Numerische Mathematik manuscript No.
(will be inserted by the editor)

A Priori Error Estimates for Space-Time Finite Element Discretization of Semilinear Parabolic Optimal Control Problems

Ira Neitzel · **Boris Vexler**

Received: date / Accepted: date

Abstract In this paper, a priori error estimates for space-time finite element discretizations of optimal control problems governed by semilinear parabolic PDEs and subject to pointwise control constraints are derived. We extend the approach from [23, 24], where linear-quadratic problems have been considered, discretizing the state equation by usual conforming finite elements in space and a discontinuous Galerkin method in time. Error estimates for controls discretized by piecewise constant functions in time and cellwise constant functions in space are derived in detail and we explain how error estimate for further discretization approaches, e. g., cellwise linear discretization in space, the postprocessing approach from [25], and the variationally discrete approach from [17] can be obtained. In addition, we derive an estimate for a setting with finitely many time-dependent controls.

Keywords optimal control · parabolic semilinear equations · error estimates · finite elements

Mathematics Subject Classification (2000) 49M25 · 65M15 · 65M60

1 Introduction

In this paper, we develop a priori error analysis for the space-time finite element discretization of optimization problems governed by semilinear parabolic equations

Supported by DFG priority program SPP1253.

I. Neitzel
Technische Universität Berlin, Institut für Mathematik, Str. des 17. Juni 136, 10623 Berlin, Germany
E-mail: neitzel@math.tu-berlin.de

B. Vexler
Technische Universität München, Zentrum Mathematik, Boltzmannstr. 3, 85748 Garching b. München, Germany
E-mail: vexler@ma.tum.de

with control q and state u, subject to pointwise inequality constraints on the control. More precisely, the model problem we will consider is given by

$$\text{Minimize } J(q,u) := \frac{1}{2}\int_0^T\int_\Omega (u(t,x)-\hat{u}(t,x))^2 dx\,dt + \frac{\nu}{2}\int_0^T\int_\Omega q(t,x)^2 dx\,dt \quad (1.1a)$$

$$\begin{aligned} \partial_t u - \Delta u + d(t,x,u) &= q \quad \text{in } (0,T)\times\Omega \\ u(0,\cdot) &= u_0 \text{ in } \Omega \\ u &= 0 \quad \text{on } (0,T)\times\partial\Omega. \end{aligned} \quad (1.1b)$$

$$q_a \le q(t,x) \le q_b \quad \text{a.e. in } (0,T)\times\Omega. \quad (1.1c)$$

The exact setting of the model problem will be described in the next section. We will use discontinuous finite element methods for the time discretization of the state equation, cf. [12,13], and usual H^1-conforming finite elements for the space discretization. In [23,24] finite element error estimates for linear-quadratic optimal control problems with and without control constraints have been developed. Our aim is to extend this approach to problems governed by semilinear parabolic equations. Only few other results have been published on error estimates for parabolic optimization problems. We mention the papers [19,20,26,29], which are concerned with linear-quadratic problems and a recent article [8], where pure convergence (without rates) of discontinuous Galerkin schemes for control problems governed by semilinear parabolic equations has been shown. On the contrary, quite a number of results are known for elliptic problems, cf., e. g., [3,6,15–17,25,18]. For parabolic optimal control problems with state constraints there are two recent contributions [22,11].

In [24], the challenges of pointwise inequality constraints on the control with respect to finite element error analysis and different ways of control discretization are discussed in detail and references are given. In addition to the difficulties posed by inequality constraints, we may encounter the existence of multiple solutions on all levels of discretization of our problem due to nonconvexity. As a consequence, we need to deal with convergence and error estimates for somehow corresponding solutions. We use a technique introduced in [7] for elliptic problems, using auxiliary problems that admit unique discrete optimal controls close to a local solution of the continuous problem.

The involved spaces have to be chosen such that on the one hand error estimates can be shown, and on the other hand differentiability properties of the control-to-state operators hold, from which quadratic growth conditions can be derived and convergence follows. Due to the presence of pointwise L^∞-bounds on the control, it is reasonable to choose $L^\infty((0,T)\times\Omega)$ as a control-space. In this space, differentiability of the control-to-state operator is guaranteed. However, since convergence of discrete solutions towards the continuous solution in the L^∞-norm is a delicate issue, we will consider local solutions in the sense of L^2. Our analysis involves a discussion of the regularity of semidiscrete and discrete solutions under usual local Carathéodory-type assumptions on the nonlinearities. The main challenge of this work lies in the issues mentioned above that are independent of the type of control discretization. We will therefore discuss in detail the concept of cellwise constant controls, and briefly point out how the results from [24] can be extended to other types of control discretization, such as piecewise linear controls in space, the variationally discrete concept originally introduced in [17], and the post processing approach from [25].

2 Optimization problem

In this section we discuss a precise formulation of the model problem with respect to existence of solutions, first and second order optimality conditions, as well as some auxiliary stability results. Throughout, we employ the usual notation for Sobolev spaces. We first lay out the principal assumptions on the data in Problem (1.1a)-(1.1c).

Assumption 2.1 *Throughout this paper, let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary $\partial\Omega$, and let $T > 0$ be a given real number that defines the time interval $I := (0,T)$. Moreover, $\nu \in \mathbb{R}$ is a positive, fixed parameter, the bounds q_a, q_b are real numbers with $q_a < q_b$. The desired state $\hat{u}$ is a function from $L^\infty(I\times\Omega)$ and the initial state u_0 is a function from $H_0^1(\Omega)\cap L^\infty(\Omega)$.*
The nonlinearity $d = d(t,x,u)\colon I\times\Omega\times\mathbb{R} \to \mathbb{R}$ is measurable with respect to $(t,x) \in I\times\Omega$ for all $u \in \mathbb{R}$, and at least twice continuously differentiable with respect to u for almost all $(t,x) \in I\times\Omega$. For $u = 0$, d and its derivatives up to order two are uniformly bounded by a constant $K > 0$, i. e.

$$\|d(\cdot,\cdot,0)\|_{L^\infty(I\times\Omega)} + \|\partial_u d(\cdot,\cdot,0)\|_{L^\infty(I\times\Omega)} + \|\partial_{uu} d(\cdot,\cdot,0)\|_{L^\infty(I\times\Omega)} \le K.$$

Moreover, they are uniformly Lipschitz continuous on bounded sets with respect to u, i. e. for each $S > 0$ there exists an $L(S) > 0$ such that

$$\|\partial_{uu} d(\cdot,\cdot,u_1) - \partial_{uu} d(\cdot,\cdot,u_2)\|_{L^\infty(I\times\Omega)} \le L(S)\,|u_1 - u_2|$$

is satisfied for all $u_1, u_2 \in \mathbb{R}$ with $|u_i| \le S$, $i = 1,2$. Last, d fulfills the monotonicity condition

$$\partial_u d(\cdot,\cdot,u) \ge 0 \quad \text{for a.a. } (t,x) \in I\times\Omega \text{ and for all } u \in \mathbb{R}.$$

It will be helpful to make use of the following short notation for inner products and norms on the spaces $L^2(\Omega)$, $L^\infty(\Omega)$, as well as $L^2(I\times\Omega)$ and $L^\infty(I\times\Omega)$:

$$\begin{aligned}
(v,w) &:= (v,w)_{L^2(\Omega)}, & \|v\| &:= \|v\|_{L^2(\Omega)}, & \|v\|_\infty &:= \|v\|_{L^\infty(\Omega)}\\
(v,w)_I &:= (v,w)_{L^2(I\times\Omega)}, & \|v\|_I &:= \|v\|_{L^2(I\times\Omega)}, & \|v\|_{\infty,\infty} &:= \|v\|_{L^\infty(I\times\Omega)}
\end{aligned}$$

In some estimates, we will also need the norm $\|v\|_{L^\infty(I,L^2(\Omega))}$, which we will denote by $\|v\|_{\infty,2}$. For convenience, let us also agree that throughout the paper, C pertains to a positive constant related to error- or stability estimates, whereas $c > 0$ is a generic auxiliary constant. Moreover, in order to find a weak formulation of the state equation (1.1b) and the optimal control problem (1.1a)-(1.1c), we introduce the state space X, the control space Q, and the set of admissible controls Q_{ad},

$$\begin{aligned}
X &:= W(0,T) = \{v \mid v \in L^2(I,H_0^1(\Omega)) \text{ and } \partial_t v \in L^2(I,H_0^1(\Omega)^*)\},\\
Q &:= L^\infty(I\times\Omega),\\
Q_{\text{ad}} &:= \{q \in Q \mid q_a \le q \le q_b \text{ a.e. in } I\times\Omega\},
\end{aligned}$$

as well as a bilinear form $\mathbf{b}(\cdot,\cdot)$ for $u, \varphi \in X$ by

$$\mathbf{b}(u,\varphi) := \int_0^T \langle \partial_t u, \varphi \rangle \, dt + (\nabla u, \nabla \varphi)_I,$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and its dual space.

Then, a weak formulation of the state equation (1.1b) for a fixed control $q \in Q$ and fixed initial state $u_0 \in H_0^1(\Omega)$ is to find a state $u \in X$ that satisfies

$$\begin{aligned} \mathbf{b}(u,\varphi) + (d(\cdot,\cdot,u),\varphi)_I &= (q,\varphi)_I \ \forall \varphi \in X, \\ u(0,\cdot) &= u_0. \end{aligned} \tag{2.1}$$

Proposition 2.1 *For fixed control $q \in Q$ and fixed initial state $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ there exists a unique solution $u \in X$ of the weak state equation* (2.1). *Moreover, the solution exhibits the improved regularity*

$$u \in L^2(I, H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(I \times \Omega) \cap H^1(I, L^2(\Omega)) \hookrightarrow C(\bar{I}, H_0^1(\Omega))$$

and the stability estimates

$$\|u\|_{\infty,\infty} \leq C(\|q\|_{\infty,\infty} + \|d(\cdot,\cdot,0)\|_{\infty,\infty} + \|u_0\|_\infty) \tag{2.2}$$

$$\|\partial_t u\|_I + \|u\|_I + \|\nabla u\|_I + \|\nabla^2 u\|_I \leq C(\|q\|_I + \|d(\cdot,\cdot,0)\|_I + \|\nabla u_0\|) \tag{2.3}$$

$$\|u\|_{\infty,2} + \|\nabla u\|_{\infty,2} \leq C(\|q\|_I + \|d(\cdot,\cdot,0)\|_I + \|\nabla u_0\|) \tag{2.4}$$

are satisfied for a constant $C > 0$.

Proof The existence of a solution $u \in X \cap L^\infty(I \times \Omega)$ and estimate (2.2) follow as in [4]. From the standard result

$$\|u\|_I + \|\nabla u\|_I \leq C\left(\|q\|_I + \|d(\cdot,\cdot,0)\|_I + \|u_0\|\right),$$

which is obtained due to the monotonicity of d from testing (2.1) with $\varphi = u$, the assertion follows by applying [14, Theorem 5] to the linear equation

$$\mathbf{b}(u,\varphi) = (q - d(\cdot,\cdot,u),\varphi)_I \quad \forall \varphi \in X, \quad u(0,\cdot) = u_0,$$

by the fact that $\|d(\cdot,\cdot,u)\|_I = \|d(\cdot,\cdot,u) - d(\cdot,\cdot,0) + d(\cdot,\cdot,0)\|_I \leq L\|u\|_I + \|d(\cdot,\cdot,0)\|_I$ due to the boundedness of u.

Notice that due to the boundedness of Q_{ad}, the constant C can be chosen independently of $q \in Q_{\text{ad}}$ in later estimates. It is now justified to introduce the control-to-state mapping $G\colon Q \to X \cap L^\infty(I \times \Omega)$, $G(q) = u$, which leads to the reduced objective function

$$j : Q \to \mathbb{R}_0^+, \quad q \mapsto J(q, G(q)).$$

This makes the optimal control problem (1.1a)–(1.1c) equivalent to

$$\text{Minimize } j(q) \text{ subject to } q \in Q_{\text{ad}}. \tag{2.5}$$

The following existence result is obtained by standard arguments, since the set of admissible controls is not empty by Assumption 2.1.

