})$ satisfying the conditions of Theorem 4. Then $\bar{u}$ admits the regularity $\bar{u} \in W_0^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$.*

With the $W^{1,\sigma}$-regularity of the optimal control, we can use the following theorem to obtain higher interior regularity of the optimal state $\bar{y}$:

***Theorem* 5.** *Let $u \in W^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$, be given, and let Ω_1 denote an interior subdomain of Ω, i.e. $\bar{\Omega}_1 \subset \Omega$. Then the state $y = G(u)$ admits the interior regularity*

$$y \in W^{2,p}(\Omega_1), \quad p = \frac{n\sigma}{n-\sigma}$$

and the a priori estimate

$$\|\nabla^2 y\|_{L^p(\Omega_1)} \leq c(\|u\|_{L^p(\Omega)} + \|d(\cdot,0)\|_{L^p(\Omega)})$$

is satisfied with a constant $c > 0$.

Proof. Note that due to Theorem 1 and the assumptions on the nonlinearity d we have $d(\cdot, y) \in L^\infty(\Omega)$. By a standard Sobolev embedding theorem, see e.g. [1], we further deduce $u \in L^p(\Omega)$ for all $p \leq (n\sigma)/(n-\sigma)$, i.e. $p < \infty$ if $n = 2$ and $p < 3$ if $n = 3$. Thus, applying regularity results from [22, Chapter 2] to the linear equation

$$y \in V: \quad (\nabla y, \nabla\varphi) = (u - d(\cdot, y), \varphi) \quad \forall \varphi \in V$$

with right-hand-side in $L^p(\Omega)$, p as above, the assertion is obtained. □

***Corollary* 6.** *Let $\bar{u} \in W^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$, be a locally optimal control of $(\mathbb{P})$ satisfying the conditions of Theorem 4 and let Ω_1 denote an interior subdomain of Ω containing the set Ω_0 where the state constraints are fulfilled, i.e.*

$$\bar{\Omega}_0 \subset \Omega_1, \quad \bar{\Omega}_1 \subset \Omega.$$

The optimal state $\bar{y} = G(\bar{u})$ admits the interior regularity

$$\bar{y} \in W^{2,p}(\Omega_1), \quad p = \frac{n\sigma}{n-\sigma}$$

and the a priori estimate

$$\|\nabla^2 \bar{y}\|_{L^p(\Omega_1)} \leq c(\|\bar{u}\|_{L^p(\Omega)} + \|d(\cdot,0)\|_{L^p(\Omega)})$$

is satisfied with a constant $c > 0$.

For later use in our convergence error estimates we require $W^{1,\sigma}$-regularity not only for a locally optimal control $\bar{u}$ but also for the so-called Slater point u_γ from Lemma 1. Let us therefore make the following

***Assumption* 3.** *The Slater point u_γ from Lemma 1 is an element of $W^{1,\sigma}(\Omega)$ for any $\sigma < n/(n-1)$.*

***Remark* 2.** *We point out that this Slater point can be approximated by an arbitrarily smooth function which is then a Slater point itself.*

In order to discuss sufficient optimality conditions, we introduce the reduced Lagrangian

$$\mathcal{L}\colon L^2(\Omega) \times \mathcal{M}(\Omega) \to \mathbb{R}, \quad \mathcal{L}(u,\mu) = f(u) + \langle y_c - G(u), \mu\rangle. \tag{2.12}$$

It is clear that due to the differentiability properties of the control-to-state mapping and the chain rule, $\mathcal{L}$ is of class C^2 with respect to u. The second derivative of the Lagrangian is given by

$$\mathcal{L}''(u,\mu)(v_1,v_2) := \frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)(v_1,v_2) = f''(u)(v_1,v_2) - \langle G''(u)(v_1,v_2), \mu\rangle. \tag{2.13}$$

With (2.13) at hand, we proceed with the formulation of the second order sufficient optimality conditions, that guarantee a control $\bar{u}$ satisfying the first order optimality conditions of Theorem 4 to be a local minimum of Problem ($\mathbb{P}$).

***Assumption* 4.** *Let $\bar{u} \in U_{feas}$ be a control satisfying the first order necessary optimality conditions from Theorem 4 with associated Lagrange multiplier $\bar{\mu}$. We assume that there exists a constant $\alpha > 0$, such that*

$$\mathcal{L}''(\bar{u},\bar{\mu})v^2 \geq \alpha\|v\|^2$$

is valid for all $v \in L^2(\Omega)$.

It is a standard result in the optimal control theory that the coercivity condition of Assumption 4 yields the quadratic growth condition for Problem ($\mathbb{P}$). This is true under even weaker conditions, cf. [9].

***Theorem* 6.** *Let $\bar{u} \in U_{feas}$ be a control satisfying the first order necessary optimality conditions from Theorem 4. Additionally, let $\bar{u}$ fulfill Assumption 4. Then there exist constants $\beta > 0$ and $\delta > 0$ such that*

$$f(u) \geq f(\bar{u}) + \beta\|u - \bar{u}\|^2 \tag{2.14}$$

for all controls $u \in U_{feas}$ with $\|u - \bar{u}\| \leq \delta$. Consequently, $\bar{u}$ is a locally optimal control of Problem ($\mathbb{P}$).

3 The Discretized Problem ($\mathbb{P}_h$)

To discretize Problem ($\mathbb{P}$), we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$ without hanging nodes, consisting of nonoverlapping triangles $T \in \mathcal{T}_h$ such that

$$\bigcup_{T \in \mathcal{T}_h} \bar{T} = \bar{\Omega}.$$

Associated with the given triangulation $\mathcal{T}_h$, we introduce the discrete state space

$$V_h = \{v_h \in C_0(\bar{\Omega}) \mid v_{h|_T} \in \mathcal{P}_1(T) \ \ \forall T \in \mathcal{T}_h\},$$

as well as the discrete control space

$$U_h = \{u_h \in C(\bar{\Omega}) \mid u_{h|_T} \in \mathcal{P}_1(T) \ \ \forall T \in \mathcal{T}_h\},$$

where $\mathcal{P}_1(T)$ denotes the set of affine real-valued functions defined on T. Note that both the controls and the states are thus discretized by piecewise linear functions, but the states have to fulfill homogeneous Dirichlet boundary conditions. For our error estimates, we will rely on usual regularity conditions for the finite element mesh. Therefore, let us introduce for each triangle $T \in \mathcal{T}_h$ the outer diameter $\rho_o(T)$ of T, and the diameter $\rho_i(T)$ of the largest circle contained in T. Moreover, we define the mesh size h by $h = \max_{T \in \mathcal{T}_h} \rho_o(T)$. Then, we make the following assumption, cf. for instance [14], which we implicitly rely on:

***Assumption* 5.** *There exist positive constants ρ_o and ρ_i such that*

$$\frac{\rho_o(T)}{\rho_i(T)} \le \rho_i \ \text{ and } \ \frac{h}{\rho_o(T)} \le \rho_o, \qquad \forall\, T \in \mathcal{T}_h,$$

are fulfilled for all $h > 0$.

3.1 The discrete control-to-state operator

We begin with a discussion of the discrete state equation. We will collect solvability results, a priori estimates, and finite element error estimates for uncontrolled equations. By means of a discrete control-to-state operator we then obtain a discrete analogue to Problem ($\mathbb{P}$), which will be discussed further.

The discretized version of the state equation (2.3) for a given $f \in L^2(\Omega)$ reads as follows:

$$\text{Find } y_h \in V_h \text{ s.t. } (\nabla y_h, \nabla \varphi_h) + (d(\cdot, y_h), \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.1}$$

***Theorem* 7.** *Let a function $f \in L^2(\Omega)$ be given and denote by $y = G(f)$ the solution of the continuous state equation* (2.3). *Then, the discrete state equation* (3.1) *admits a unique solution $y_h \in V_h$, and there exists a constant $c > 0$ independent of h such that the a priori estimate*

$$\|\nabla y_h\| \le c(\|f\| + \|d(\cdot, 0)\|) \tag{3.2}$$

and the error estimates

$$\|y_h - y\| \le ch^2(\|f\| + \|d(\cdot,0)\|), \tag{3.3}$$

$$\|y_h - y\|_\infty \le ch^{2-n/2}(\|f\| + \|d(\cdot,0)\|) \tag{3.4}$$

are satisfied for all h sufficiently small.

Proof. The proof of existence and estimate (3.2) follow by standard arguments using the monotonicity of the nonlinearity d. For the error estimate (3.3), we refer for instance to [10] . The L^∞-error estimate (3.4) then follows in a standard way applying inverse estimates. □

From the boundedness of y in $L^\infty(\Omega)$ stated in Theorem 1 combined with the L^∞-error estimate (3.4) from the last theorem, we directly obtain a uniform boundedness result for the discrete states independent of h.

Corollary 7. *Let the assumptions of Theorem 7 be satisfied. There exists a constant $c > 0$ independent of h, such that*

$$\|y_h\|_\infty \le c(\|f\| + \|d(\cdot,0)\|)$$

is satisfied.

Due to the higher interior regularity of the state functions guaranteed by Theorem 5, it is possible to derive a convergence result of higher order in the L^∞-norm, which will be crucial for our error analysis for the control functions. We will use a result for linear equations from [33] and extend the result to our semilinear PDE with techniques from [30]. We point out that under our assumptions we can write down the following specific setting of Theorem 5.1 in [33]:

Corollary 2. *Let Ω_1 be an inner subset of Ω as in Theorem 5. Moreover, let $y \in H^2(\Omega) \cap W^{2,p}(\Omega_1)$, $p \ge 2$, and $y_h \in V_h$ satisfy*

$$(\nabla(y - y_h), \nabla\varphi_h) = 0 \quad \forall \varphi_h \in V_h.$$

Then the estimate

$$\|y - y_h\|_{\infty,\Omega_0} \le ch^2(h^{-n/p}|\ln h|\|\nabla^2 y\|_{L^p(\Omega_1)} + \|\nabla^2 y\|) \tag{3.5}$$

is satisfied for all sufficiently small $h > 0$ with a constant c that is independent of h, y, y_h, and Ω_0.