Lemma 2.1 *Let Assumption 2.1 be satisfied. The optimal control problem* (2.5) *admits at least one optimal control $\bar{q} \in Q_{ad}$ with associated optimal state $\bar{u} = G(\bar{q})$.*

We refer to [28] for more details. We point out, though, that j need not be convex due to the nonlinearity of the control-to-state operator and introduce the notation of local solutions in the sense of $L^2(I \times \Omega)$.

Definition 2.1 A control $\bar{q} \in Q_{\mathrm{ad}}$ is called a local solution of (2.5) in the sense of $L^2(I \times \Omega)$ if there exists a constant $\varepsilon > 0$, such that the inequality

$$j(q) \geq j(\bar{q})$$

is satisfied for all $q \in Q_{\mathrm{ad}}$ with $\|\bar{q} - q\|_I \leq \varepsilon$.

We proceed by discussing first order necessary and second order sufficient optimality conditions for the optimal control problem. We point out that the control-to-state-operator G is of class C^2 with respect to the $L^\infty(I \times \Omega)$-topology, see, e. g., [28]. With $\delta q, \delta q_1, \delta q_2 \in Q$ and $u = G(q)$, its first and second order derivatives are given by $\tilde{u} := G'(q)\delta q$ and $\tilde{w} := G''(q)\delta q_1 \delta q_2$ being the solutions of

$$\mathbf{b}(\tilde{u}, \varphi) + (\partial_u d(\cdot,\cdot,u)\tilde{u}, \varphi)_I = (\delta q, \varphi)_I \quad \forall \varphi \in X, \quad \tilde{u}(0,\cdot) = 0, \tag{2.6}$$

and

$$\mathbf{b}(\tilde{w}, \varphi) + (\partial_u d(\cdot,\cdot,u)\tilde{w}, \varphi)_I = (-\partial_{uu} d(\cdot,\cdot,u)\tilde{u}_1 \tilde{u}_2, \varphi)_I \quad \forall \varphi \in X, \quad \tilde{w}(0,\cdot) = 0, \tag{2.7}$$

where $\tilde{u}_1 = G'(q)\delta q_1$ and $\tilde{u}_2 = G'(q)\delta q_2$. Note that similar estimates as in Proposition 2.1 hold for $\tilde{u}$ and $\tilde{w}$, since $\partial_u d(\cdot,\cdot,u)$ is bounded by Assumption 2.1. We will especially make use of the boundedness estimates

$$\|\tilde{u}\|_I \leq c\|\delta q\|_I \tag{2.8}$$

$$\|\tilde{u}\|_{\infty,\infty} \leq c\|\delta q\|_{\infty,\infty} \tag{2.9}$$

$$\|\tilde{u}\|_{\infty,2} + \|\nabla \tilde{u}\|_{\infty,2} \leq c\|\delta q\|_I. \tag{2.10}$$

The reduced objective function j is also of class C^2 and we can formulate standard first order necessary optimality conditions with the help of a variational inequality.

Lemma 2.2 *Let $\bar{q} \in Q_{ad}$ be a local solution of* (2.5) *in the sense of Definition 2.1. Then the following variational inequality holds:*

$$j'(\bar{q})(p - \bar{q}) \geq 0 \quad \forall p \in Q_{ad}. \tag{2.11}$$

For a proof, we refer, e. g., to [28]. For a control $q \in Q$ with associated state $u = u(q) \in X$ we define the adjoint state $z = z(q) \in X$ with the help of the weak adjoint equation

$$\mathbf{b}(\varphi, z) + (\varphi, \partial_u d(\cdot,\cdot,u)z)_I = (\varphi, u - \hat{u})_I \quad \forall \varphi \in X, \quad z(T,\cdot) = 0, \tag{2.12}$$

for which the following existence and regularity result is applicable:

Proposition 2.2 *For every $d_0 \in L^\infty(I\times\Omega)$, $g \in L^p(I\times\Omega)$, $p>2$, and $z_T \in H_0^1(\Omega)\cap L^\infty(\Omega)$ there exists a unique solution $z \in X\cap L^\infty(I\times\Omega)$ of the equation*

$$\boldsymbol{b}(\varphi,z)+(\varphi,d_0 z)_I=(\varphi,g)_I \quad \forall\varphi\in X, \quad z(T,\cdot)=z_T.$$

Moreover, the solution exhibits the improved regularity

$$z\in L^2(I,H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(I,L^2(\Omega))\cap L^\infty(I\times\Omega)\hookrightarrow C(\bar I,H_0^1(\Omega))$$

and the stability estimates

$$\|z\|_{\infty,\infty}\le C\left(\|g\|_{L^p(I\times\Omega)}+\|z_T\|_\infty\right)$$

$$\|\partial_t z\|_I+\|z\|_I+\|\nabla z\|_I+\|\nabla^2 z\|_I+\|z\|_{\infty,2}+\|\nabla z\|_{\infty,2}\le C(\|g\|_I+\|\nabla z_T\|)$$

are satisfied for a constant $C>0$.

This is a straightforward extension of Proposition 2.1 and Proposition 2.1 in [23].With the help of the adjoint state $z(q)$ we can write the first order optimality conditions from Lemma 2.2 in the form

$$(\nu\bar q+z(\bar q),p-\bar q)_I\ge 0 \quad \forall p\in Q_{\mathrm{ad}}.$$

Then, using the pointwise projection on the admissible set,

$$P_{Q_{\mathrm{ad}}}\colon L^2(I,L^2(\Omega))\to Q_{\mathrm{ad}}, \quad P_{Q_{\mathrm{ad}}}(r)(t,x)=\max(q_a,\min(q_b,r(t,x))),$$

the optimality condition simplifies further to

$$\bar q=P_{Q_{\mathrm{ad}}}\left(-\frac{1}{\nu}z(\bar q)\right).$$

The projection $P_{Q_{\mathrm{ad}}}$ satisfies the regularity properties

$$\|\nabla(P_{Q_{\mathrm{ad}}}(v)(t))\|_{L^p(\Omega)}\le\|\nabla v(t)\|_{L^p(\Omega)} \ \ \forall v\in L^2(I,W^{1,p}(\Omega)),\ 1\le p\le\infty,$$

for almost all $t\in I$, and therefore we obtain the following regularity result:

Proposition 2.3 *Let $\bar q$ be a local solution of the optimization problem* (2.5)*, and $\bar z = z(\bar q)$ denote the corresponding adjoint state. Then*

$$\bar u,\bar z\in L^2(I,H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(I,L^2(\Omega))\cap L^\infty(I\times\Omega)\cap C(\bar I,H_0^1(\Omega)),$$
$$\bar q\in L^2(I,W^{1,p}(\Omega))\cap H^1(I,L^2(\Omega))\cap L^\infty(I\times\Omega)$$

holds for any $p<\infty$.

The following Lipschitz stability results will be helpful in the sequel:

Lemma 2.3 *Let $p,q\in Q_{ad}$ and $\delta q\in Q$ be given. Then there exists a constant $C>0$ such that*

$$\|G(p)-G(q)\|_I\le C\|p-q\|_I \tag{2.13}$$
$$\|G'(p)\delta q-G'(q)\delta q\|_I\le C\|p-q\|_I\|\delta q\|_I \tag{2.14}$$

is fulfilled.

Proof Denote in short $u := G(p)$ and $v := G(q)$. The difference $y := u - v$ fulfills the equation

$$\mathbf{b}(y,\varphi) + (d(\cdot,\cdot,u) - d(\cdot,\cdot,v),\varphi)_I = (p-q,\varphi)_I, \quad \forall \varphi \in X, \quad y(0,\cdot) = 0,$$

which is equivalent to

$$\mathbf{b}(y,\varphi) + (d_\xi y,\varphi)_I = (p-q,\varphi)_I, \quad \forall \varphi \in X, \quad y(0,\cdot) = 0,$$

with $d_\xi(t,x) = \int_0^1 \partial_u d(t,x,u(t,x) + \xi(v(t,x) - u(t,x)))d\xi$, which is bounded in $L^\infty(I \times \Omega)$ independently from u and v by the boundedness of Q_{ad}. The first assertion then follows from Proposition 2.1 and the boundedness of Q_{ad}. Note that we also obtain

$$\|G(p) - G(q)\|_{L^\infty(I,H_0^1(\Omega))} \leq c\|p-q\|_I. \tag{2.15}$$

To prove (2.14), we set $\tilde{u} := G'(p)\delta q$ and $\tilde{v} := G'(q)\delta q$, and note that $\tilde{y} := \tilde{u} - \tilde{v}$ fulfills

$$\begin{aligned} \mathbf{b}(\tilde{y},\varphi) + (\partial_u d(\cdot,\cdot,u)\tilde{y},\varphi)_I &= -((\partial_u d(\cdot,\cdot,u) - \partial_u d(\cdot,\cdot,v))\tilde{v},\varphi)_I \quad \forall \varphi \in X, \\ \tilde{y}(0,\cdot) &= 0. \end{aligned}$$

Hence from the local Lipschitz continuity of $\partial_u d$, the boundedness of u and v, and estimates (2.15) and (2.10) we obtain the last estimate,

$$\begin{aligned} \|\tilde{y}\|_I \leq c\|(\partial_u d(\cdot,\cdot,u) - \partial_u d(\cdot,\cdot,v))\tilde{v}\|_I &\leq c\|u-v\|_{L^4(I\times\Omega)}\|\tilde{v}\|_{L^4(I\times\Omega)} \\ &\leq c\|u-v\|_{L^\infty(I,H_0^1(\Omega))}\|\tilde{v}\|_{L^\infty(I,H_0^1(\Omega))} \leq c\|p-q\|_I\|\delta q\|_I \end{aligned}$$

by the embedding $L^\infty(I,H_0^1(\Omega)) \hookrightarrow L^4(I\times\Omega)$.

For the adjoint state, we will need a Lipschitz stability result in the L^2-norm, which is proven next.

Lemma 2.4 *Let $p,q \in Q_{ad}$ be given. Then there exists a constant $C > 0$ such that the estimate*

$$\|z(p) - z(q)\|_I \leq C\|p-q\|_I \tag{2.16}$$

is fulfilled.

Proof The difference $y := z(p) - z(q)$ fulfills the equation

$$\begin{aligned} \mathbf{b}(\varphi,y) + (\partial_u d(\cdot,\cdot,u)y,\varphi)_I &= -((\partial_u d(\cdot,\cdot,u) - \partial_u d(\cdot,\cdot,v))z(q) + v - u,\varphi)_I \ \forall \varphi \in X, \\ y(T,\cdot) &= 0 \end{aligned}$$

with $u := G(q)$ and $v := G(p)$. Proposition 2.2 and Lemma 2.3, the Lipschitz continuity of $\partial_u d$ and the boundedness of $q \in Q_{\text{ad}}$ guarantee

$$\begin{aligned} \|y\|_I \leq c(\|(\partial_u d(\cdot,\cdot,u) - \partial_u d(\cdot,\cdot,v))z(q)\|_I + \|u-v\|_I) \\ \leq c(\|z(q)\|_{\infty,\infty} + 1)\|u-v\|_I \leq c\|p-q\|_I. \end{aligned}$$

For a second-order analysis of our model problem we formulate the following second-order sufficient condition.