Proof. Let $\tilde\Omega_1$ be a subset of Ω with $\Omega_0 \subset\subset \tilde\Omega_1 \subset\subset \Omega_1$ and let $d = \mathrm{dist}(\partial\Omega_0, \partial\tilde\Omega_1)$. Note that $\tilde\Omega_1$ can be chosen such that d is independent of h. Using Theorem 5.1 of [33] for h small enough, i.e.,

$$\|y - y_h\|_{\infty,\Omega_0} \le c\left\{|\ln h| \inf_{v\in V_h} \|y - v\|_{\infty,\tilde\Omega_1} + d^{-n/2}\|y - y_h\|\right\},$$

and employing standard estimates for the interpolation and finite element error, the desired result follows immediately. □

Theorem 8. *Denote by y the solution of the state equation* (2.3) *and by y_h the solution of the discrete state equation* (3.1) *with right-hand-side $u \in W^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$. Then there exists a constant $c > 0$ independent of h such that the interior L^∞-error estimate*

$$\|y - y_h\|_{\infty,\Omega_0} \le ch^{3-n/\sigma} |\ln h| (\|u\|_W + \|d(\cdot,0)\|_{L^p(\Omega)})$$

is satisfied.

Proof. Note that the state $y = G(u)$ satisfies

$$y \in V\colon \quad (\nabla y, \nabla \varphi_h) = (u, \varphi_h) - (d(\cdot, y), \varphi_h) \quad \forall \varphi_h \in V_h, \tag{3.6}$$

the discrete state $y_h = G_h(u)$ satisfies

$$y_h \in V_h\colon \quad (\nabla y_h, \nabla \varphi_h) = (u, \varphi_h) - (d(\cdot, y_h), \varphi_h) \quad \forall \varphi_h \in V_h, \tag{3.7}$$

and define an auxiliary discrete state z_h satisfying

$$z_h \in V_h\colon \quad (\nabla z_h, \nabla \varphi_h) = (u, \varphi_h) - (d(\cdot, y), \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.8}$$

Splitting the error into

$$\|y - y_h\|_{\infty,\Omega_0} \le \|y - z_h\|_{\infty,\Omega_0} + \|z_h - y_h\|_{\infty,\Omega_0}, \tag{3.9}$$

we can apply Lemma 2 to the first term on the right-hand-side of (3.9). With the regularity result from Theorem 5 as well as Lemma 2 we obtain

$$\|y - z_h\|_{\infty,\Omega_0} \le ch^2 (h^{-n/p} |\ln h| \|\nabla^2 y\|_{L^p(\Omega_1)} + \|\nabla^2 y\|) \tag{3.10}$$

with $p = (n\sigma)/(n-\sigma)$. Note that $\sigma \ge \underline{\sigma}$ by Assumption 2. By the a priori results from Theorems 1 and 5, combined with $3 - n/\sigma < 2$ due to $\sigma < n/(n-1)$ we further deduce

$$\begin{aligned} \|y - z_h\|_{\infty,\Omega_0} &\le c(h^{3-n/\sigma} |\ln h| (\|u\|_{L^p(\Omega)} + \|d(\cdot,0)\|_{L^p(\Omega)}) + h^2(\|u\| + \|d(\cdot,0)\|)) \\ &\le ch^{3-n/\sigma} |\ln h| (\|u\|_W + \|d(\cdot,0)\|_{L^p(\Omega)}). \end{aligned} \tag{3.11}$$

To estimate the second term in (3.9) we use a known duality argument. We define $w_h := z_h - y_h \in V_h$ and note that it fulfills

$$(\nabla w_h, \nabla \varphi_h) = (d(\cdot, y_h) - d(\cdot, y), \varphi_h) \quad \forall \varphi_h \in V_h.$$

Defining $w \in V$ as the continuous analogue of w_h satisfying

$$(\nabla w, \nabla \varphi) = (d(\cdot, y_h) - d(\cdot, y), \varphi) \quad \forall \varphi \in V, \tag{3.12}$$

we can apply Theorems 1 and 7 and obtain

$$\|z_h - y_h\|_{\infty,\Omega_0} \le \|w_h - w\|_{\infty,\Omega_0} + \|w\|_{\infty,\Omega_0} \le c\|d(\cdot, y_h) - d(\cdot, y)\|. \tag{3.13}$$

From the Lipschitz continuity of d we deduce

$$\|z_h - y_h\|_{\infty,\Omega_0} \le c\|y_h - y\|. \tag{3.14}$$

With the L^2-error estimate from Theorem 7 we finally obtain

$$\|z_h - y_h\|_{\infty,\Omega_0} \le ch^2\|\nabla^2 y\| \le ch^2(\|u\| + \|d(\cdot,0)\|), \tag{3.15}$$

where the last inequality follows from Theorem 1. Inserting (3.10) and (3.15) in (3.9) yields the assertion. □

***Remark* 3.** *For the purpose of readability, we define*

$$\delta(h,\sigma) := h^{2+n-2n/\sigma}, \quad \alpha(h,\sigma) := h^{3-n/\sigma}.$$

Simple calculations show that

$$\alpha(h,\sigma)|\ln h| = \delta(h,\sigma)h^{1-n+n/\sigma}|\ln h| \le c_\sigma\delta(h,\sigma),$$

since we have $\sigma < n/(n-1)$. We will therefore apply the last theorem in the form

$$\|y - y_h\|_{\infty,\Omega_0} \le c\delta(h,\sigma)$$

in the sequel. Moreover, the reader may easily verify for future use that

$$h^{2-n/2}\sqrt{\delta(h,\sigma)} = \alpha(h,\sigma) \le c_\sigma\delta(h,\sigma).$$

Analogously to Section 2 we are now able to define a discrete control-to-state-operator

$$G_h\colon L^2(\Omega) \to V_h,$$

which assigns a unique discrete state $y_h = G_h(u)$ to each $u \in L^2(\Omega)$. By applying the same technique as for the continuous control-to-state mapping, see e.g. [35], we can show that the mapping G_h is of class C^2. This has also been used for semilinear elliptic control-constrained problems in [8]. For $u \in L^2(\Omega)$ and an arbitrary elements $v \in L^2(\Omega)$, the first derivative $y_h^v = G_h'(u)v \in V_h$ is given by the unique solution of

$$\text{Find } y_h^v \in V_h \text{ s.t. } (y_h^v, \varphi_h) + (d_y(\cdot, y_h)y_h^v, \varphi_h) = (v, \varphi_h) \quad \forall\varphi_h \in V_h \tag{3.16}$$

with $y_h = G_h(u)$, and the second derivative $y_h^{v_{1,2}} = G_h''(u)(v_1,v_2) \in V_h$ is the unique solution of

$$\text{Find } y_h^{v_{1,2}} \in V_h \text{ s.t. } (y_h^{v_{1,2}}, \varphi_h) + (d_y(\cdot, y_h)y_h^{v_{1,2}}, \varphi_h) = -(d_{yy}(\cdot, y_h)y_h^{v_1}y_h^{v_2}, \varphi_h) \quad \forall\varphi_h \in V_h \tag{3.17}$$

with $y_h = G_h(u)$ and $y_h^{v_i} = G_h'(u)v_i$, for $v_i \in L^2(\Omega)$, $i = 1, 2$.

***Remark* 4.** *As an analogue to Remark 1 on the continuous level, we observe that the continuity result from Theorem 7 holds for* (3.16) *and* (3.17) *if modified accordingly. In particular, we obtain*

$$\|\nabla y_h^v\| \le c\|v\|$$

as well as

$$\|\nabla y_h^{v_{1,2}}\| \le c\|v_1\|\|v_2\|$$

Again, the constant c may depend on u.

3.2 Auxiliary results

Let us now provide some auxiliary results, starting with finite element error estimates for linearized state equations.

***Corollary* 3.** *Let $u, v \in L^2(\Omega)$ be given and denote by $y_v := G'(u)v$ and $y_h^v := G_h'(u)v$ the solutions of the linearized state equations* (2.5) *and* (3.16) *with right-hand-side v. Then there exists a constant $c > 0$ independent of h such that the L^∞-error estimate*

$$\|y_v - y_h^v\|_\infty \le c h^{2-n/2} \|v\|$$

is fulfilled with a constant $c > 0$ not depending on h.

Proof. We define the auxiliary function $z_v \in V$ as the unique weak solution of

$$(\nabla z_v, \nabla \varphi) + (d_y(\cdot, y_h) z_v, \varphi) = (v, \varphi) \quad \forall \varphi \in V$$

with $y_h = G_h(u)$. Recall that $y_v \in V$ fulfills the equation

$$(\nabla y_v, \nabla \varphi) + (d_y(\cdot, y) y_v, \varphi) = (v, \varphi), \quad \forall \varphi \in V$$

with $y = G(u)$, and $y_h^v \in V_h$ fulfills

$$(\nabla y_h^v, \nabla \varphi_h) + (d_y(\cdot, y_h) y_h^v, \varphi_h) = (v, \varphi_h), \quad \forall \varphi_h \in V_h.$$

We point out that $d_y(\cdot, y)$ as well as $d_y(\cdot, y_h)$ are bounded independently of h. We split the error into

$$\|y_v - y_h^v\|_\infty \le \|y_v - z_v\|_\infty + \|z_v - y_h^v\|_\infty. \tag{3.18}$$

Then, it is clear that the first term in (3.18) accounts for the linearization of d at different states y and y_h, and that the second term in (3.18) is a pure discretization error for linear equations. For the first term in (3.18), we observe that the difference $y_v - z_v \in V$ fulfills the equation

$$(\nabla(y_v - z_v), \nabla \varphi) + (d_y(\cdot, y)(y_v - z_v), \varphi) = ((d_y(\cdot, y_h) - d_y(\cdot, y)) z_v, \varphi) \quad \forall \varphi \in V,$$

which results in

$$\|y_v - z_v\|_\infty \le c \|y_h - y\| \|z_v\|_\infty \le c h^2 \|v\| \tag{3.19}$$

by Theorems 1 and 7, the Lipschitz continuity of d_y, and the boundedness of y and y_h. The estimate

$$\|z_v - y_h^v\|_\infty \le c h^{2-n/2} \|v\| \tag{3.20}$$

follows by applying Theorem 7, combined with the regularity and stability estimate for elliptic equations from Theorem 1. Combining (3.18) - (3.19) yields the assertion. □

We can also prove a higher order error estimate in the interior of Ω, analogously to Theorem 8.

***Corollary* 4.** *Let $u, v \in W^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$, be given and denote by $y_v := G'(u)v$ and $y_h^v := G_h'(u)v$ the solutions of the linearized state equations (2.5) and (3.16) with right-hand-side v. Then there exists a constant $c > 0$ independent of h such that the L^∞-error estimate*

$$\|y_v - y_h^v\|_{\infty,\Omega_0} \le ch^{3-n/\sigma}|\ln h|\|v\|_W$$

is fulfilled with a constant $c > 0$ not depending on h.

Proof. We again split the error into

$$\|y_v - y_h^v\|_{\infty,\Omega_0} \le \|y_v - z_v\|_{\infty,\Omega_0} + \|z_v - y_h^v\|_{\infty,\Omega_0},$$

where z_v is defined as in the proof of Lemma 3. As therein, we obtain

$$\|y_v - z_v\|_{\infty,\Omega_0} \le \|y_v - z_v\|_\infty \le ch^2\|v\|,$$

cf. equation (3.19). The estimate

$$\|z_v - y_h^v\|_{\infty,\Omega_0} \le ch^{3-n/\sigma}|\ln h|\|v\|_W$$

follows by applying Lemma 2 or the results from [33] combined with the regularity and stability estimate for elliptic equations from Theorem 1. Combining both estimates yields the assertion. □

***Remark* 5.** *Analogously to Remark 3, we will estimate the error estimate from the last lemma in the form*

$$\|y_v - y_h^v\|_{\infty,\Omega_0} \le c\delta(h,\sigma)$$

in the sequel.

Last, let us introduce the L^2-projection onto the space of piecewise linear functions,

$$\Pi_h\colon V \to V_h, \quad (\Pi_h v - v, w_h) = 0 \quad \forall\, w_h \in V_h,$$

and prove some auxiliary estimates based on the properties

$$\|v - \Pi_h v\| \le c\sqrt{\delta(h,\sigma)}\|v\|_W \quad \forall v \in W^{1,\sigma}(\Omega), \quad \sigma < n/(n-1), \tag{3.21}$$

$$\|v - \Pi_h v\|_{W^*} \le c\delta(h,\sigma)\|v\|_W \quad \forall v \in W^{1,\sigma}(\Omega), \quad \sigma < n/(n-1), \tag{3.22}$$

cf. [31, Lemma 4].

***Corollary* 5.** *Let $\underline{\sigma} < \sigma < n/(n-1)$ as in Theorem 2 and $\tilde{u} \in L^2(\Omega)$ as well as $u, v \in W^{1,\sigma}(\Omega)$ be given. Then, the following estimates are satisfied with a constant $c > 0$ independent of h:*

$$\|G_h'(\tilde{u})(v - \Pi_h v)\|_\infty \le c\delta(h,\sigma)\|v\|_W, \tag{3.23}$$

$$\|G_h(\Pi_h u) - G_h(u)\|_\infty \le c\delta(h,\sigma)\|u\|_W, \tag{3.24}$$

$$\|G_h'(\Pi_h u)\Pi_h v - G'(u)v\|_{\infty,\Omega_0} \le c\delta(h,\sigma)\|v\|_W. \tag{3.25}$$

Proof. We begin by proving the first estimate analogously to the linear-quadratic setting with smooth boundary in [31, Lemma 5]. We split

$$\|G_h'(\tilde{u})(v-\Pi_h v)\|_\infty \le \|(G_h'(\tilde{u})-G'(\tilde{u}))(v-\Pi_h v)\|_\infty \\ + \|G'(\tilde{u})(v-\Pi_h v)\|_\infty. \quad (3.26)$$

The first term in (3.26) can be estimated by

$$\|(G_h'(\tilde{u})-G'(\tilde{u}))(v-\Pi_h v)\|_\infty \le ch^{2-n/2}\|v-\Pi_h v\| \le c\delta(h,\sigma)\|v\|_W$$

by Lemma 3 and the projection error estimate (3.21), combined with the calculations from Remark 3. For the second term in (3.26) we have

$$\|G'(\tilde{u})(v-\Pi_h v)\|_\infty \le c\|v-\Pi_h v\|_{W^*} \le c\delta(h,\sigma)\|v\|_W$$

by Corollary 1 and the projection error estimate (3.22). Thus, (3.23) is proven. Estimate (3.24) is a direct consequence of (3.23) since

$$\|G_h(\Pi_h u)-G_h(u)\|_\infty \le c\|G_h'(\tilde{u})(u-\Pi_h u)\|_\infty \quad (3.27)$$

with some $\tilde{u} = u+\xi(\Pi_h u-u)$, $0<\xi<1$, which is bounded in $L^2(\Omega)$ independent of h. For proving the last auxiliary estimate we observe

$$\|G_h'(\Pi_h u)\Pi_h v - G'(u)v\|_{\infty,\Omega_0} \le \|(G_h'(\Pi_h u)(\Pi_h v - v)\|_{\infty,\Omega_0} \\ + \|((G_h'(\Pi_h u)-G'(\Pi_h u))v\|_{\infty,\Omega_0} \\ + \|(G'(\Pi_h u)-G'(u))v\|_{\infty,\Omega_0}.$$

Estimate (3.23), Lemma 4, as well as Corollary 3 combined with estimate (3.22) yield the assertion. □

3.3 The discrete reduced optimal control problem

With the discrete reduced objective function

$$f_h\colon L^2(\Omega)\to\mathbb{R}, \quad f_h(u) := J(G_h(u),u),$$

we formulate the discrete problem in a convenient, reduced way:

$$\text{Minimize } f_h(u_h) \text{ subject to } u_h\in U_h, \quad G_h(u)(x)\ge y_c \quad \forall x\in\bar{\Omega}_0. \quad (\mathbb{P}_h)$$

Note that the state constraints are still prescribed in infinitely many points. Due to the linear discretization of the states and the constant bounds this can be achieved by prescribing the constraints in the nodes of all triangles or tetrahedrons that are at least partially contained in Ω_0. As for the continuous Problem ($\mathbb{P}$), we therefore introduce the notation of feasible controls and local solutions to ($\mathbb{P}_h$).

Definition 2. *A control $u_h\in U_h$ is called feasible if the associated state $y_h = G_h(u)$ fulfills the state constraints $y_h(x)\ge y_c$ in $\bar{\Omega}_0$. The set of all feasible discrete controls will be denoted by $U_{h,feas}$.*

***Definition* 3.** *A feasible control $\bar{u}_h \in U_{h,feas}$ is called a local solution of $(\mathbb{P}_h)$, if there exists a positive real number ρ such that*

$$f_h(\bar{u}_h) \leq f_h(u_h)$$

holds for all feasible controls $u_h \in U_{h,feas}$ of $(\mathbb{P}_h)$ with $\|u_h - \bar{u}_h\| \leq \rho$.

We will now not directly discuss Problem $(\mathbb{P}_h)$ analogously to the continuous setting, but introduce an auxiliary problem which we will use to prove convergence and thus indirectly obtain existence and convergence results on local solutions of $(\mathbb{P}_h)$, as well as first order necessary optimality conditions. Second order sufficient conditions will be discussed based on these convergence results.

For completeness and later use, let us at this point only mention that by the differentiability properties of G_h, we obviously also have differentiability of f_h up to order two, with the first and second order derivatives of f_h being given by

$$f_h'(u)v = (y_h - y_d, y_h^v) + \nu(u, v) \tag{3.28}$$

as well as

$$f_h''(u)(v_1, v_2) = (y_h^{v_{1,2}}, y_h - y_d) + (y_h^{v_1}, y_h^{v_2}) + \nu(v_1, v_2), \tag{3.29}$$

where for any $u, v, v_1, v_2 \in L^2(\Omega)$ we use again the notation $y_h := G_h(u)$, $y_h^v := G_h'(u)v$, as well as $y_h^{v_i} := G_h'(u)v_i$, $i = 1, 2$, and $y_h^{v_{1,2}} := G_h''(u)(v_1, v_2)$.

4 Convergence Analysis

We will now prove our convergence result for the discrete optimal controls. In the linear-quadratic setting, the Slater point u_γ has been used to construct auxiliary feasible controls that were used as test functions in the variational inequalities for the continuous and the discrete optimal control, cf. [18] or [31]. Now, we construct feasible auxiliary functions and use arguments involving the quadratic growth condition in the neighborhood of $\bar{u}$, cf. also [34].

To adequately deal with local solutions we apply a meanwhile well-known localization argument from [13]. For a given locally optimal control $\bar{u} \in L^2(\Omega)$ of Problem $(\mathbb{P})$ satisfying the first order necessary condition of Theorem 4 and the second order sufficient condition of Assumption (4), let the set

$$U^r := \{u \in L^2(\Omega) \colon \|u - \bar{u}\| \leq r\}$$

be given, with $r > 0$ small enough such that the quadratic growth condition (6) is satisfied for all $u \in U^r$. Moreover, we define

$$U^r_{\text{feas}} := \{u \in U^r \colon \; G(u)(x) \geq y_c \quad \forall\, x \in \bar{\Omega}_0\}.$$

Then, consider the discrete auxiliary sets

$$U_h^r := U^r \cap U_h,$$

as well as

$$U^r_{h,\text{feas}} := \{u_h \in U^r_h \colon\ G_h(u) \geq y_c \quad \forall\, x \in \bar{\Omega}_0\},$$

and the auxiliary problem

$$\text{Minimize } f_h(u_h) \text{ subject to } u_h \in U^r_{h,\text{feas}}. \tag{$\mathbb{P}^r_h$}$$

We proceed as follows:

- We prove that Problem ($\mathbb{P}^r_h$) admits at least one global solution.
- For any such solution, we prove - in a first step - convergence (of low order) by means of the so-called two-way-feasibility.
- The results obtained for the auxiliary Problem ($\mathbb{P}^r_h$) are then transferred to the discrete Problem ($\mathbb{P}_h$). Optimality conditions and higher regularity are developed, and the order of convergence is improved.