Assumption 2.2 *Let $\bar{q} \in Q_{ad}$ fulfill the first-order necessary optimality conditions* (2.11). *We assume that there exists a constant $\gamma > 0$ such that*

$$j''(\bar{q})(\delta q, \delta q) \geq \gamma \|\delta q\|_I^2 \quad \forall\, \delta q \in Q$$

In this assumption, we have used the differentiability of j in $Q = L^\infty(I \times \Omega)$. We will however show in the next lemma, that the second derivative of j is Lipschitz continuous in $L^2(I \times \Omega)$. Eventually, this will allow to prove a quadratic growth condition without two-norm-discrepancy. For convenience, we point out that for $q \in Q_{\text{ad}}$ and $\delta q_1, \delta q_2 \in Q$ the second derivative of j is given by

$$j''(q)(\delta q_1, \delta q_2) = \int_0^T \int_\Omega (\tilde{u}_1 \tilde{u}_2 - z(q) \partial_{uu} d(\cdot,\cdot,u) \tilde{u}_1 \tilde{u}_2 + \nu \delta q_1 \delta q_2)\, dx\, dt,$$

with $u = G(q)$, $\tilde{u}_i = G'(q)\delta q_i$, $i = 1,2$. We refer to, e. g., [28] for details.

Remark 2.1 Let us point out that Assumption 2.2 is a rather strong second-order sufficient condition. Weaker conditions including (strongly) active sets are known, cf. e. g. [28] for a detailed discussion of elliptic and parabolic control constrained problems, and have been used in the error analysis of elliptic problems, cf. e. g. [7].

Lemma 2.5 *There exists a constant C depending only on the data $\hat{u}$ and d, such that for all $p, q \in Q_{ad}$ and all $\delta q \in Q$*

$$|j''(p)(\delta q, \delta q) - j''(q)(\delta q, \delta q)| \leq C \|p - q\|_I \|\delta q\|_I^2$$

is satisfied.

Proof Let $p, q \in Q_{\text{ad}}$ and $\delta q \in Q$ be given and consider the auxiliary functions

$$u := G(p), \quad \tilde{u} := G'(p)\delta q, \quad v := G(q), \quad \tilde{v} := G'(q)\delta q.$$

Moreover, let $z_p := z(p)$ be the adjoint state associated with p, as well as $z_q := z(q)$ denote the adjoint state associated with q. Direct calculations show that

$$\begin{aligned}
&|j''(p)(\delta q, \delta q) - j''(q)(\delta q, \delta q)| \\
&\quad = \left| \int_0^T \int_\Omega (\tilde{u}^2 - \tilde{v}^2 - z_p \partial_{uu} d(\cdot,\cdot,u) \tilde{u}^2 + z_q \partial_{uu} d(\cdot,\cdot,v) \tilde{v}^2) dx\, dt \right| \\
&\quad \leq \int_0^T \int_\Omega \big| (\tilde{u} + \tilde{v})(\tilde{u} - \tilde{v}) + (z_q - z_p) \partial_{uu} d(\cdot,\cdot,u) \tilde{u}^2 \\
&\qquad - z_q \partial_{uu} d(\cdot,\cdot,u)(\tilde{u} - \tilde{v})(\tilde{u} + \tilde{v}) - z_q (\partial_{uu} d(\cdot,\cdot,u) - \partial_{uu} d(\cdot,\cdot,v)) \tilde{v}^2 \big|\, dx\, dt
\end{aligned}$$

from which we obtain that

$$
\begin{aligned}
&|j''(p)(\delta q,\delta q)-j''(q)(\delta q,\delta q)| \\
&\quad \le (\|\tilde u\|_I+\|\tilde v\|_I)\|\tilde u-\tilde v\|_I + c\|\partial_{uu}d(\cdot,\cdot,u)\|_{\infty,\infty}\|z_p-z_q\|_I\|\tilde u\|^2_{L^4(I\times\Omega))} \\
&\qquad + c\|z_q\|_{\infty,\infty}(\|\partial_{uu}d(\cdot,\cdot,u)\|_{\infty,\infty}(\|\tilde u\|_I+\|\tilde v\|_I)\|\tilde u-\tilde v\|_I+\|u-v\|_I\|\tilde v\|^2_{L^4(I\times\Omega)}).
\end{aligned}
$$

In the last line, we have used the Lipschitz continuity of $\partial_{uu}d$. By the embedding $L^\infty(I,H^1_0(\Omega))\hookrightarrow L^4(I\times\Omega)$ and the stability estimate (2.10) we obtain

$$
\|\tilde u\|_{L^4(I\times\Omega)}\le c\|\delta q\|_I,\quad \|\tilde v\|_{L^4(I\times\Omega)}\le c\|\delta q\|_I.
$$

With the boundedness of Q_{ad} and $\partial_{uu}d$, as well as estimates (2.8), (2.13), (2.14), and (2.16), we obtain the assertion.

Lemma 2.6 *Let $\bar q$ satisfy Assumption 2.2. There exists $\varepsilon>0$ such that*

$$
j''(q)(\delta q,\delta q)\ge \frac{\gamma}{2}\|\delta q\|_I^2
$$

is satisfied for all $\delta q\in Q$ and all $q\in Q_{ad}$ with $\|q-\bar q\|_I\le\varepsilon$.

Proof This follows from Assumption 2.2 and Lemma 2.5 noting

$$
\begin{aligned}
j''(q)(\delta q,\delta q) &= j''(\bar q)(\delta q,\delta q)+(j''(q)(\delta q,\delta q)-j''(\bar q)(\delta q,\delta q)) \\
&\ge \gamma\|\delta q\|_I^2-c\|q-\bar q\|_I\|\delta q\|_I^2.
\end{aligned}
$$

Let us now state a quadratic growth condition in an L^2-neighborhood of a local solution.

Theorem 2.1 *Let Assumption 2.2 be satisfied, and let $\bar q\in Q_{ad}$ fulfill the necessary first order optimality conditions* (2.11)*. Then there are constants $\varepsilon,\sigma>0$ such that the quadratic growth condition*

$$
j(q)\ge j(\bar q)+\sigma\|q-\bar q\|_I^2
$$

is satisfied for all $q\in Q_{ad}$ with $\|q-\bar q\|_I\le\varepsilon$.

Proof We proceed by Taylor expansion of j in $\bar q$. For $q\in Q_{\mathrm{ad}}$ we obtain

$$
j(q)=j(\bar q)+j'(\bar q)(q-\bar q)+\frac12 j''(\bar q^\xi)(q-\bar q,q-\bar q)
$$

with $q^\xi=\bar q+\xi(q-\bar q)$ for a $\xi\in(0,1)$ due to the differentiability of G in $L^\infty(I\times\Omega)$. With the variational inequality (2.11), Assumption 2.2, as well as Lemma 2.5 we obtain

$$
\begin{aligned}
j(q) &= j(\bar q)+j'(\bar q)(q-\bar q)+\frac12 j''(\bar q)(q-\bar q,q-\bar q)+\frac12(j''(q^\xi)-j''(\bar q))(q-\bar q,q-\bar q) \\
&\ge j(\bar q)+\frac{\gamma}{2}\|q-\bar q\|_I^2-c\|q^\xi-\bar q\|_I\|q-\bar q\|_I^2
\end{aligned}
$$

The assertion follows noting that $\|q^\xi-\bar q\|_I=\xi\|q-\bar q\|_I$.

3 The time-discrete problem

In this section we present the discretization of the state equation in time by a discontinuous Galerkin scheme along the lines of [23]. In a first step, we explain the discretization scheme and analyze the time-discrete optimal control problem with respect to existence and regularity of solutions as well as optimality conditions. Then, we derive error estimates for the appearing uncontrolled equations and also develop some auxiliary results.

3.1 Semidiscretization in time

Let a partitioning of the time interval $\bar{I} = [0,T]$ be given as

$$\bar{I} = \{0\} \cup I_1 \cup I_2 \cup \ldots \cup I_M$$

with subintervals $I_m = (t_{m-1}, t_m]$ of size k_m defined by time points

$$0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T.$$

The discretization is characterized by the discretization parameter k defined as a piecewise constant function by setting $k_{|_{I_m}} = k_m$ for $m = 1,2,\ldots,M$. Moreover, k also denotes the maximal size of the time steps, i. e., $k = \max k_m$. We define the semidiscrete trial and test space

$$X_k^r = \{v_k \in L^2(I, H_0^1(\Omega)) | v_{k|_{I_m}} \in \mathscr{P}_r(I_m, H_0^1(\Omega)),\ m = 1,2,\ldots,M\},$$

where $\mathscr{P}_r(I_m, H_0^1(\Omega))$ denotes the space of polynomials up to order r defined on I_m with values in $H_0^1(\Omega)$. The control space Q and the set of admissible controls Q_{ad} remains unchanged, since no control discretization is considered at this point.

Assumption 3.1 *In the sequel, let $r = 0$, i. e. we consider functions that are piecewise constant in time. The semidiscrete state space will therefore be denoted by X_k^0. We will refer to this discretization as $dG(0)$.*

For functions $v_k \in X_k^0$ we define

$$\begin{aligned} v_{k,m}^+ &:= \lim_{t \to 0^+} v_k(t_m + t) = v_k(t_{m+1}) =: v_{k,m+1}, \\ v_{k,m}^- &:= \lim_{t \to 0^+} v_k(t_m - t) = v_k(t_m) =: v_{k,m}, \\ [v_k]_m &:= v_{k,m}^+ - v_{k,m}^- = v_{k,m+1} - v_{k,m} \end{aligned}$$

and introduce the short notation

$$(v,w)_{I_m} := (v,w)_{L^2(I_m, L^2(\Omega))},\ \|v\|_{I_m} := \|v\|_{L^2(I_m, L^2(\Omega))}.$$

The semidiscrete version of the bilinear form $\mathbf{b}(\cdot,\cdot)$ for $u_k, \varphi \in X_k^0$ is given by

$$\begin{aligned} \mathbf{B}(u_k, \varphi) :=& \sum_{m=1}^{M} (\partial_t u_k, \varphi)_{I_m} + (\nabla u_k, \nabla \varphi)_I + \sum_{m=2}^{M} ([u_k]_{m-1}, \varphi_{m-1}^+) + (u_{k,0}^+, \varphi_0^+) \\ =& \ (\nabla u_k, \nabla \varphi)_I + \sum_{m=2}^{M} (u_{k,m} - u_{k,m-1}, \varphi_m) + (u_{k,1}, \varphi_1), \end{aligned}$$

and the $dG(0)$ semidiscretization of the state equation (2.1) for fixed control $q \in Q$ reads as follows: Find a state $u_k = u_k(q) \in X_k^0$ such that

$$\mathbf{B}(u_k, \varphi) + (d(\cdot,\cdot,u_k),\varphi)_I = (q,\varphi)_I + (u_0,\varphi_1) \quad \forall \varphi \in X_k^0. \tag{3.1}$$

A crucial part in the analysis of the semidiscrete problem is to show boundedness of semidiscrete solutions $u_k = u_k(q) \in X_k^0$ in $L^\infty(I \times \Omega)$ independently of the discretization parameter k.

Theorem 3.1 *For every fixed control $q \in Q$ and initial state $u_0 \in H_0^1(\Omega) \cap L^\infty(I \times \Omega)$, the semidiscrete state equation* (3.1) *admits a unique semidiscrete solution $u_k \in X_k^0 \cap L^\infty(I \times \Omega)$ satisfying the boundedness result*

$$\|u_k\|_{\infty,\infty} \le c(\|q\|_{L^p(I\times\Omega)} + \|d(\cdot,\cdot,0)\|_{L^p(I\times\Omega)} + \|u_0\|_\infty) \tag{3.2}$$

for every $p > 2$ and a constant c independent of the discretization parameter k.