***Remark* 6.** *At this point, we would like to mention that due to the boundedness of $U^r_{h,feas}$ any constant in e.g. the a priori estimates that may depend on the L^2-norm of u can in fact be estimated by an upper bound independent of the control.*

4.1 Auxiliary results

Let us first show that we can safely assume that the Slater point u_γ lies in the $r/2$-neighborhood of $\bar{u}$. Indeed, choosing

$$u^r_\gamma = \bar{u} + t(u_\gamma - \bar{u}), \quad t = \min\left\{1, \frac{r}{2\|u_\gamma - \bar{u}\|}\right\},$$

fulfills this closeness condition, and the Slater point property

$$\begin{aligned} G(\bar{u}) + G'(\bar{u})(u^r_\gamma - \bar{u}) =&(1-t)G(\bar{u}) + t(G(\bar{u}) + G'(\bar{u})(u_\gamma - \bar{u})) \\ \geq&(1-t)y_c + t(y_c + \gamma) \geq y_c + \gamma_r \end{aligned} \tag{4.1}$$

is fulfilled in $\bar{\Omega}_0$ with a distance parameter $\gamma_r = t\gamma$. For h small enough, this distance estimate ensures that the L^2-projection $\Pi_h u_\gamma$ lies in an r-neighborhood of $\bar{u}$ as well. For later purposes, we point out that γ_r depends only linearly on r. Consequently, it is reasonable to formulate the following assumption, which we rely on without explicit further notice.

***Assumption* 6.** *Let $r > 0$ be small enough such that the quadratic growth condition from Theorem 6 is fulfilled for all $u \in U_{feas} \cap U^r$. Suppose the Slater point u_γ from Lemma 1 fulfills the regularity condition from Assumption 3 as well as the distance estimate*

$$\|u_\gamma - \bar{u}\| \leq \frac{r}{2}.$$

We provide some auxiliary results, following ideas from the linear-quadratic setting in [31], which we extend to the nonlinear case by e.g. Taylor-type arguments. The first result is in essence also used in [23] for proving plain convergence of perturbed solutions.

***Corollary* 6.** *Let $\bar{u}$ be a locally optimal control of Problem $(\mathbb{P})$, satisfying the first order optimality conditions of Theorem 4, and let u_γ be the Slater point from Assumption 6. There exists a sequence $\{u_{t(h)}\}_{t(h)>0}$ of controls that are feasible for $(\mathbb{P}_h^r)$ for h and r sufficiently small, and that converge to $\bar{u}$ strongly in $W^{-1,\sigma'}(\Omega)$ with order $\mathcal{O}(\delta(h,\sigma))$, as h tends to zero.*

Proof. Consider

$$u_t := \Pi_h\bar{u} + t(\Pi_h u_\gamma - \Pi_h\bar{u})$$

with $t = t(h)$ tending to zero as h tends to zero. Obviously, $\{u_{t(h)}\}_{t(h)}$ converges to $\bar{u}$ as h tends to zero, and the order of convergence is defined by $t = t(h)$ and the projection error $\|\Pi_h\bar{u} - \bar{u}\|_{W^*} \le c\delta(h,\sigma)$, cf. (3.22). By the properties of Π_h, it is also clear that $\|\bar{u} - u_t\| \le r$ if h is sufficiently small, thus $u_t \in U_h^r$. To prove feasibility of u_t for $(\mathbb{P}_h^r)$, we proceed as follows:

$$\begin{aligned} G_h(u_t) = &\underbrace{(1-t)G(\bar{u}) + t(G(\bar{u}) + G'(\bar{u})(u_\gamma - \bar{u}))}_{\text{(I)}} \\ &+ \underbrace{G_h(\Pi_h\bar{u}) - G_h(\bar{u})}_{\text{(II)}} + \underbrace{G_h(\bar{u}) - G(\bar{u})}_{\text{(III)}} \\ &+ t\underbrace{(G_h'(\Pi_h\bar{u})(\Pi_h u_\gamma - \Pi_h\bar{u}) - G'(\bar{u})(u_\gamma - \bar{u}))}_{\text{(IV)}} \\ &+ \frac{1}{2}\underbrace{G_h''(u_\xi)(u_t - \Pi_h\bar{u})^2}_{\text{(V)}}, \end{aligned} \tag{4.2}$$

where we applied Taylor expansion to G_h with some

$$u_\xi = \Pi_h\bar{u} + \xi(u_t - \Pi_h\bar{u}) = \Pi_h\bar{u} + t\xi(\Pi_h u_\gamma - \Pi_h\bar{u}), \quad 0 < \xi < 1.$$

We consider the first term, (I), of (4.2) in $\bar{\Omega}_0$, and obtain

$$(1-t)G(\bar{u}) + t(G(\bar{u}) + G'(\bar{u})(u_\gamma - \bar{u})) \ge (1-t)y_c + t(y_c + \gamma) = y_c + t\gamma \tag{4.3}$$

by the feasibility of $\bar{y} = G(\bar{u})$ for $(\mathbb{P})$, and the Slater point property of u_γ. The second term, (II), can be estimated with the help of Lemma 5. We obtain

$$\|G_h(\Pi_h\bar{u}) - G_h(\bar{u})\|_{\infty,\Omega_0} \le c\delta(h,\sigma)\|\bar{u}\|_W. \tag{4.4}$$

The third term, (III), is a finite element discretization error in Ω_0, where we can apply Theorem 8. This yields

$$\|G_h(\bar{u}) - G(\bar{u})\|_{\infty,\Omega_0} \le c\delta(h,\sigma)\|\bar{u}\|_W. \tag{4.5}$$

Applying Lemma 5 to (IV) yields

$$\|G_h'(\Pi_h\bar u)(\Pi_h u_\gamma - \Pi_h \bar u) - G'(\bar u)(u_\gamma - \bar u)\|_{\infty,\Omega_0} \le c\delta(h,\sigma)(\|\bar u\|_W + \|u_\gamma\|_W). \quad (4.6)$$

Finally, for the last term, we observe

$$\begin{aligned}\|G_h''(u_\xi)(u_t - \Pi_h\bar u)^2\|_{\infty,\Omega_0} \le \|G_h''(u_\xi)(u_t - \Pi_h\bar u)^2\|_\infty &\le c\|u_t - \Pi_h \bar u\|^2\\ &\le ct^2\|\Pi_h u_\gamma - \Pi_h \bar u\|^2\\ &\le ct^2 r^2,\end{aligned} \quad (4.7)$$

since we have $\Pi_h u_\gamma, \Pi_h \bar u \in U_h^r$ for all h sufficiently small due to property (3.21). Collecting all estimates and inserting them in (4.2), we obtain

$$G_h(u_t) \ge y_c - c_1\delta(h,\sigma) + t(\gamma - c_2\delta(h,\sigma) - c_3 t r^2) \quad (4.8)$$

in $\bar\Omega_0$. Note again that γ may depend linearly on r by (4.1). Still, choosing

$$t(h) = \frac{c_1\delta(h,\sigma)}{\gamma - c_2\delta(h,\sigma) - c_3 r^2},$$

we have $0 < t < 1$ for r and h small and obtain

$$G_h(u_t) \ge y_c \text{ in } \bar\Omega_0,$$

and obviously $t(h) = \mathcal{O}(\delta(h,\sigma))$. □

As a side effect of Lemma 6, we can deduce a solvability result for the discrete auxiliary problem $(\mathbb{P}_h^r)$.

Corollary 7. *Let $\bar u$ denote a locally optimal control of Problem $(\mathbb{P})$, and let $u_\gamma \in W^{1,\sigma}(\Omega)$ be a Slater point fulfilling Assumption 3. Then there exists at least one globally optimal control $\bar u_h^r \in U_{h,feas}^r$ with associated discrete optimal state $\bar y_h^r = G_h(\bar u_h^r)$ for Problem $(\mathbb{P}_h^r)$ for all r sufficiently small.*

Proof. Existence of solutions follows by standard arguments, since Lemma 6 guarantees that the set $U_{h,\text{feas}}^r$ is not empty. □

With the existence of an optimal control to Problem $(\mathbb{P}_h^r)$ verified, the next step towards an error estimate is the construction of a continuous analogue to u_t, i.e. an auxiliary control sequence $\{u_{\tau(h)}\}_{\tau(h)} \subset U_{\text{feas}}^r$ that converges to $\bar u_h^r$. To obtain that, we first prove that the projection of the Slater point u_γ from Assumption 1 is also a Slater point for the discrete problem.

Corollary 8. *For all sufficiently small $r, h > 0$ the Slater point u_γ from Assumption 3 satisfies*

$$G_h(\bar u_h^r) + G_h'(\bar u_h^r)(\Pi_h u_\gamma - \bar u_h^r) \ge y_c + \frac{\gamma}{2} \quad \text{in } \bar\Omega_0.$$

Proof. We proceed similarly to the proof of Lemma 6. In $\bar{\Omega}_0$, we observe

$$
\begin{aligned}
G_h(\bar{u}_h^r)+&G_h'(\bar{u}_h^r)(\Pi_h u_\gamma - \bar{u}_h^r) \\
&= \underbrace{G(\bar{u}) + G'(\bar{u})(u_\gamma - \bar{u})}_{\text{(I)}} \\
&\quad + \underbrace{G_h(\bar{u}_h^r) - G(\bar{u}_h^r) + (G_h'(\bar{u}_h^r) - G'(\bar{u}_h^r))(\Pi_h u_\gamma - \bar{u}_h^r)}_{\text{(II)}} \\
&\quad + \underbrace{G(\bar{u}_h^r) - G(\bar{u}) - G'(\bar{u})(\bar{u}_h^r - \bar{u})}_{\text{(III)}} \\
&\quad + \underbrace{(G'(\bar{u}_h^r) - G'(\bar{u}))(\Pi_h u_\gamma - \bar{u}_h^r) + G'(\bar{u})(\Pi_h u_\gamma - u_\gamma)}_{\text{(IV)}} .
\end{aligned} \tag{4.9}
$$

Now, we can estimate (I) by means of the Slater point property for the continuous problem from Lemma 1,

$$G(\bar{u}) + G'(\bar{u})(u_\gamma - \bar{u}) \geq y_c + \gamma, \tag{4.10}$$

the second term, (II), by the L^∞-error estimates from Theorem 7 and Lemma 3,

$$
\begin{aligned}
G_h(\bar{u}_h^r) - G(\bar{u}_h^r) + (G_h'(\bar{u}_h^r) - G'(\bar{u}_h^r))(\Pi_h u_\gamma - \bar{u}_h^r)& \\
\geq -ch^{2-n/2}\|\bar{u}_h^r\| - h^{2-n/2}\|\Pi_h u_\gamma - \bar{u}_h^r\|) \geq -c_1 h^{2-n/2}.&
\end{aligned} \tag{4.11}
$$

Here, we point out that $\|\bar{u}_h^r\|$ and $\|\Pi_h u_\gamma - \bar{u}_h^r\|$ are clearly bounded independent of h for all h sufficiently small. The third term, (III), is estimated by usual elliptic regularity results, i.e.

$$G(\bar{u}_h^r) - G(\bar{u}) - G'(\bar{u})(\bar{u}_h^r - \bar{u}) = \frac{1}{2}G''(u_\xi)(\bar{u}_h^r - \bar{u})^2 \geq -c_2 r^2, \tag{4.12}$$

which follows from Remark 1 by Taylor expansion with $u_\xi = \bar{u} + \xi(\bar{u}_h^r - \bar{u})$ for some $\xi \in (0,1)$. Finally, for (IV), we observe

$$(G'(\bar{u}_h^r) - G'(\bar{u}))(\Pi_h u_\gamma - \bar{u}_h^r) \geq -c_3\|\bar{u}_h^r - \bar{u}\|\|\Pi_h u_\gamma - \bar{u}_h^r\| \geq -c_3 r^2 \tag{4.13}$$

according to Corollary 3 and embeddings in classical Sobolev spaces, as well as

$$G'(\bar{u})(\Pi_h u_\gamma - u_\gamma) \geq -c_4\|\Pi_h u_\gamma - u_\gamma\|_{W*} \geq -c_4\delta(h,\sigma)\|u_\gamma\|_W \tag{4.14}$$

by means of Corollary 1 and (3.22). Insertion of (4.10)-(4.14) into (4.9) yields

$$G_h(\bar{u}_h^r) + G_h'(\bar{u}_h^r)(\Pi_h u_\gamma - \bar{u}_h^r) \geq y_c + \frac{\gamma}{2} \tag{4.15}$$

for r, h sufficiently small. Again, we point out that γ may depend on r, but only linearly. □

We can now proceed to construct the auxiliary sequence $\{u_{\tau(h)}\}_{\tau(h)>0}$, which is feasible for Problem $(\mathbb{P})$, but close to the discrete solution $\bar{u}_h^r$. In contrast to Lemma 6, we do not yet obtain the order $\mathcal{O}(\delta(h,\sigma))$, since uniform boundedness of $\bar{u}_h^r \in W^{1,\sigma}(\Omega)$ is not yet guaranteed.

Corollary 9. *Let $r > 0$ be given sufficiently small, let $\bar{u}$ be a locally optimal control of $(\mathbb{P})$, and let $\bar{u}_h^r$ be any globally optimal control of $(\mathbb{P}_h^r)$. Moreover, let u_γ be the Slater point from Assumption 3. There exists a sequence $\{u_{\tau(h)}\}_{\tau(h)>0}$ of controls that are feasible for $(\mathbb{P})$ and that converge to $\bar{u}_h^r$ strongly in $L^2(\Omega)$ with order $\mathcal{O}(h^{2-n/2})$ as h tends to zero.*

Proof. The existence of $\{u_{\tau(h)}\}_{\tau(h)>0}$ follows similar to Lemma 6. Consider

$$u_\tau := \bar{u}_h^r + \tau(u_\gamma - \bar{u}_h^r)$$

with $\tau = \tau(h)$ tending to zero as h tends to zero. Obviously, u_τ converges to $\bar{u}_h^r$ as h tends to zero, and the order of convergence is determined by $\tau(h)$. To prove feasibility of u_τ for $(\mathbb{P})$, note that in $\bar{\Omega}_0$, we observe

$$\begin{aligned} G(u_\tau) = &\underbrace{(1-\tau)G_h(\bar{u}_h^r) + \tau(G_h(\bar{u}_h^r) + G_h'(\bar{u}_h^r)(\Pi_h u_\gamma - \bar{u}_h^r))}_{\text{(I)}} \\ &+ \underbrace{G(u_\tau) - G_h(u_\tau)}_{\text{(II)}} + \tau \underbrace{G_h'(\bar{u}_h^r)(u_\gamma - \Pi_h u_\gamma)}_{\text{(III)}} \\ &+ \frac{1}{2}\underbrace{G_h''(u_\xi)(u_\tau - \bar{u}_h^r)^2}_{\text{(IV)}} \end{aligned} \tag{4.16}$$

which follows by Taylor expansion of $G_h(u_\tau)$ at $\bar{u}_h^r$ with a $u_\xi = \bar{u}_h^r + \xi(u_\gamma - \bar{u}_h^r)$, $\xi \in (0,1)$. The first term, (I), can be estimated by means of feasibility and the Slater point property as

$$(1-\tau)G_h(\bar{u}_h^r) + \tau(G_h(\bar{u}_h^r) + G_h'(\bar{u}_h^r)(\Pi_h u_\gamma - \bar{u}_h^r)) \geq y_c + \tau\frac{\gamma}{2}$$

by Lemma 8. To estimate (II), we apply Theorem 7 and obtain

$$G(u_\tau) - G_h(u_\tau) \geq -c_1 h^{2-n/2}. \tag{4.17}$$

The third term is estimated by means of Lemma 5. This yields

$$G_h'(\bar{u}_h^r)(u_\gamma - \Pi_h u_\gamma) \geq -c_2\delta(h,\sigma).$$

Finally, (IV) is estimated by

$$G_h''(u_\xi)(u_\tau - \bar{u}_h^r)^2 \geq -c\|u_\tau - \bar{u}_h^r\|^2 = -c\tau^2\|u_\gamma - \bar{u}_h^r\|^2 \geq -c_3\tau^2 r^2.$$

Collecting all estimates yields

$$G(u_\tau) \geq y_c - c_1 h^{2-n/2} + \tau(\gamma/2 - c_2\delta(h,\sigma) - c_3\tau r^2). \tag{4.18}$$

Choosing

$$\tau = \tau(h) = \frac{c_1 h^{2-n/2}}{\frac{\gamma}{2} - c_2\delta(h,\sigma) - c_3 r^2} \tag{4.19}$$

yields $0 < \tau < 1$ for h, r sufficiently small, as well as

$$G(u_\tau) \geq y_c \quad \text{in } \bar{\Omega}_0.$$

Clearly, $\tau(h) = \mathcal{O}(h^{2-n/2})$, as $h \to 0$. □

4.2 Convergence result and error estimate

With the results of the last subsection, we can now state a convergence result for the auxiliary discrete problem.

***Corollary* 10.** *Let $\bar{u}$ be a local solution of Problem $(\mathbb{P})$ satisfying the quadratic growth condition (2.14), and let $\{h\}_{>0}$ be an arbitrary sequence of positive mesh sizes converging to zero. Moreover, let $\{\bar{u}_h^r\}_{h>0}$ be any sequence of globally optimal controls for $(\mathbb{P}_h^r)$ with $r > 0$ fixed and small enough such that the quadratic growth condition (6) as well as Lemmas 6, 8, and 9 hold. Then the sequence $\{\bar{u}_h^r\}_{h>0}$ converges strongly in $L^2(\Omega)$ to $\bar{u}$ with order $\mathcal{O}(h^{1-n/4})$.*

Proof. The proof resembles the one in [34] for finitely many state constraints. Let h be a sequence of positive mesh sizes converging to zero. Let $u_t := u_{t(h)} \in U_{h,\text{feas}}^r$ and $u_\tau := u_{\tau(h)} \in U_{\text{feas}}^r$ be the controls from Lemmas 6 and 9, respectively. We split the error

$$\|\bar{u} - \bar{u}_h^r\| \leq \|\bar{u} - u_\tau\| + \|u_\tau - \bar{u}_h^r\| = \underbrace{\|\bar{u} - u_\tau\|}_{\text{(I)}} + \underbrace{\tau(h)\|u_\gamma - \bar{u}_h^r\|}_{\text{(II)}}. \tag{4.20}$$

Clearly, by Lemma 9 the second term in (4.20) converges to zero as h tends to zero, since it is easily estimated by

$$\tau(h)\|u_\gamma - \bar{u}_h^r\| \leq ch^{2-n/2} \tag{4.21}$$

due to $\tau(h) = \mathcal{O}(h^{2-n/2})$. To estimate the first term, we point out that u_τ is feasible for Problem $(\mathbb{P})$, and we may apply the quadratic growth condition (2.14). We obtain

$$\begin{aligned} \beta\|\bar{u} - u_\tau\|^2 &\leq f(u_\tau) - f(\bar{u}) \\ &\leq f(u_\tau) - f_h(\bar{u}_h^r) + f_h(\bar{u}_h^r) - f_h(u_t) + f_h(u_t) - f(\bar{u}) \\ &\leq f(u_\tau) - f_h(\bar{u}_h^r) + f_h(u_t) - f(\bar{u}), \end{aligned} \tag{4.22}$$

where the last inequality follows from the fact that u_t is feasible and $\bar{u}_h^r$ is globally optimal for $(\mathbb{P}_h^r)$. We continue by estimating the term $f(u_\tau) - f_h(\bar{u}_h^r)$ by direct calculations. We observe

$$f(u_\tau) - f_h(\bar{u}_h^r) \leq \frac{1}{2}\|G(u_\tau) + G_h(\bar{u}_h^r) - 2y_d\|\|G(u_\tau) - G_h(\bar{u}_h^r))\| + \frac{\nu}{2}\|u_\tau + \bar{u}_h^r\|\|u_\tau - \bar{u}_h^r\|. \tag{4.23}$$

By the fact that both $u_\tau, \bar{u}_h^r \in U^r$ combined with the a priori results for G and G_h from Theorems 1 and 7, we obtain, applying the triangle inequality,

$$f(u_\tau) - f_h(\bar{u}_h^r) \leq c\left(\|G(u_\tau) - G(\bar{u}_h^r)\| + \|G(\bar{u}_h^r) - G_h(\bar{u}_h^r)\| + \|u_\tau - \bar{u}_h^r\|\right). \tag{4.24}$$

By well-known Lipschitz properties of G with respect to the L^2-norm, we obtain

$$f(u_\tau) - f_h(\bar{u}_h^r) \leq c\left(\|G(\bar{u}_h^r) - G_h(\bar{u}_h^r)\| + \|u_\tau - \bar{u}_h^r\|\right). \tag{4.25}$$