Proof The existence of a solution follows by applying standard arguments from elliptic theory to the system of semilinear elliptic PDEs for each time interval obtained after semidiscretization in time. For $r = 0$, only one elliptic equation remains. It is sufficient to prove the assertion on the interval I_1. We obtain the formulation

$$k_1(\nabla u_{k,1}, \nabla \varphi_1) + (u_{k,1},\varphi_1) + (d(t,x,u_{k,1}),\varphi_1)_{I_1} = (q,\varphi_1)_{I_1} + (u_0,\varphi_1),$$

where q and u_0 are given.
With $\tilde{d}(x,u_{k,1}) := \int_{I_1} d(t,x,u_{k,1}(x))dt$ and $\tilde{q}(x) = \int_{I_1} q(t,x)dt \in L^\infty(\Omega)$, this yields

$$k_1(\nabla u_{k,1}, \nabla \varphi_1) + (u_{k,1},\varphi_1) + (\tilde{d}(\cdot,u_{k,1}),\varphi_1) = (\tilde{q},\varphi_1) + (u_0,\varphi_1),$$

where the monotonicity and Lipschitz properties of d remain valid for $\tilde{d}$. It is known from elliptic theory that for each $u_0, \tilde{q} \in L^2(\Omega)$ this equation admits a unique solution $u_{k,1} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, c.f. for example [28]. This implies the existence of a semidiscrete solution $u_k \in X_k^0 \cap L^\infty(I \times \Omega)$ to (3.1).

To obtain a stability estimate in the $L^\infty(I \times \Omega)$-norm independent of the discretization parameter k, we apply Stampacchia's method and follow closely the proof of [28, Theorem 4.5] for elliptic equations. Let therefore $b \ge \|u_0\|_\infty$ be a real number and choose a test-function $v_k \in X_k^0$ such that

$$v_{k,m} = \begin{cases} u_{k,m} - b \text{ on } \Omega_m^+(b) := \{x \in \Omega : u_{k,m}(x) > b\} \\ u_{k,m} + b \text{ on } \Omega_m^-(b) := \{x \in \Omega : u_{k,m}(x) < -b\} \\ 0 \qquad\quad\; \text{on } \Omega \setminus (\Omega_m^+(b) \cup \Omega_m^-(b)). \end{cases}$$

For brevity, we will write Ω_m^+ instead of $\Omega_m^+(b)$ as well as Ω_m^- instead of $\Omega_m^-(b)$.

It is easily verified that

$$(d(\cdot,\cdot,u_k) - d(\cdot,\cdot,0), v_k)_{I_m} \ge 0 \quad \forall m = 1,\dots,M \tag{3.3}$$

as well as

$$\|\nabla v_{k,m}\|^2 = (\nabla v_{k,m}, \nabla u_{k,m}) \quad \forall m = 1,\dots,M, \tag{3.4}$$

hold. Now observe that for $m = 2, \dots, M$, we have

$$(u_{k,m} - u_{k,m-1}, v_{k,m}) = (v_{k,m} + b - u_{k,m-1}, v_{k,m})_{\Omega_m^+} + (v_{k,m} - b - u_{k,m-1}, v_{k,m})_{\Omega_m^-},$$

where the index Ω_m^+ or Ω_m^- denotes L^2 inner product on the respective set. On Ω_m^+, we obtain

$$\begin{aligned}
&(v_{k,m} + b - u_{k,m-1}, v_{k,m})_{\Omega_m^+} \\
&\quad = (v_{k,m} + b - u_{k,m-1}, v_{k,m})_{\Omega_m^+ \cap \Omega_{m-1}^+} + (v_{k,m} + b - u_{k,m-1}, v_{k,m})_{\Omega_m^+ \cap \Omega_{m-1}^-} \\
&\qquad\qquad + (v_{k,m} + b - u_{k,m-1}, v_{k,m})_{\Omega_m^+ \setminus (\Omega_{m-1}^- \cup \Omega_{m-1}^+)} \\
&\quad \geq (v_{k,m} - v_{k,m-1}, v_{k,m})_{\Omega_m^+ \cap \Omega_{m-1}^+} + (v_{k,m} - v_{k,m-1} + 2b, v_{k,m})_{\Omega_m^+ \cap \Omega_{m-1}^-} \\
&\qquad\qquad + (v_{k,m}, v_{k,m})_{\Omega_m^+ \setminus (\Omega_{m-1}^+ \cup \Omega_{m-1}^-)} \\
&\quad \geq (v_{k,m} - v_{k,m-1}, v_{k,m})_{\Omega_m^+}, \qquad (3.5)
\end{aligned}$$

since $v_{k,m-1} = 0$ on $\Omega_m^+ \setminus (\Omega_{m-1}^+ + \Omega_{m-1}^-)$. With similar calculations on Ω_m^-, we arrive at

$$(u_{k,m} - u_{k,m-1}, v_{k,m}) \geq (v_{k,m} - v_{k,m-1}, v_{k,m}) \quad \forall m = 2, \dots, M. \qquad (3.6)$$

For $m = 1$, we proceed similarly, taking into account the term $(u_0, v_{k,1})$ from the right-hand-side of (3.1). On Ω_1^+, we observe

$$(u_{k,1} - u_0, v_{k,1})_{\Omega_1^+} = (v_{k,1} + b - u_0, v_{k,1})_{\Omega_1^+} \geq \|v_{k,1}\|_{\Omega_1^+}^2,$$

where the last inequality follows from the fact that $b \geq \|u_0\|_\infty$ and $v_{k,1} > 0$ on Ω_1^+. Together with analogous calculations on Ω_1^-, this yields

$$(u_{k,1} - u_0, v_{k,1}) \geq \|v_{k,1}\|^2. \qquad (3.7)$$

From (3.3)–(3.7), we obtain

$$\mathbf{B}(v_k, v_k) \leq \mathbf{B}(u_k, v_k) - (u_0, v_{k,1}) \leq (q - d(\cdot,\cdot,0), v_k)_I,$$

which in particular implies

$$\|v_k\|^2_{L^\infty(I, L^2(\Omega)) \cap L^2(I, H_0^1(\Omega))} \leq (q - d(\cdot,\cdot,0), v_k)_I,$$

as in the proof of Theorem 3.2. Note that we have the embeddings $H_0^1(\Omega) \hookrightarrow L^\sigma(\Omega)$, as well as $L^\infty(I, L^2(\Omega)) \hookrightarrow L^\sigma(I, L^2(\Omega))$ for all $1 < \sigma < \infty$ since Ω is two dimensional. Then, known interpolation error estimates, cf. [27], yield

$$L^2(I, H_0^1(\Omega)) \cap L^\sigma(I, L^2(\Omega)) \hookrightarrow L^{r_s}(I, [H_0^1(\Omega), L^2(\Omega)]_s) \hookrightarrow L^{r_s}(I, L^{q_s}(\Omega))$$

with

$$\frac{1}{r_s} = \frac{1-s}{2} + \frac{s}{\sigma}, \quad \frac{1}{q_s} = \frac{1-s}{\sigma} + \frac{s}{2}.$$

Choosing $s=\frac{1}{2}$ yields $r_s=q_s=\frac{4\sigma}{2+\sigma}$, which is monotonically increasing in σ. Hence, we deduce that

$$L^\infty(I,L^2(\Omega))\cap L^2(I,H_0^1(\Omega))\hookrightarrow L^\tau(I\times\Omega)$$

is satisfied for any positive real number $\tau<4$. Then, for any $p>2$ there exist $\lambda>1$ and $p'>0$ such that $\frac{1}{p}+\frac{1}{p'}=1$ and $p'=\frac{\tau}{2\lambda}$ is satisfied and the estimate

$$\begin{aligned}\|v_k\|^2_{L^\tau(I\times\Omega)} &\le c\|q-d(\cdot,\cdot,0)\|_{L^p(I\times\Omega)}\|v_k\|_{L^{p'}(I\times\Omega)}\\ &\le c\|q-d(\cdot,\cdot,0)\|_{L^p(I\times\Omega)}\|v_k\|_{L^{2p'}(I\times\Omega)}|\mathscr{I}(b)|^{\frac{1}{2p'}}\end{aligned}$$

is true. Here, $\mathscr{I}(b)\subset I\times\Omega$ is given by

$$\mathscr{I}(b):=\{(t,x)\in I\times\Omega\mid |u_k(t,x)|>b\}.$$

With the chosen parameters $p,p',\tau>0$ we hence obtain

$$\begin{aligned}\|v_k\|^2_{L^\tau(I\times\Omega)} &\le c\|q-d(\cdot,\cdot,0)\|_{L^p(I\times\Omega)}\|v_k\|_{L^\tau(I\times\Omega)}|\mathscr{I}(b)|^{\frac{1}{2p'}}\\ &\le c\|q-d(\cdot,\cdot,0)\|^2_{L^p(I\times\Omega)}|\mathscr{I}(b)|^{\frac{1}{p'}}+c_\varepsilon\|v_k\|^2_{L^\tau(I\times\Omega)},\end{aligned}$$

by Young's inequality. Choosing c_ε sufficiently small, we obtain

$$\|v_k\|^2_{L^\tau(I\times\Omega)}\le c\|q-d(\cdot,\cdot,0)\|^2_{L^p(I\times\Omega)}|\mathscr{I}(b)|^{\frac{2}{\tau}\lambda}.$$

By the definition of v_k, this yields

$$\left(\int\limits_{\mathscr{I}(b)}(|u_k|-b)^\tau dx\,dt\right)^{\frac{2}{\tau}}\le c\|q-d(\cdot,\cdot,0)\|^2_{L^p(I\times\Omega)}|\mathscr{I}(b)|^{\frac{2}{\tau}\lambda}.$$

For every $\tilde{b}>b$ we know that $|\mathscr{I}(\tilde{b})|\le|\mathscr{I}(b)|$, and we can estimate

$$\left(\int\limits_{\mathscr{I}(b)}(|u_k|-b)^\tau dx\,dt\right)^{\frac{2}{\tau}}\ge\left(\int\limits_{\mathscr{I}(\tilde{b})}(\tilde{b}-b)^\tau\right)^{\frac{2}{\tau}}\ge(\tilde{b}-b)^2|\mathscr{I}(\tilde{b})|^{\frac{2}{\tau}}.$$

This finally yields

$$(\tilde{b}-b)^2|\mathscr{I}(\tilde{b})|^{\frac{2}{\tau}}\le c\|q-d(\cdot,\cdot,0)\|^2_{L^p(I\times\Omega)}|\mathscr{I}(b)|^{\frac{2}{\tau}\lambda}.$$

Applying Lemma 7.5 from [28] yields that $|\mathscr{I}(b)|=0$ for b large enough, i.e.

$$u_k(t,x)\le b\quad\text{a.e. in } I\times\Omega.$$

The desired estimate (3.2) also follows from [28, Lemma 7.5].

Corollary 3.1 *As an immediate consequence of the last theorem we obtain that for all controls $q\in Q_{ad}$ the associated states $u_k(q)$ are uniformly bounded in $L^\infty(I\times\Omega)$ independent of k, and by the boundedness of Q_{ad} also independent of q.*

Now, we complete the semidiscrete analogue to Proposition 2.1 by a stability estimate that we will essentially rely on when proving our error estimates.

Theorem 3.2 *For the solution $u_k \in X_k^0$ of the dG(0) semidiscretized state equation* (3.1) *with right-hand-side $q \in Q$ and initial condition $u_0 \in H_0^1(\Omega)$, the stability estimate*

$$\|u_k\|_I^2 + \|u_k\|_{\infty,2} + \|\nabla u_k\|_I^2 + \|\nabla u_k\|_{\infty,2} + \|\Delta u_k\|_I^2 + \sum_{m=1}^{M} k_m^{-1}\|[u_k]_{m-1}\|^2 \\ \leq C\{\|q\|_I^2 + \|d(\cdot,\cdot,0)\|_I^2 + \|\nabla u_0\|^2\}$$

holds. The constant C depends only on the domain Ω. The jump term $[u_k]_0$ at $t = 0$ is defined as $u_{k,1} - u_0$.