By Lemma 9 and the finite element error estimate from Theorem 7 in $L^2(\Omega)$, we finally obtain

$$f(u_\tau) - f_h(\bar{u}_h^r) \leq c(h^2 + h^{2-n/2}) \leq ch^{2-n/2} \tag{4.26}$$

as $h \to 0$. The term $f_h(u_t) - f(\bar{u})$ is estimated similarly. In anticipation of being able to prove $\tau(h) = \mathcal{O}(\delta(h,\sigma))$ in Lemma 9, we make use of a duality pairing between $W^{1,\sigma}(\Omega)$ and $W^{-1,\sigma'}(\Omega)$ in some estimates instead of using L^2-norms, which eventually allows to obtain higher order estimates. We point out that $u_\gamma, \bar{u} \in W^{1,\sigma}(\Omega)$, and hence $\Pi_h u_\gamma, \Pi_h \bar{u}$ are bounded in $W^{-1,\sigma'}(\Omega)$ independent of h due to (3.22). By direct calculations, we observe:

$$f_h(u_t) - f(\bar{u}) \leq \frac{1}{2}\|G_h(u_t) + G(\bar{u}) - 2y_d\|\|G_h(u_t) - G(\bar{u})\| + \frac{\nu}{2}\|u_t + \bar{u}\|_W \|u_t - \bar{u}\|_{W^*}. \tag{4.27}$$

Analogously to (4.24) and (4.25), having regard to Corollary 2, we obtain

$$f_h(u_t) - f(\bar{u}) \leq c\left(\|G_h(u_t) - G(u_t)\| + \|u_t - \bar{u}\|_{W^*}\right). \tag{4.28}$$

Applying Theorem 7 and Lemma 6, we deduce

$$f_h(u_t) - f(\bar{u}) \leq c(h^2 + \delta(h,\sigma)) \leq c\delta(h,\sigma). \tag{4.29}$$

Inserting (4.26) and (4.29) into (4.22) yields with $\beta > 0$, and after taking the square root:

$$\|\bar{u} - u_\tau\| \leq ch^{1-n/4}. \tag{4.30}$$

This and (4.21), inserted in (4.20), yield

$$\|\bar{u} - \bar{u}_h^r\| \leq ch^{1-n/4}$$

as h tends to zero. □

Theorem 9. *Let $\bar{u}$ be a local solution of Problem ($\mathbb{P}$) satisfying the quadratic growth condition (2.14). Moreover, let $\{h\}_{>0}$ be an arbitrary sequence of positive mesh sizes converging to zero. Then there exists a sequence $\{\bar{u}_h\}$ of local solutions of Problem ($\mathbb{P}_h$) such that $\bar{u}_h$ converges strongly in $L^2(\Omega)$ to $\bar{u}$.*

Moreover, there exist a regular Borel measure $\bar{\mu}_h \in \mathcal{M}(\Omega)$ and an adjoint state $\bar{p}_h \in V_h$, such that with $\bar{y}_h := G_h(\bar{u}_h)$ the following optimality system is satisfied:

$$(\nabla \bar{y}_h, \nabla \varphi_h) + (d(\cdot, \bar{y}_h), \varphi_h) = (\bar{u}_h, \varphi_h) \quad \forall \varphi_h \in V_h, \tag{4.31}$$

$$(\nabla \bar{p}_h, \nabla \varphi_h) + (d_y(\cdot, \bar{y}_h)\bar{p}_h, \varphi_h) = (\bar{y}_h - y_d, \varphi_h) - \langle \varphi_h, \bar{\mu}_h \rangle \quad \forall \varphi_h \in V_h, \quad (4.32)$$

$$\nu \bar{u}_h + \bar{p}_h = 0, \quad (4.33)$$

$$\langle y_c - \bar{y}_h, \bar{\mu}_h \rangle = 0, \quad \bar{y}_h(x) \geq y_c \quad \forall\, x \in \bar{\Omega}_0, \quad \bar{\mu}_h \geq 0. \quad (4.34)$$

Proof. The existence of the sequence $\{\bar{u}_h\}$ follows directly from Lemma 10, noting that global solutions of Problem $(\mathbb{P}_h^r)$ are local solutions of Problem $(\mathbb{P}_h)$, since due to the convergence result of Lemma 10 the constraint $\|\bar{u}_h^r - \bar{u}\| \leq r$ is not active for sufficiently small $h > 0$. Then, the optimality conditions for Problem $(\mathbb{P}_h)$ can be formulated analogously to the continuous problem, since the existence of a discrete Slater point has been verified in Lemma 8. □

***Remark* 7.** *Note that $\bar{\mu}_h$ is in fact a finite-dimensional element, but can be identified with an element of $\mathcal{M}(\Omega)$ when interpreting it as a sum of Dirac measures located in the mesh points.*

We immediately obtain a convergence result for the discrete optimal states.

***Corollary* 8.** *Under the assumptions of Lemma 10, the sequence of optimal discrete states $\bar{y}_h$ associated with $\bar{u}_h$ converges uniformly to the continuous optimal state $\bar{y}$ associated with $\bar{u}$. Moreover, there exists a constant $c > 0$ independent of h, such that*

$$\|\bar{y} - \bar{y}_h\|_\infty \leq c h^{1-n/4}$$

is satisfied.

Proof. This follows immediately from the Lipschitz continuity of G, Lemma 10 as well as the finite element error estimate from Theorem 7 by considering

$$\|\bar{y} - \bar{y}_h\|_\infty \leq \|G(\bar{u}) - G(\bar{u}_h)\|_\infty + \|G(\bar{u}_h) - G_h(\bar{u}_h)\|_\infty.$$

□

Moreover, we obtain for the linearized discrete states:

***Corollary* 9.** *Under the assumptions of Lemma 10, the sequence of linearized discrete states $\bar{y}_h^v := G_h'(\bar{u}_h)v$ converges uniformly to the continuous linearized state $\bar{y}_v := G'(\bar{u})v$ associated with $\bar{u}$. Furthermore, there exists a constant $c > 0$ such that*

$$\|\bar{y}_h^v - \bar{y}_v\|_\infty \leq c h^{1-n/4} \|v\|$$

is fulfilled.

Proof. We split the error into

$$\|G_h'(\bar{u}_h)v - G'(\bar{u})v\|_\infty \leq \|G_h'(\bar{u}_h)v - G'(\bar{u}_h)v\|_\infty + \|G'(\bar{u}_h)v - G'(\bar{u})v\|_\infty.$$

Applying the error estimate of Lemma 3 and well-known Lipschitz results for the linearized control-to-state operator combined with the convergence result of Lemma 10 yields the assertion. □

***Corollary* 11.** *The sequence of Lagrange multipliers $\{\bar{\mu}_h\}$ associated to the state constraints of $(\mathbb{P}_h)$ is uniformly bounded in $\mathcal{M}(\Omega)$.*

Proof. This follows directly from meanwhile standard computations involving the gradient equation. Note that (4.33) implies

$$(\nu\bar{u}_h + \bar{p}_h, \Pi_h u_\gamma - \bar{u}_h) = 0, \tag{4.35}$$

where u_γ is the Slater point from Lemma 1 fulfilling Assumption 3. Hence, Lemma 8 is applicable in the following. We reformulate (4.35) and obtain

$$\begin{aligned}
0 =&(\nu\bar{u}_h, \Pi_h u_\gamma - \bar{u}_h) + (\bar{y}_h - y_d, G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h)) - \langle G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h), \bar{\mu}_h\rangle \\
=&(\nu\bar{u}_h, \Pi_h u_\gamma - \bar{u}_h) + (\bar{y}_h - y_d, G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h)) \\
&- \langle G_h(\bar{u}_h) + G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h) - y_c, \bar{\mu}_h\rangle + \langle G_h(\bar{u}_h) - y_c, \bar{\mu}_h\rangle \\
\leq&(\nu\bar{u}_h, \Pi_h u_\gamma - \bar{u}_h) + (\bar{y}_h - y_d, G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h)) - \langle \gamma/2, \bar{\mu}_h\rangle,
\end{aligned}$$

where we used the Slater point property from Lemma 8 as well as the complementary slackness conditions (4.34). Reformulation of the inequality then yields

$$\begin{aligned}
\frac{\gamma}{2}\int_{\Omega_0} 1 d\bar{\mu}_h &\leq (\nu\bar{u}_h, \Pi_h u_\gamma - \bar{u}_h) + (\bar{y}_h - y_d, G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h)) \\
&\leq \nu\|\bar{u}_h\|\|\Pi_h u_\gamma - \bar{u}_h\| + \|\bar{y}_h - y_d\|\|G_h'(\bar{u}_h)(\Pi_h u_\gamma - \bar{u}_h)\|
\end{aligned}$$

which implies the assertion by the boundedness of the right-hand-side. This boundedness follows from the convergence result of Lemma 10 as well as the stability estimates for G_h and G_h' from Corollaries 8 and 9 which obviously imply boundedness in $L^2(\Omega)$, noting that $\|\Pi_h u_\gamma\|$ remains bounded due to estimate (3.21). □

As a direct consequence of Theorem 9 we obtain:

***Corollary* 12.** *The sequence of discrete locally optimal solutions $\{\bar{u}_h\}_{h>0}$ from Theorem 9 is uniformly bounded in $W^{1,\sigma}(\Omega)$.*

Proof. Let $p^h \in L^2(\Omega)$ denote the solution of

$$-(p^h, \Delta\varphi) + (d_y(\cdot, \bar{y}_h)p^h, \varphi) = (\bar{y}_h - y_d, \varphi) - \langle \varphi, \bar{\mu}_h\rangle \quad \forall \varphi \in H^2(\Omega) \cap V.$$

Due to the boundedness results for $\bar{y}_h$ and $\bar{\mu}_h$ from Corollary 7 and Lemma 11, we deduce

$$\|p^h\|_W \leq c(\|\bar{y}_h\| + \|y_d\| + \|\bar{\mu}_h\|_{\mathcal{M}(\Omega)}) \leq c$$

according to Theorem 1 of [4]. Moreover, arguing as at the beginning of Section 8.5 of [3] we obtain

$$\|\bar{p}_h\|_W \leq c\|p^h\|_W.$$

Thus, the assertion follows from the gradient equation (4.33). □

Now, we can prove our main result:

***Theorem* 10.** *Let $\bar{u}$ be a local solution of Problem $(\mathbb{P})$ satisfying the quadratic growth condition (2.14). Moreover, let $\{h\}_{>0}$ be an arbitrary sequence of positive mesh sizes converging to zero. Then there exists a sequence $\{\bar{u}_h\}$ of local solutions of Problem $(\mathbb{P}_h)$ such that $\bar{u}_h$ converges strongly in $L^2(\Omega)$ to $\bar{u}$. Moreover, the error estimate*

$$\|\bar{u} - \bar{u}_h\| \le ch^{2-n/2-\varepsilon}$$

is satisfied for an arbitrarily small $\varepsilon > 0$.