Proof For all $\varphi \in \mathscr{P}_0(I_m, H_0^1(\Omega))$, the solution $u_k \in X_k^0$ of (3.1) satisfies the following system of equations:

$$(\nabla u_k, \nabla\varphi)_{I_m} + (u_{k,m} - u_{k,m-1}, \varphi_m) + (d(\cdot,\cdot,u_k),\varphi)_{I_m} = (q,\varphi)_{I_m}, \quad m = 1,\dots,M. \tag{3.8}$$

Testing this with u_k yields for all $m = 1,\dots,M$

$$\begin{aligned}(q,u_k)_{I_m} &= \|\nabla u_k\|_{I_m}^2 + (d(\cdot,\cdot,u_k),u_k)_{I_m} + ([u_k]_{m-1},u_{k,m}),\\ &= \|\nabla u_k\|_{I_m}^2 + (d(\cdot,\cdot,u_k),u_k)_{I_m} + \frac{1}{2}(\|u_{k,m}\|^2 + \|[u_k]_{m-1}\|^2 - \|u_{k,m-1}\|^2),\end{aligned}$$

and by summation over all $m = 1,\dots,M$, we obtain

$$\|\nabla u_k\|_I^2 + (d(\cdot,\cdot,u_k),u_k)_I + \frac{1}{2}\|u_{k,M}\|^2 + \sum_{i=1}^{M}\|[u_k]_{m-1}\|^2 = (q,u_k)_I + \frac{1}{2}\|u_0\|^2,$$

implying

$$\|\nabla u_k\|_I^2 + (d(\cdot,\cdot,u_k) - d(\cdot,\cdot,0),u_k)_I \leq \frac{1}{2}\|u_0\|^2 + (q - d(\cdot,\cdot,0),u_k)_I.$$

By the monotonicity of d and Young's inequality we arrive at

$$\|\nabla u_k\|_I^2 \leq c(\|q\|_I^2 + \|d(\cdot,\cdot,0)\|_I^2 + \|\nabla u_0\|^2). \tag{3.9}$$

Then, by Poincaré's inequality we also obtain

$$\|u_k\|_I^2 \leq c(\|q\|_I^2 + \|d(\cdot,\cdot,0)\|_I^2 + \|\nabla u_0\|^2.$$

The terms $\|\Delta u_k\|_I^2$ and $\sum_{m=1}^{M} k_m^{-1}\|[u_k]_{m-1}\|^2$ are estimated by applying [23, Theorem 4.1] to the linear equation

$$\mathbf{B}(u_k,\varphi) = (q - d(\cdot,\cdot,u_k),\varphi)_I + (u_0,\varphi_1), \quad \forall\varphi \in X_k^0,$$

utilizing

$$\|q - d(\cdot,\cdot,u_k)\|_I^2 \leq c(\|q\|_I^2 + \|d(\cdot,\cdot,0)\|_I^2 + \|d(\cdot,\cdot,0) - d(\cdot,\cdot,u_k)\|_I^2)$$

noting that

$$\|d(\cdot,\cdot,0)-d(\cdot,\cdot,u_k)\|_I^2 \le L\|u_k\|_I^2 \le c(\|q\|_I^2+\|d(\cdot,\cdot,0)\|_I^2)+\|\nabla u_0\|^2.$$

since d is locally Lipschitz continuous and u_k is uniformly bounded for all $q \in Q$ due to Theorem 3.1. The proof in [23, Theorem 4.1] also gives rise to the stability estimate in $L^\infty(I,H_0^1(\Omega))$, because integrating (3.8) by parts and testing by $\varphi = -\Delta u_k$ yields

$$\|\Delta u_k\|_{I_m}^2 + ([\nabla u_k]_{m-1}, \nabla u_{k,m}) = (q-d(\cdot,\cdot,0)+d(\cdot,\cdot,0)-d(\cdot,\cdot,u_k), -\Delta u_k)_{I_m}$$

for all $m=1,\dots,M$. Using

$$([\nabla u_k]_{m-1}, \nabla u_{k,m}) = \frac{1}{2}(\|\nabla u_{k,m}\|^2 + \|[\nabla u_k]_{m-1}\|^2 - \|\nabla u_{k,m-1}\|^2)$$

as well as the Lipschitz property of d, we obtain after summation over all m

$$\frac{1}{2}\|\nabla u_{k,M}\|^2 + \|\Delta u_k\|_I^2 \le (q-d(\cdot,\cdot,0), -\Delta u_k)_I + c\|u_k\|_I\|\Delta u_k\|_I + \frac{1}{2}\|\nabla u_0\|_I^2$$

With Young's inequality and (3.9) we obtain the remaining estimates.

With the existence and regularity results for the semidiscrete state equation, it is now possible to define a semidiscrete control-to-state mapping $G_k \colon Q \to X_k^0$, $q \mapsto G_k(q)$, where $G_k(q)$ is the solution of (3.1). Note that also G_k is of class C^2, with first and second order derivatives being the semi-discretized versions of (2.6) and (2.7), given by

$$\mathbf{B}(\tilde{u}_k,\varphi) + (\partial_u d(\cdot,\cdot,u_k)\tilde{u}_k,\varphi)_I = (\delta q,\varphi)_I \;\forall \varphi \in X_k^0, \tag{3.10}$$

as well as

$$\mathbf{B}(\tilde{w}_k,\varphi) + (\partial_u d(\cdot,\cdot,u_k)\tilde{w}_k,\varphi)_I = (-\partial_{uu} d(\cdot,\cdot,u_k)\tilde{u}_{k,1}\tilde{u}_{k,2},\varphi)_I \;\forall \varphi \in X_k^0, \tag{3.11}$$

where $u_k = G_k(q)$, $\tilde{u}_{k,1} = G_k'(q)\delta q_1$ and $\tilde{u}_{k,2} = G_k'(q)\delta q_2$. Again, it is quite obvious that the stability estimates of Theorems 3.1 and 3.2 are also valid for linearized semidiscrete state equations, and we obtain

$$\|\tilde{u}_k\|_I \le c\|\delta q\|_I \tag{3.12}$$

$$\|\tilde{u}_k\|_{\infty,\infty} \le c\|\delta q\|_{L^p(I\times\Omega)} \tag{3.13}$$

$$\|\tilde{u}_k\|_{\infty,2} + \|\nabla \tilde{u}_k\|_{\infty,2} \le c\|\delta q\|_I \tag{3.14}$$

for any $p > 2$. Introducing the semidiscrete reduced objective function

$$j_k : Q \to \mathbb{R}_0^+, \quad q \mapsto J(q, G_k(q)),$$

which is also of class C^2, we obtain the reduced semidiscrete problem formulation

$$\min_{q_k \in Q_{\mathrm{ad}}} j_k(q_k). \tag{3.15}$$

By standard arguments, it is clear that there exists at least one solution $(\bar{q}_k, \bar{u}_k) \in Q_{\mathrm{ad}} \times X_k^0$. Similarly to the continuous case, we define a semidiscrete local solution.

Definition 3.1 A control $\bar{q}_k \in Q_{\mathrm{ad}}$ is called a semidiscrete local solution to (3.15) in the sense of $L^2(I \times \Omega)$ if there exists an $\varepsilon > 0$ such that for all $q \in Q_{\mathrm{ad}}$ with $\|q - \bar{q}_k\|_I \leq \varepsilon$

$$j_k(q) \geq j_k(\bar{q}_k)$$

is satisfied.

The semidiscrete first order necessary optimality condition for $\bar{q}_k \in Q_{\mathrm{ad}}$ reads

$$j_k'(\bar{q}_k)(p - \bar{q}_k) \geq 0 \quad \forall p \in Q_{\mathrm{ad}}. \tag{3.16}$$

We define the semidiscrete adjoint state $z_k = z_k(q) \in X_k^0$ as the solution of the semidiscrete adjoint equation

$$\mathbf{B}(\varphi, z_k) + (\varphi, \partial_u d(\cdot,\cdot,u_k(q))z_k) = (\varphi, u_k(q) - \hat{u})_I \quad \forall \varphi \in X_k^0. \tag{3.17}$$

With arguments very similar to the proof of Theorems 3.1 and 3.2, as well as the proofs in [23] we can show existence as well as stability estimates for the adjoint equation. For a more general right-hand-side $g \in L^\infty(I \times \Omega)$ and terminal condition $z_T \in H_0^1(\Omega)$ we consider the semidiscretized dual equation

$$\mathbf{B}(\varphi, z_k) + (\varphi, \partial_u d(\cdot,\cdot,u_k(q))z_k)_I = (\varphi, g)_I + (\varphi_M, z_T) \quad \forall\, \varphi \in X_k^0. \tag{3.18}$$

and obtain the following result applicable to (3.17).

Corollary 3.2 *For each right-hand-side $g \in L^p(I \times \Omega)$, $p > 2$, and terminal condition $z_T \in H_0^1(\Omega) \cap L^\infty(\Omega)$, there exists a unique solution $z_k \in X_k^0 \cap L^2(I, H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(I \times \Omega)$ of the semidiscrete dual equation* (3.18) *such that the estimates*

$$\|z_k\|_{\infty,\infty} \leq C(\|g\|_{L^p(I\times\Omega)} + \|z_T\|_\infty)$$

$$\|z_k\|_{\infty,2} + \|\nabla z_k\|_{\infty,2} \leq C(\|g\|_I + \|z_T\|)$$

$$\|z_k\|_I^2 + \|\nabla z_k\|_I^2 + \|\Delta z_k\|_I^2 + \sum_{m=1}^{M} k_m^{-1} \|[z_k]_m\|^2 \leq C(\|g\|_I^2 + \|\nabla z_T\|^2)$$

hold with a constant $C > 0$. Here, the jump term $[z_k]_M$ at $t = T$ is defined as $z_T - z_{k,M}$.

It is clear that for each $q \in Q_{\mathrm{ad}}$ and $g := u_k(q) - \hat{u}$ as well as $z_T = 0$ the solution z_k of (3.17) is uniformly bounded independent of k, i. e. we have the estimate

$$\|z_k(q)\|_{\infty,\infty} \leq c$$

for a constant $c > 0$ independent of the discretization parameter k.

Similar to the continuous problem, we can express the derivative of j_k at $\bar{q}_k$ in the direction $p - \bar{q}_k$ as

$$j_k'(\bar{q}_k)(p - \bar{q}_k) = (\nu \bar{q}_k + z_k(\bar{q}_k), p - \bar{q}_k)_I,$$

and also make use of the projection formula

$$\bar{q}_k = P_{Q_{\mathrm{ad}}}\left(-\frac{1}{\nu} z_k(\bar{q}_k)\right)$$

on this level of discretization. This implies in particular that any optimal control $\bar{q}_k$ is piecewise constant in time.

Lemma 3.1 *Let $p, q \in Q_{ad}$ and $\delta q \in Q$ be given. Then there exists a constant $C > 0$ independent of k such that*

$$\begin{aligned}\|G_k(p) - G_k(q)\|_I \leq & C\|p-q\|_I \\ \|G_k'(p)\delta q - G_k'(q)\delta q\|_I \leq & C\|p-q\|_I\|\delta q\|_I\end{aligned}$$

is fulfilled.

Proof The proof is exactly the same as in the continuous case from Lemma 2.3, making use of the boundedness result of Theorem 3.1, the stability estimates of Theorem 3.2, the corresponding estimates for linearized equations, as well as the boundedness of Q_{ad}.

Proposition 3.1 *Let $p, q \in Q_{ad}$ be given. Then there exists a constant $C > 0$ such that the estimate*

$$\|z_k(p) - z_k(q)\|_I \leq C\|p-q\|_I$$

is satisfied.

Proof Again, the proof follows as in the continuous case making use of the appropriate semidiscrete existence and stability results.

We point out that the uniform boundedness results for the state equation and linearized state equation, which are independent of the discretization parameter, plays a central role in the last results.

3.2 Analysis of the temporal discretization error of uncontrolled equations

In this section, we provide error estimates for the uncontrolled semidiscrete state equation, associated linearized equations, as well as the adjoint equation. We begin with an estimate for the state.

Let $u \in X$ be the solution of the state equation (2.1) for a fixed $q \in Q$, and let $u_k \in X_k^0$ be the solution of the corresponding semidiscretized equation (3.1). We emphasize that the solutions $u \in X$ and $u_k \in X_k^0$ possess the regularity $\partial_t u, \nabla^2 u \in L^2(I \times \Omega)$ as well as $\nabla^2 u_k \in L^2(I \times \Omega)$, which is guaranteed by Proposition 2.1 and Theorem 3.2. We will estimate the temporal discretization error

$$e_k := u - u_k,$$

and begin by defining the semidiscrete projection $\pi_k \colon C(\bar{I}, H_0^1(\Omega)) \to X_k^0$ for $m = 1, 2, \ldots, M$ by $\pi_k u_{|I_m} \in \mathscr{P}_0(I_m, H_0^1(\Omega))$,

$$\pi_k u(t_m) = u(t_m).$$

The projection is applicable to the state u, since it belongs to $C(\bar{I}, H_0^1(\Omega))$. For short notation, we introduce the abbreviations $\eta_k := u - \pi_k u$ and $\xi_k := \pi_k u - u_k$ and split the error $e_k = \eta_k + \xi_k$. Before estimating the error e_k, let us point out a helpful result. Noting that both the exact solution $u = u(q) \in X$ and the semidiscrete solution $u_k = u_k(q) \in X_k^0$ satisfy the identities

$$\begin{aligned} \mathbf{B}(u,\varphi) + (d(\cdot,\cdot,u),\varphi)_I &= (q,\varphi)_I + (u_0,\varphi_1) \quad \forall \varphi \in X_k^0, \\ \mathbf{B}(u_k,\varphi) + (d(\cdot,\cdot,u_k),\varphi)_I &= (q,\varphi)_I + (u_0,\varphi_1) \quad \forall \varphi \in X_k^0, \end{aligned}$$

we obtain

$$\mathbf{B}(\xi_k + \eta_k, \varphi) = \mathbf{B}(e_k,\varphi) = -(d(\cdot,\cdot,u) - d(\cdot,\cdot,u_k),\varphi)_I \quad \forall\, \varphi \in X_k^0 \tag{3.19}$$

as an analogue to Galerkin orthogonality for linear state equations. We will make use of (3.19) quite frequently in the sequel.

Theorem 3.3 *For the error $e_k := u - u_k$ between the continuous solution $u \in X$ of* (2.1) *and the dG(0) semidiscretized solution $u_k \in X_k^0$ of* (3.1) *the error estimate*

$$\|e_k\|_I \le Ck\|\partial_t u\|_I$$

holds with a constant C that is independent of the size of the time steps k.

Proof Following an idea from [5], see also [21], we introduce a function $\tilde{d}$ defined by

$$\tilde{d}(t,x) = \begin{cases} \frac{d(t,x,u(t,x)) - d(t,x,u_k(t,x))}{u(t,x) - u_k(t,x)} & \text{if } u(t,x) \neq u_k(t,x) \\ 0 & \text{else.} \end{cases}$$

Note that $\|\tilde{d}\|_{\infty,\infty} \le c$ for a $c > 0$ due to the boundedness of Q_{ad}. In addition, we define $\tilde{z}_k \in X_k^0$ to be the solution of the auxiliary dual equation

$$\mathbf{B}(\varphi,\tilde{z}_k) + (\varphi,\tilde{d}\tilde{z}_k)_I = (\varphi,e_k)_I \quad \forall\, \varphi \in X_k^0. \tag{3.20}$$

Testing (3.20) with ξ_k and making use of (3.19) yields

$$\begin{aligned} (\xi_k,e_k)_I &= \mathbf{B}(\xi_k,\tilde{z}_k) + (\xi_k,\tilde{d}\tilde{z}_k) \\ &= -\mathbf{B}(\eta_k,\tilde{z}_k) - (d(\cdot,\cdot,u) - d(\cdot,\cdot,u_k),\tilde{z}_k)_I + (e_k - \eta_k,\tilde{d}\tilde{z}_k). \end{aligned}$$

By the definition of $\tilde{d}$ and [23, Lemma 9], that guarantees the identity

$$\mathbf{B}(\eta_k,\varphi) = (\nabla\eta_k,\nabla\varphi)_I \quad \forall\, \varphi \in X_k^0,$$

we obtain

$$(\xi_k,e_k)_I = -(\nabla\eta_k,\nabla\tilde{z}_k)_I - (\eta_k,\tilde{d}\tilde{z}_k) = (\eta_k,\Delta\tilde{z}_k)_I - (\eta_k,\tilde{d}\tilde{z}_k).$$

Then, the Lipschitz continuity of d as well as Corollary 3.2 yield

$$\|e_k\|_I^2 \le \|\eta_k\|_I\|\Delta\tilde{z}_k\|_I + \|\tilde{d}\|_{\infty,\infty}\|\eta_k\|_I\|\tilde{z}_k\|_I + \|\eta_k\|_I\|e_k\|_I \le c\|\eta_k\|_I\|e_k\|_I.$$

With the well known estimate $\|\eta_k\|_{I_m} \le Ck_m\|\partial_t u\|_{I_m}$ we obtain the assertion.

We prove a similar estimate for linearized equations, which is needed to prove a coercivity result for the second derivative of j_k.

Proposition 3.2 *For fixed $q \in Q_{ad}$ let $u = u(q) \in X$ be given by the solutions of the state equation* (2.1) *and let $\tilde{u} := G'(q)\delta q$ for $\delta q \in L^\infty(I \times \Omega)$. Moreover, let $u_k \in X_k^0$ be determined as solution of the semidiscrete state equation* (3.1)*, and let $\tilde{u}_k \in X_k^0$ denote the solution $\tilde{u}_k := G_k'(q)\delta q$. Then the error estimate*

$$\|\tilde{u} - \tilde{u}_k\|_I \leq Ck\|\delta q\|_I$$

is satisfied for a constant $C > 0$ independent of the time discretization.

Proof We observe first that

$$\|\tilde{u} - \tilde{u}_k\|_I \leq \|\tilde{u} - \tilde{w}\|_I + \|\tilde{w} - \tilde{u}_k\|_I, \tag{3.21}$$

where $\tilde{w} \in X$ is defined by

$$\mathbf{b}(\tilde{w}, \varphi) + (\partial_u d(\cdot,\cdot,u_k)\tilde{w}, \varphi)_I = (\delta q, \varphi)_I \quad \forall \varphi \in X, \quad \tilde{w}(0,\cdot) = 0.$$

Noting that $\tilde{u}$ fulfills

$$\mathbf{b}(\tilde{u}, \varphi) + (\partial_u d(\cdot,\cdot,u)\tilde{u}, \varphi)_I = (\delta q, \varphi)_I \quad \forall \varphi \in X, \quad \tilde{u}(0,\cdot) = 0.$$

we obtain for the first term in (3.21) that

$$\mathbf{b}(\tilde{u} - \tilde{w}, \varphi) + (\partial_u d(\cdot,\cdot,u)(\tilde{u} - \tilde{w}), \varphi)_I = ((\partial_u d(\cdot,\cdot,u_k) - \partial_u d(\cdot,\cdot,u))\tilde{w}, \varphi)_I \tag{3.22}$$

is satisfied for all $\varphi \in X$. Moreover, we define the auxiliary adjoint state $\tilde{z} \in X$ as the solution of

$$\mathbf{b}(\varphi, \tilde{z}) + (\varphi, \partial_u d(\cdot,\cdot,u_k)\tilde{z})_I = (\varphi, \tilde{u} - \tilde{w})_I \quad \forall \varphi \in X.$$

Then, we obtain

$$\|\tilde{u} - \tilde{w}\|_I^2 = \mathbf{b}(\tilde{u} - \tilde{w}, \tilde{z}) + (\tilde{u} - \tilde{w}, \partial_u d(\cdot,\cdot,u_k)\tilde{z})_I,$$

which, combined with (3.22) yields

$$\begin{aligned}
\|\tilde{u} - \tilde{w}\|_I^2 &\leq ((\partial_u d(\cdot,\cdot,u_k) - \partial_u d(\cdot,\cdot,u))\tilde{u}, \tilde{z})_I \\
&\leq \|\partial_u d(\cdot,\cdot,u_k) - \partial_u d(\cdot,\cdot,u)\|_I \|\tilde{u}\|_{L^4(I\times\Omega)} \|\tilde{z}\|_{L^4(I\times\Omega)} \\
&\leq c\|u - u_k\|_I \|\tilde{u}\|_{L^\infty(I,H_0^1(\Omega))} \|\tilde{z}\|_{L^\infty(I,H_0^1(\Omega))} \\
&\leq ck\|\partial_t u\|_I \|\delta q\|_I \|\tilde{u} - \tilde{w}\|_I,
\end{aligned}$$

which implies

$$\|\tilde{u} - \tilde{w}\|_I \leq ck\|\delta q\|_I$$

by the boundedness of Q_{ad}, the Lipschitz continuity of $\partial_u d$, estimate (2.10) and analogous results for adjoint equations from Proposition 2.2, as well as the discretization error estimate from Theorem 3.3. The second term in 3.21 is a pure discretization error for a linear state equation, where the results of [23] can be applied with only minor adaptation due to the term $\partial_u d(\cdot,\cdot,u_k(q))$.

Finally, we also estimate the discretization error for the adjoint equation.

Proposition 3.3 *For fixed $q \in Q_{ad}$ let $u \in X$ and $z \in X$ be given by the solutions of the state equation* (2.1) *and the adjoint equation* (2.12)*, respectively. Moreover, let $u_k \in X_k^0$ and $z_k \in X_k^0$ be determined as solutions of the semidiscrete state equation* (3.1) *and adjoint equation* (3.17)*. Then the error estimate*

$$\|z - z_k\|_I \le Ck\{\|\partial_t u\|_I + \|\partial_t z\|_I\}$$

is satisfied for a constant $C > 0$ independent of the time discretization.

Proof The proof is similar to the one of Proposition 3.2, introducing an auxiliary adjoint $\hat{z} \in X$, defined as the solution of

$$\mathbf{b}(\varphi, \hat{z}) + (\varphi, \partial_u d(\cdot,\cdot,u_k(q))\hat{z})_I = (\varphi, u_k - \hat{u})_I \quad \forall \varphi \in X, \quad \hat{z}(T,\cdot) = 0.$$

We split the error into

$$\|z - z_k\|_I \le \|z - \hat{z}\|_I + \|\hat{z} - z_k\|_I$$

and observe

$$\mathbf{b}(\varphi, z - \hat{z}) + (\varphi, \partial_u d(\cdot,\cdot,u)(z - \hat{z}))_I = (\varphi, (\partial_u d(\cdot,u_k) - \partial_u d(\cdot,\cdot,u))\hat{z})_I,$$

which yields

$$\|z - \hat{z}\|_I \le c\|\hat{z}\|_{\infty,\infty}\|u - u_k\|_I \le ck\|\partial_t u\|_I,$$

making use of Theorem 3.3, Proposition 2.2, and the boundedness of Q_{ad}. The second term to estimate is again a pure discretization error, which can be handled by arguments similar to [23].

Last, we state some properties of j_k''. In the sequel, we will need coercivity of the second derivative of the semidiscrete reduced objective function in the neighborhood of $\bar{q}$ as well as $\bar{q}_k$.

Lemma 3.2 *Let $\bar{q}$ be a local solution of* (2.5) *and let Assumption 2.2 be valid. Then there exists an $\varepsilon > 0$, such that for all $q \in Q_{ad}$ with $\|q - \bar{q}\|_I \le \varepsilon$ and all $p \in L^\infty(I \times \Omega)$*

$$j_k''(q)(p,p) \ge \frac{\gamma}{4}\|p\|_I^2$$

holds for k sufficiently small.