Proof. The existence of $\{\bar{u}_h\}_{h>0}$ follows directly from Theorem 9, it remains to prove the error estimate. We first point out that the boundedness of $\{\bar{u}_h\}$ in $W^{1,\sigma}(\Omega)$ allows to prove that the sequence $\{u_\tau(h)\}_{\tau(h)} > 0$ from Lemma 9 converges to $\bar{u}_h$ with order $\mathcal{O}(\delta(h,\sigma))$, since now (4.17) can be estimated by

$$G(u_\tau) - G_h(u_\tau) \ge -c_1\delta(h,\sigma),$$

making use of Theorem 8. Adapting (4.18) and (4.19) yields $\tau(h) = \mathcal{O}(\delta(h,\sigma))$. This can be used in the proof of Lemma 10 to estimate the second term, (II), in (4.20) by

$$\tau(h)\|u_\gamma - \bar{u}_h^r\| \le c\delta(h,\sigma).$$

The first term in (4.20) can be estimated with the following changes in (4.26). Replacing it by

$$f(u_\tau) - f_h(\bar{u}_h) \le c\delta(h,\sigma),$$

eventually leads to

$$\beta\|\bar{u} - u_\tau\|^2 \le c\delta(h,\sigma),$$

which then applies the assertion after taking the square root. □

***Remark* 8.** *We point out that the error estimates of Corollaries 8 and 9 can be improved by means of the convergence results of Theorem 10. However, plain convergence rather than optimal convergence rates of the FE-discretizations to state and linearized states are sufficient for the purpose of transferring the second order sufficient conditions to the discrete level, which is subject of the next section.*

5 Second Order Sufficient Conditions for $(\mathbb{P}_h)$

In this section we prove a stability result for the second order sufficient optimality conditions from Assumption 4 with respect to discretization, i.e. we show that they can be carried over from the continuous Problem $(\mathbb{P})$ to the discretized Problem $(\mathbb{P}_h)$.

For the remainder of this paper, we agree upon the following assumption, without explicit further notice.

Assumption 7. *Let $\{h\}_{>0}$ be an arbitrary sequence of positive mesh sizes converging to zero. In accordance with Theorem 10 let $\{\bar{u}_h\}$ and $\{\bar{y}_h\}$ be sequences of local solutions of $(\mathbb{P}_h)$ converging to $\bar{u}$ and $\bar{y}$, respectively. By $\bar{p}_h, \bar{\mu}_h$, we denote the associated (unique) discrete adjoints and multipliers, respectively.*

For our further calculations, let us point out that the following representation of the second derivative of the Lagrangian using the adjoint state introduced in Theorem 4 is well known:

$$\mathcal{L}''(u,\mu)(v_1,v_2) = \int_\Omega (y_{v_1}y_{v_2} + \nu v_1 v_2 - d_{yy}(x,y)py_{v_1}y_{v_2})\,dx \tag{5.1}$$

with $y = G(u)$, $y_{v_i} = G'(u)v_i$, $i = 1,2$, and $p \in W_0^{1,\sigma}(\Omega)$, $\sigma < n/(n-1)$, is the solution of

$$-(p, \Delta\varphi) + (d_y(\cdot, y)p, \varphi) = (y - y_d, \varphi) - \langle\varphi, \mu\rangle \quad \forall \varphi \in H^2(\Omega) \cap V.$$

More details can again be found in e.g. [35]. Its discrete counterpart can be formulated as

$$\mathcal{L}_h''(u,\mu)(v_1,v_2) = \int_\Omega (y_h^{v_1}y_h^{v_2} + \nu v_1 v_2 - d_{yy}(x,y_h)p_h y_h^{v_1}y_h^{v_2})\,dx \tag{5.2}$$

with $y_h = G_h(u)$, $y_h^{v_i} = G_h'(u)v_i$, $i = 1,2$, and p_h is the solution of

$$(\nabla p_h, \nabla\varphi_h) + (d_y(\cdot, y_h)p_h, \varphi_h) = (y_h - y_d, \varphi_h) - \langle\varphi_h, \mu\rangle \quad \forall \varphi_h \in V_h.$$

When aiming at proving second order sufficient optimality conditions for the discrete problem, it is therefore necessary to develop convergence results not only for the primal but the adjoint state as well.

Corollary 13. *The sequence of adjoint states $\bar{p}_h$ associated to Problem $(\mathbb{P}_h)$ converges strongly in $L^2(\Omega)$ to the limit $\bar{p} \in L^2(\Omega)$ which is the solution of the continuous adjoint equation (2.8). There exists a constant $c > 0$ independent of h such that*

$$\|\bar{p} - \bar{p}_h\| \le ch^{2-n/2-\varepsilon}$$

holds for all h sufficiently small.

Proof. This follows immediately from Theorem 10, since the optimality conditions for Problems $(\mathbb{P})$ and $(\mathbb{P}_h)$ imply

$$\|\bar{p} - \bar{p}_h\| = \nu\|\bar{u} - \bar{u}_h\| \le ch^{2-n/2-\varepsilon}.$$

□

We can now prove the main result of this section.

***Theorem* 11.** *Let $\bar{u}$ be an optimal control of Problem ($\mathbb{P}$) satisfying Assumption 4. Furthermore, let $\bar{u}_h$ be a discrete optimal control in the vicinity of $\bar{u}$, satisfying the discrete first order necessary optimality conditions from Theorem 9. Then, there exists a constant $\alpha' > 0$, such that*

$$\mathcal{L}_h''(\bar{u}_h, \bar{\mu}_h)v^2 \geq \alpha' \|v\|^2 \tag{5.3}$$

is valid for all $v \in L^2(\Omega)$, provided that $h > 0$ is chosen sufficiently small.

Proof. The proof follows by direct calculations, using the discretization error estimates for the state the linearized states as well as the adjoint states. We estimate

$$\begin{aligned}
&\mathcal{L}_h''(\bar{u}_h, \bar{\mu}_h)v^2 = \mathcal{L}''(\bar{u}, \bar{\mu})v^2 + \mathcal{L}_h''(\bar{u}_h, \bar{\mu}_h)v^2 - \mathcal{L}''(\bar{u}, \bar{\mu})v^2 \\
&= \mathcal{L}''(\bar{u}, \bar{\mu})v^2 + \|\bar{y}_h^v\|^2 - \|\bar{y}_v\|^2 + \int_\Omega \left(d_{yy}(x, \bar{y})(\bar{y}_v)^2 \bar{p} - d_{yy}(x, \bar{y}_h)(\bar{y}_h^v)^2 \bar{p}_h\right)\, dx
\end{aligned} \tag{5.4}$$

with $\bar{y}_v = G'(\bar{u})v$ and $\bar{y}_h^v = G_h'(\bar{u}_h)v$. With the help of the continuity of G' and G_h', cf. Remarks 1 and 4, as well as the error estimate from Corollary 9, noting that $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\|\bar{y}_h^v\|^2 - \|\bar{y}_v\|^2 \leq \|\bar{y}_h^v + \bar{y}_v\| \|\bar{y}_v - \bar{y}_h^v\| \leq c_1 h^{1-n/4} \|v\|^2. \tag{5.5}$$

Further, we observe

$$\begin{aligned}
&\int_\Omega \left(d_{yy}(x, \bar{y}_h)(\bar{y}_h^v)^2 \bar{p}_h - d_{yy}(x, \bar{y})(\bar{y}_v)^2 \bar{p}\right)\, dx = \\
&\underbrace{\int_\Omega \left(d_{yy}(x, \bar{y}_h) - d_{yy}(x, \bar{y})\right)(\bar{y}_h^v)^2 \bar{p}_h\, dx}_{\text{(I)}} + \underbrace{\int_\Omega d_{yy}(x, \bar{y}) \left((\bar{y}_h^v)^2 - (\bar{y}_v)^2\right) \bar{p}_h\, dx}_{\text{(II)}} \\
&\qquad + \underbrace{\int_\Omega d_{yy}(x, \bar{y})(\bar{y}_v)^2 \left(\bar{p}_h - \bar{p}\right)\, dx}_{\text{(III)}}.
\end{aligned} \tag{5.6}$$

We continue by estimating the terms (I)-(III) separately. For (I), we obtain using Hölder's inequality, the Lipschitz continuity of d_{yy}, the boundedness of $\bar{p}_h$ in $L^2(\Omega)$, which can be deduce from Lemma 13, the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, cf. [1], and the L^∞-error estimate for the discrete state from Corollary 8:

$$\begin{aligned}
\int_\Omega \left(d_{yy}(x, \bar{y}_h) - d_{yy}(x, \bar{y})\right)(\bar{y}_h^v)^2 \bar{p}_h\, dx &\leq \|d_{yy}(\cdot, \bar{y}_h) - d_{yy}(\cdot, \bar{y})\|_\infty \|\bar{p}_h\| \|\bar{y}_h^v\|_{L^4(\Omega)}^2 \\
&\leq c\|\bar{y}_h - \bar{y}\|_\infty \|\nabla \bar{y}_h^v\|^2 \leq c h^{1-n/4} \|v\|^2.
\end{aligned} \tag{5.7}$$

To estimate the second term, (II), we apply the boundedness of $d_{yy}(\cdot,\bar y)$ in $L^\infty(\Omega)$, the boundedness of $\bar p_h$ in $L^2(\Omega)$, as well as Remarks 1 and 4, and the discretization error estimate from Corollary 9. This leads to

$$\int_\Omega d_{yy}(x,\bar y)((\bar y_h^v)^2-(\bar y_v)^2)\bar p_h\,dx \\ \le \|d_{yy}(\cdot,\bar y)\|_\infty\|\bar p_h\|\|\bar y_h^v+\bar y_v\|\|\bar y_h^v-\bar y_v\|_\infty \le ch^{1-n/4}\|v\|^2. \quad (5.8)$$