Proof With the definition of j'' and j_k'' we obtain

$$\begin{aligned}
&|j_k''(q)(p,p) - j''(q)(p,p)| \\
&\quad = \left| \int_0^T \int_\Omega \left(\tilde{u}_k^2 - \tilde{u}^2 - z_k \partial_{uu} d(\cdot,\cdot,u_k)\tilde{u}_k^2 + z\partial_{uu} d(\cdot,\cdot,u)\tilde{u}^2\right) dx\, dt \right| \\
&\quad \le (\|\tilde{u}_k\|_I + \|\tilde{u}\|_I)\|\tilde{u} - \tilde{u}_k\|_I + c\|z - z_k\|_I \|\tilde{u}_k\|^2_{L^\infty(I,H_0^1(\Omega))} \\
&\qquad + c\|z\|_{\infty,\infty}(\|\tilde{u}_k\|_I + \|\tilde{u}\|_I)\|\tilde{u} - \tilde{u}_k\|_I + c\|z\|_{\infty,\infty}\|\tilde{u}_k\|^2_{L^\infty(I,H_0^1(\Omega))}\|u_k - u\|_I \\
&\quad \le ck\|p\|_I^2
\end{aligned}$$

by Theorem 3.1, which guarantees $d(\cdot,\cdot,u_k)$ to be uniformly bounded, as well as Theorem 3.3 and Propositions 3.2 and 3.3. Since this tends to zero as k tends to zero, the assertion is obtained by Lemma 2.6, observing that

$$j_k''(q)(p,p) = j''(q)(p,p) + j_k''(q)(p,p) - j''(q)(p,p) \geq \frac{\gamma}{2}\|p\|_I^2 - ck\|p\|_I^2.$$

Lemma 3.3 *There exists a constant $C > 0$ depending only on the data $\hat{u}$ and d, such that for all $p,q \in Q_{ad}$ and all $\delta q \in Q$*

$$|j_k''(p)(\delta q,\delta q) - j_k''(q)(\delta q,\delta q)| \leq c\|p-q\|_I\|\delta q\|_I^2$$

is satisfied.

Proof This follows exactly as in Lemma 2.5, utilizing Lemma 3.1 as well as Corollary 3.2.

4 The discrete problem formulation

4.1 Discretization in space

Now, we introduce the spatial discretization of the optimal control problem. We consider two-dimensional shape regular and quasi-uniform meshes; see, e. g., [9], consisting of quadrilateral cells K, which constitute a nonoverlapping cover of the computational domain Ω. We denote the mesh by $\mathscr{T}_h = \{K\}$ and define the discretization parameter h as a cellwise constant function by setting $h_{|_K} = h_K$ with the diameter h_K of the cell K. We use the symbol h also for the maximal cell size, i. e., $h = \max h_K$.

On the mesh $\mathscr{T}_h$ we construct a conforming finite element space $V_h \subset H_0^1(\Omega)$ in the standard way

$$V_h^s = \{v \in H_0^1(\Omega) | v_{|_K} \in \mathscr{Q}_s(K) \text{ for } K \in \mathscr{T}_h\},$$

where $\mathscr{Q}_s(K)$ consists of shape functions obtained via bilinear transformations of polynomials in $\widehat{\mathscr{Q}_s}(\hat{K})$ defined on the reference cell $\hat{K} = (0,1)^2$; cf. also Section 3.2 in [23].

Assumption 4.1 *In the following, we will assume $s = 1$ in addition to $r = 0$, i. e. we consider functions that are piecewise constant in time and piecewise linear in space.*

Then, we utilize the space-time finite element space

$$X_{k,h}^{0,1} = \{v_{kh} \in L^2(I,V_h^1) | v_{kh|_{I_m}} \in \mathscr{P}_0(I_m,V_h^1),\ m = 1,2,\ldots,M\} \subset X_0^1.$$

The so-called cG(1)dG(0) discretization of the state equation for given control $q \in Q$ has the following form: Find a state $u_{kh} = u_{kh}(q) \in X_{k,h}^{0,1}$ such that

$$\mathbf{B}(u_{kh},\varphi) + (d(\cdot,\cdot,u_{kh}),\varphi)_I = (q,\varphi)_I + (u_0,\varphi_1) \quad \forall \varphi \in X_{k,h}^{0,1}. \tag{4.1}$$

Just like in the semidiscrete case, we proceed by showing an existence- as well as a uniform boundedness result for the solution u_{kh} of the fully discrete state equation for all $q \in Q$ and $u_0 \in H_0^1(\Omega)$. Let u_k be the solution of the semidiscrete state equation (3.1). We define the projection $\pi_h : X_k^0 \to X_{k,h}^{0,1}$ by means of the spatial L^2-projection $\Pi_h \colon H_0^1(\Omega) \to V_h^1$ pointwise in time as

$$(\pi_h u_k)(t) = \Pi_h u_k(t).$$

Assuming for the moment that a solution u_{kh} to (4.1) exists, we define

$$e_h := u_k - u_{kh} = \eta_h + \xi_h,$$

where the errors η_h and ξ_h are defined by

$$\eta_h := u_k - \pi_h u_k, \quad \xi_h := \pi_h u_k - u_{kh}. \tag{4.2}$$

Theorem 4.1 *Let Assumption 2.1 be satisfied. Then, for each $q \in Q$ and initial state $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, there exists a unique solution $u_{kh} \in X_{k,h}^{0,1} \cap L^\infty(I \times \Omega)$ of equation* (4.1). *Moreover, for each $q \in Q$, there is a constant $C > 0$ independent of k and h such that*

$$\|u_{kh}\|_{\infty,\infty} \leq C.$$

is satisfied. Moreover, u_{kh} satisfies the stability estimate

$$\|u_{kh}\|_I^2 + \|u_{kh}\|_{\infty,2} + \|\nabla u_{kh}\|_I^2 + \|\nabla u_{kh}\|_{\infty,2} + \|\Delta_h u_{kh}\|_I^2 + \sum_{m=1}^{M} k_m^{-1} \|[u_{kh}]_{m-1}\|^2 \\ \leq C\{\|q\|_I^2 + \|d(\cdot,\cdot,0)\|_I^2 + \|\Pi_h \nabla u_0\|^2\}$$

Here and in the following, $\Delta_h \colon V_h^1 \to V_h^1$ is defined by

$$(\Delta_h u, \varphi) = -(\nabla u, \nabla \varphi) \quad \forall \varphi \in V_h^1.$$

Proof The existence of u_{kh} is a consequence of Brouwer's fixed point theorem using the monotonicity and Lipschitz continuity of d. To obtain the uniform boundedness, we first prove that $\pi_h u_k$ is uniformly bounded. We consider the pointwise in time interpolant $(i_h u_k)(t) = I_h u_k(t)$, where I_h denotes the Clément interpolant, cf. [10]. Then, by the boundedness of u_k in $L^\infty(I \times \Omega)$ as well as $\nabla u_k \in L^\infty(I, L^2(\Omega))$, we obtain

$$\begin{aligned} \|\pi_h u_k\|_{\infty,\infty} &\leq \|i_h u_k\|_{\infty,\infty} + \|i_h u_k - \pi_h u_k\|_{\infty,\infty} \\ &\leq \|u_k\|_{\infty,\infty} + ch^{-1}\|i_h u_k - \pi_h u_k\|_{\infty,2} \\ &\leq c + ch^{-1}(\|i_h u_k - u_k\|_{\infty,2} + \|\pi_h u_k - u_k\|_{\infty,2}) \\ &\leq c + c\|\nabla u_k\|_{\infty,2} \\ &\leq c \end{aligned}$$

by an inverse estimate and the boundedness of Q_{ad}. Then, note that

$$\mathbf{B}(u_k - u_{kh}, \varphi) + (d(\cdot,\cdot,u_k) - d(\cdot,\cdot,u_{kh}), \varphi)_I = 0 \quad \forall \varphi \in X_{k,h}^{0,1},$$

which with $\varphi = \xi_h$ yields

$$\mathbf{B}(e_h, \xi_h) + (d(\cdot,\cdot,u_k) - d(\cdot,\cdot,\pi_h u_k), \xi_h)_I + (d(\cdot,\cdot,\pi_h u_k) - d(\cdot,\cdot,u_{kh}), \xi_h)_I = 0.$$

From that, we obtain

$$\begin{aligned}\mathbf{B}(\xi_h, \xi_h) &\leq -\mathbf{B}(\eta_h, \xi_h) - (d(\cdot,\cdot,u_k) - d(\cdot,\cdot,\pi_h u_k), \xi_h)_I \\ &\leq \|\nabla \eta_h\|_I \|\nabla \xi_h\|_I + c\|\eta_h\|_I \|\xi_h\|_I\end{aligned}$$

by the monotonicity and Lipchitz continuity of d as well as the properties of $\mathbf{B}$ and π_h, where we used in particular the boundedness of u_k and $\pi_h u_k$. Similarly to [23, Lemma 5.7], one obtains

$$\|\xi_h\|_{\infty,2}^2 + \|\nabla \xi_h\|_I^2 \leq \mathbf{B}(\xi_h, \xi_h),$$

which combined with the previous estimate yields

$$\|\xi_h\|_{\infty,2}^2 + \|\nabla \xi_h\|_I^2 \leq c\|\nabla \eta_h\|_I^2 + c\|\eta_h\|_I^2 \leq ch^2\|q\|_I^2 + ch^4\|q\|_I^2.$$

by Young's inequality and the fact that $\|\xi_h\|_I \leq c\|\nabla \xi_h\|_I$. By an inverse estimate, we obtain $\|\xi_h\|_{\infty,\infty} \leq c\|q\|_I$, which implies boundedness of u_{kh} by the boundedness of u_k as well as $\pi_h u_k$. The second estimate can be proven similarly to Theorem 3.2.

Remark 4.1 We point out that the boundedness of Q_{ad} guarantees uniform boundedness of the discrete states $u_{kh} = u_{kh}(q)$ for all $q \in Q_{\text{ad}}$ by a constant C independent of $k, h,$ and q.

We now proceed as in the semidiscrete setting and introduce the discrete control-to-state operator $G_{kh}\colon Q \to X_{k,h}^{0,1}$, which is also of class C^2 with derivatives that fulfill (3.10) and (3.11) for test functions in $X_{k,h}^{0,1}$. The stability estimates of Theorem 4.1 are valid for linearized equations. In particular, for $q \in Q_{\text{ad}}$ and $\delta q \in Q$ we obtain that the linearized state $\tilde{u}_{kh} := G'_{kh}(q)\delta q$ satisfies

$$\|\tilde{u}_{kh}\|_I \leq c\|\delta q\|_I \tag{4.3}$$

$$\|\tilde{u}_{kh}\|_{\infty,\infty} \leq c \tag{4.4}$$

$$\|\tilde{u}_{kh}\|_{\infty,2} + \|\nabla \tilde{u}_k\|_{\infty,2} \leq c\|\delta q\|_I \tag{4.5}$$

for a constant $c > 0$. Let us also state a Lipschitz continuity result for G_{kh} and its derivatives, which can be shown as in the semidiscrete setting of Lemma 3.1.

Lemma 4.1 *Let $p, q \in Q_{ad}$ and $\delta q \in Q$ be given. Then there exists a constant $C > 0$ independent of k and h such that*

$$\begin{aligned}\|G_{kh}(p) - G_{kh}(q)\|_I &\leq C\|p - q\|_I \\ \|G'_{kh}(p)\delta q - G'_{kh}(q)\delta q\|_I &\leq C\|p - q\|_I \|\delta q\|_I\end{aligned}$$

is fulfilled.