Last, for (III) note that the boundedness of d_{yy}, the stability result for the linearized state equation from Remark 1, and the discretization error estimate for the adjoint state from Lemma 13 yield

$$\int_\Omega d_{yy}(x,\bar y)(\bar y_v)^2(\bar p_h-\bar p)\,dx \le \|d_{yy}(\cdot,\bar y)\|_\infty\|\bar p-\bar p_h\|\|\bar y_v\|^2_{L^4(\Omega)} \\ \le c\|\bar p-\bar p_h\|\|\nabla\bar y_v\|^2 \le ch^{2-n/2-\varepsilon}\|v\|^2. \quad (5.9)$$

Collecting and inserting (5.7)-(5.9) into (5.6) yields

$$\int_\Omega \left(d_{yy}(x,\bar y_h)(\bar y_h^v)^2\bar p_h\,dx - d_{yy}(x,\bar y)(\bar y_v)^2\bar p\right)\,dx \le c_2h^{1-n/4}\|v\|^2. \quad (5.10)$$

Clearly, (5.5) and (5.10) inserted into (5.4) yield

$$\mathcal{L}_h''(\bar u_h,\bar\mu_h)v^2 \ge \mathcal{L}''(\bar u,\bar\mu)v^2-(c_1+c_2)h^{1-n/4}\|v\|^2.$$

Using Assumption 4, we obtain

$$\mathcal{L}_h''(\bar u_h,\bar\mu_h)v^2 \ge (\alpha-(c_1+c_2)h^{1-n/4})\|v\|^2.$$

Obviously, for h small enough there exists a constant $\alpha'=(\alpha-(c_1+c_2)h^{1-n/4})>0$ such that the assertion is obtained. □

6 Generalizations

Before we address the numerical verification of our theoretical results, let us give a brief outlook to possible generalizations of our theory.

6.1 Modification of the state equation

Let us first mention that it is possible to generalize the proven theory to elliptic operators in divergence form with regular coefficients, as well as e.g. homogeneous Neumann or generalized Neumann boundary conditions instead of Dirichlet boundary conditions.

6.2 Modification of the discretization

We emphasize that for the considered problem without control constraints, full discretization and variational discretization generate the same discrete solutions, see [31].

6.3 Modification of the state constraint

For a polygonal subdomain Ω_0 the constraint $y_h(x) \geq y_c$ can equivalently be expressed by restrictions in the nodes of the finite element discretization. This can be done in different ways. We recommend to require the restriction in a slightly larger set of point such that Ω_0 is contained in a set Ω_1 of triangles generated by such nodes. The discussion of the error estimate can be done in the same way. Because of the continuity of $\bar{y}$ the linearized Slater condition is satisfied on a slightly larger set with a parameter $\gamma' < \gamma$. The construction of the feasible control for the discrete problem remains the same and there only small modifications are necessary to obtain the same error estimate.

In practical problems the quantity y_c may be a function. Our theory remains valid if y_c is a continuous function on $\bar{\Omega}_0$. Often such a function y_c is replaced by an interpolate $I_h y_c$. Then one needs that the pointwise interpolation error is of the same order as the pointwise finite element error. Consequently, one has to require $y_c \in W^{2,\infty}(\Omega)$ in general.

Another practical important case are state constraints which are required on the whole domain Ω instead of Ω_0, which would be of interest if e.g. homogeneous Neumann or Robin boundary conditions are considered. Then one can derive error estimates with lower convergence rates because of the lower regularity of the state on the whole domain. We will discuss this in a forthcoming paper,

6.4 Regularization of the problem

The discrete state constrained problem is difficult to solve numerically. Often regularization terms are introduced. It is possible to combine the results of our paper with the results for regularized problems [27]. This discussion becomes technical but without additional difficulties.

7 Numerical experiment

In the following we present a numerical example which illustrates the theoretical findings of Theorem 10. We restrict ourselves to the consideration of an example in two space dimensions similar to one in [31]. More precisely, we are interested

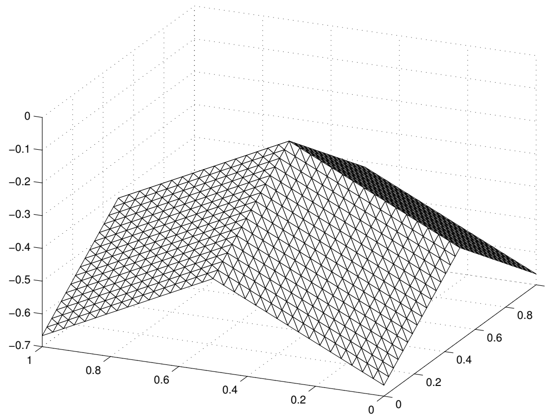


Figure 1: State constraint

in the solution of the optimal control problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & -\Delta y + y^3 = u \quad \text{in } \Omega, \\ & \qquad\quad y = 0 \quad \text{on } \Gamma, \\ \text{and} \quad & y(x) \geq y_c(x) \quad \text{for all } x \in \bar{\Omega}_0, \end{aligned}$$

where Ω denotes the unit square. The data y_d, ν and y_c are chosen as follows

$$y_d = -1, \quad \nu = 10^{-3}$$

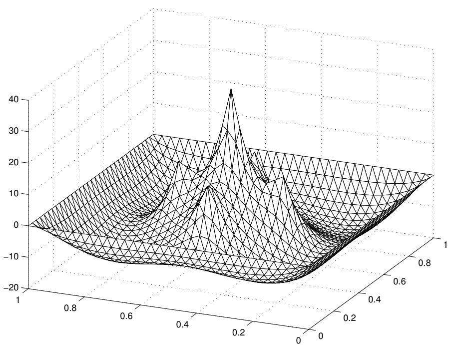
and

$$y_c(x) = -\frac{2}{3} + \min\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(1 + x_1 - x_2), \frac{1}{2}(1 - x_1 + x_2), 1 - \frac{1}{2}(x_1 + x_2)\right)$$

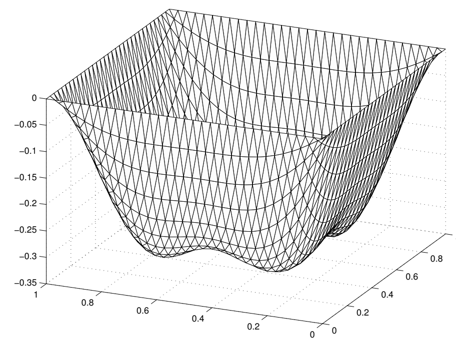
with $x = (x_1, x_2) \in \Omega$. One can find an illustration of the function y_c in Figure 1. Note that the triangulation is generated in such a way that y_c can exactly be integrated by the nodal Lagrange interpolate.

Further, we will see that the state constraint becomes only active in an inner subset of Ω. Thus, the existence of a subset Ω_0 with $\bar{\Omega}_0 \subset\subset \Omega$ as required is given.

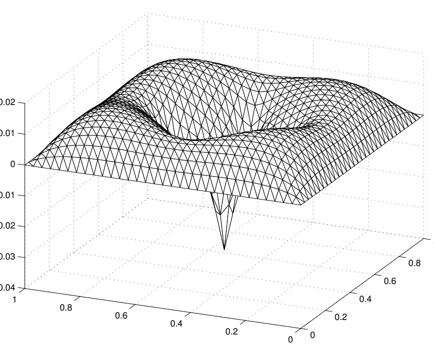
To solve this problem on the discrete level, we proceed in two steps. In the first one we apply an SQP method to reduce the nonlinear problem to a sequence of linear quadratic problems. The second one consists of solving these sub-problems. Here, we employ a quadratic penalization of the state constraints, cf. [24], combined with a primal dual active set method. Let us point out that the stopping criterion for the SQP method and the penalization parameter are chosen such that the discretization error dominates the overall error. Figures 2(a)–2(d) show exemplarily the numerical solution for $h = 0.5^5\sqrt{2}$.



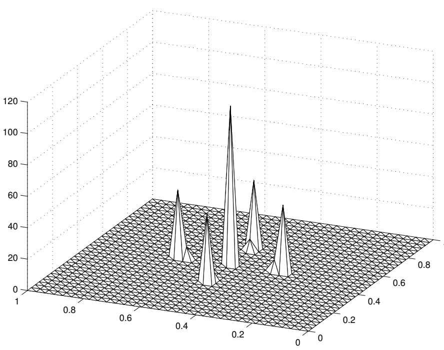
(a) Optimal control



(b) Optimal state



(c) Adjoint state



(d) Lagrange multiplier

Figure 2: Numerical solution for $h = 0.5^5\sqrt{2}$

Next, we are interested in the convergence rates of the finite element discretization. Since we are not able to state an analytic solution for our test example, we compute a reference solution $\bar{u}_{ref}$ on a mesh with mesh size $h_{ref} = 0.5^{10}\sqrt{2}$. By projection of the solutions $\bar{u}_h$ on the reference mesh we are able to calculate an approximated experimental order of convergence by

$$\text{eoc} := \frac{\ln(\|\bar{u}_{ref} - \bar{u}_{h_{i-1}}\| / \|\bar{u}_{ref} - \bar{u}_{h_i}\|)}{\ln(h_{i-1}/h_i)},$$

where h_{i-1} and h_i denote two consecutive mesh sizes. The discretization errors $\|\bar{u}_{ref} - \bar{u}_h\|$ and the corresponding experimental convergence rates can be found for different mesh sizes in Table 1. These confirm the theoretical results of Theorem 10.

$h/\sqrt{2}$	$\|\bar{u}_{ref} - \bar{u}_h\|$	eoc
0.5	$8.6354179e-00$	
0.5^2	$4.3189440e-00$	1.00
0.5^3	$1.8477157e-00$	1.22
0.5^4	$9.0732353e-01$	1.03
0.5^5	$4.5130090e-01$	1.01
0.5^6	$2.1373122e-01$	1.08
0.5^7	$9.8533988e-02$	1.12
0.5^8	$4.3223082e-02$	1.19

Table 1: Discretization errors $\|\bar{u}_{ref} - \bar{u}_h\|$ and approximated experimental convergence rates for different mesh sizes h and $h_{ref} = 0.5^{10}\sqrt{2}$

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