With the discrete reduced objective function

$$j_{kh}: Q_{\mathrm{ad}} \to \mathbb{R}_0^+, \quad q \mapsto J(q, G_{kh}(q))$$

we obtain the discrete problem formulation

$$\min_{q_{kh} \in Q_{\mathrm{ad}}} j_{kh}(q_{kh}). \tag{4.6}$$

The existence of at least one optimal solution $\bar{q}_{kh} \in Q_{\mathrm{ad}}$ is again easily obtained.

Definition 4.1 A control $\bar{q}_{kh} \in Q_{\mathrm{ad}}$ is called a discrete local solution to (4.6) in the sense of $L^2(I \times \Omega)$, if there exists an $\varepsilon > 0$ such that for all $q_{kh} \in Q_{\mathrm{ad}}$ with $\|q_{kh} - \bar{q}_{kh}\|_I \leq \varepsilon$

$$j_{kh}(q_{kh}) \geq j_{kh}(\bar{q}_{kh})$$

is satisfied.

First order necessary optimality condition for $\bar{q}_{kh} \in Q_{\mathrm{ad}}$ are given by

$$j'_{kh}(\bar{q}_{kh})(p - \bar{q}_{kh}) \geq 0 \quad \forall p \in Q_{\mathrm{ad}}, \tag{4.7}$$

where the derivative of j_{kh} can be expressed as

$$j'_{kh}(\bar{q}_{kh})(p - \bar{q}_{kh}) = (\nu \bar{q}_{kh} + z_{kh}(\bar{q}_{kh}), p - \bar{q}_{kh})_I.$$

Here, for every $q \in Q_{\mathrm{ad}}$ and $u_{kh} = G_{kh}(q)$, $z_{kh} = z_{kh}(q) \in X_{k,h}^{0,1}$ denotes the solution of the discrete adjoint equation

$$\mathbf{B}(\varphi, z_{kh}) + (\varphi, \partial_u d(\cdot,\cdot,u_{kh}) z_{kh})_I = (\varphi, u_{kh} - \hat{u})_I \quad \forall \varphi \in X_{k,h}^{0,1}. \tag{4.8}$$

Corollary 4.1 *For the solution $z_{kh} \in X_{k,h}^{0,1}$ of the discrete dual equation* (4.8) *with right-hand-side $g \in L^2(I, H)$ and terminal condition $z_T \in H_0^1(\Omega)$, the estimate*

$$\|z_{kh}\|_I^2 + \|\nabla z_{kh}\|_I^2 + \|\Delta_h z_{kh}\|_I^2 + \sum_{m=1}^{M} k_m^{-1} \|[z_{kh}]_m\|^2 \leq C\{\|g\|_I^2 + \|\Pi_h \nabla z_T\|^2\}$$

holds. Here, the jump term $[z_{kh}]_M$ at $t = T$ is defined as $z_T - z_{kh,M}^-$

As on the continuous and semidiscrete levels, we deduce a projection formula for $\bar{q}_{kh}$:

$$\bar{q}_{kh} = P_{Q_{\mathrm{ad}}}\left(-\frac{1}{\nu} z_{kh}(\bar{q}_{kh})\right).$$

Remark 4.2 Note that on this level of discretization, the controls are still not discretized.

4.2 Analysis of the spatial discretization error for uncontrolled equations

In this section, we prove error estimates for the uncontrolled discrete state and adjoint equations in the L^2-norm.

Theorem 4.2 *For the error $e_h := u_k - u_{kh}$ between the dG(0) semidiscretized solution $u_k \in X_k^0$ of* (3.1) *and the fully cG(1)dG(0) discretized solution $u_{kh} \in X_{k,h}^{0,1}$ of* (4.1)*, we have the error estimate*

$$\|e_h\|_I \leq Ch^2 \|\nabla^2 u_k\|_I,$$

where the constant C is independent of the mesh size h and the size of the time steps k.

Proof Similarly to the proof of Theorem 4.1, we obtain the existence of a constant $c > 0$ such that

$$\|\nabla \xi_h\|_I^2 + \|\xi_h\|_I^2 \leq c(\|\nabla \eta_h\|_I^2 + \|\eta_h\|_I^2), \tag{4.9}$$

with ξ_h and η_h defined as in (4.2). We define $\hat{z}_k \in X_k^0$ as the solution of the auxiliary semidiscrete dual equation

$$\mathbf{B}(\varphi, \hat{z}_k) + (\varphi, \hat{d}\hat{z}_k)_I = (\varphi, e_h)_I \quad \forall\, \varphi \in X_k^0,$$

with

$$\hat{d}(t,x) = \begin{cases} \frac{d(t,x,u_k(t,x)) - d(t,x,u_{kh}(t,x))}{u_k(t,x) - u_{kh}(t,x)} & \text{if } u_k(t,x) \neq u_{kh}(t,x) \\ 0 & \text{else.} \end{cases}$$

and test it with $\varphi = e_h$. With the abbreviation $\eta_h^* := \hat{z}_k - \pi_h \hat{z}_k$ we obtain

$$\begin{aligned} \|e_h\|_I^2 &= \mathbf{B}(e_h, \hat{z}_k) + (e_h, \hat{d}\hat{z}_k)_I \\ &= \mathbf{B}(e_h, \eta_h^*) + \mathbf{B}(e_h, \pi_h \hat{z}_k) + (e_h, \hat{d}\eta_h^*)_I + (e_h, \hat{d}\pi_h \hat{z}_k)_I \\ &= \mathbf{B}(e_h, \eta_h^*) - (d(\cdot,\cdot,u_k) - d(\cdot,\cdot,u_{kh}), \pi_h \hat{z}_k)_I + (e_h, \hat{d}\eta_h^*)_I + (e_h, \hat{d}\pi_h \hat{z}_k)_I, \end{aligned}$$

where the last equality follows from the fact that $\pi_h \hat{z}_k \in X_{k,h}^{0,1}$. By the definition of $\hat{d}$, this simplifies further to

$$\|e_h\|_I^2 = \mathbf{B}(\xi_h, \eta_h^*) + \mathbf{B}(\eta_h, \eta_h^*) + (e_h, \hat{d}\eta_h^*)_I. \tag{4.10}$$

With Lemmas 5.7 and 5.8 from [23], that guarantee

$$\begin{aligned} \mathbf{B}(\xi_h, \eta_h^*) &= (\nabla \xi_h, \nabla \eta_h^*), \\ \mathbf{B}(\eta_h, \eta_h^*) &\leq \|\nabla \eta_h\|_I \|\nabla \eta_h^*\|_I + c\|\eta_h\|_I \|e_h\|_I, \end{aligned}$$

(4.10) can be estimated by

$$\begin{aligned} \|e_h\|_I^2 \leq \|\nabla \xi_h\|_I \|\nabla \eta_h^*\|_I &+ \|\nabla \eta_h\|_I \|\nabla \eta_h^*\|_I \\ &+ c\|\eta_h\|_I \|e_h\|_I + c\|\xi_h\|_I \|\eta_h^*\|_I + c\|\eta_h\|_I \|\eta_h^*\|_I \end{aligned}$$

since u_k, u_{kh}, and $\hat{d}$ are bounded in $L^\infty(I \times \Omega)$. This yields

$$\begin{aligned} \|e_h\|_I^2 \leq c(\|\nabla \eta_h\|_I \|\nabla \eta_h^*\|_I &+ \|\eta_h\|_I \|\nabla \eta_h^*\|_I + \|\nabla \eta_h\|_I \|\eta_h^*\|_I + \\ &\|\eta_h\|_I \|\eta_h^*\|_I + \|\eta_h\|_I \|e_h\|_I). \end{aligned} \tag{4.11}$$

Well-known a priori estimates for the spatial L^2-projection π_h lead to

$$\|\eta_h\|_I \le ch^2\|\nabla^2 u_k\|_I, \quad \|\nabla\eta_h\|_I \le ch\|\nabla^2 u_k\|_I$$
$$\|\eta_h^*\|_I \le ch^2\|\nabla^2 \hat{z}_k\|_I, \quad \|\nabla\eta_h^*\|_I \le ch\|\nabla^2 \hat{z}_k\|_I.$$

With the fact that the domain Ω is polygonal and convex, we further have

$$\|\nabla^2\hat{z}_k\|_I \le c\|\Delta\hat{z}_k\|_I \le c\|e_h\|_I,$$

due to Corollary 3.2, which inserted into (4.11) yields the assertion.

Similarly to Propositions 3.3 and 3.2 we obtain a result for the spatial discretization error $\|z_k - z_{kh}\|_I$ as well as $\|G_k'(q)p - G_{kh}'(q)p\|_I$, which we state without proof.

Proposition 4.1 *For $q \in Q_{ad}$ let $u_k \in X_k^0$ be given by the solutions of the semidiscrete state equation* (3.1) *and and let $\tilde{u}_k := G_k'(q)\delta q \in X_k^0$ be the solution of the linearized state equation, respectively. Moreover, let $u_{kh} \in X_{k,h}^{0,1}$ and $\tilde{u}_{kh} := G_{kh}'(q)\delta q \in X_{k,h}^{0,1}$ denote the solutions of the discrete state equation* (4.1) *and a corresponding linearized equation. Then the error estimate*

$$\|\tilde{u}_k - \tilde{u}_{kh}\|_I \le Ch^2\|\delta q\|_I$$

is satisfied for a constant $C > 0$ independent of the time and space discretization.

Proposition 4.2 *For $q \in Q_{ad}$ let $u_k \in X_k^0$ and $z_k \in X_k^0$ be given by the solutions of the semidiscrete state equation* (3.1) *and adjoint equation* (3.17)*, respectively. Moreover, let $u_{kh} \in X_{k,h}^{0,1}$ and $z_{k,h} \in X_{k,h}^{0,1}$ denote the solutions of the discrete state equation* (4.1) *and adjoint equation* (4.8)*. Then the error estimate*

$$\|z_k - z_{kh}\|_I \le Ch^2\{\|\nabla^2 u_k\|_I + \|\nabla^2 z_k\|_I\}$$

is satisfied for a constant $C > 0$ independent of the space discretization.

5 Error Estimates

In this section we provide a priori error estimates for the optimal control problem, i. e. eventually we are interested in the error $\|\bar{q} - \bar{q}_\sigma\|_I$, where $\bar{q}_\sigma$ denotes a locally optimal control of a problem where all appearing PDEs as well as the control itself have been discretized in space and time. By δ, we denote an abstract discretization parameter for the control. We collect all discretization parameters k, h, δ into a single parameter $\sigma = (k, h, \delta)$. We will discuss in detail a setting where the control is discretized piecewise constant in time and cellwise constant in space. Then, we briefly comment on a cellwise linear discretization, a variationally discrete setting from, e. g., [17], and a post processing approach, cf. [25], as they have already been discussed in the linear-quadratic setting of [24]. The final result will be obtained in two main steps. The first step involves the estimate between $\bar{q}$ and an auxiliary semidiscrete problem, in the second step the error due to the spatial discretization of the PDE as well as the controls is estimated. We formally divide the error into

$$\|\bar{q} - \bar{q}_\sigma\|_I \le \|\bar{q} - \bar{q}_k\|_I + \|\bar{q}_k - \bar{q}_\sigma\|_I, \tag{5.1}$$

but auxiliary problems that guarantee closeness of the different solutions have to be taken into account. Note that a local solution $\bar{q}_k$ of (3.15) is piecewise constant in time, and hence the second term in (5.1) only contains the spatial discretization error for the PDE as well as the control. We point out that all auxiliary problems to be considered in the following fulfill first order necessary optimality conditions similar in form of variational inequalities like (2.11), (3.16), or (4.7), respectively, if the admissible sets are chosen appropriately.

5.1 Error estimates for the semidiscrete optimal control problem

We now discuss the error due to time discretization. Let $\bar{q}$ be a local solution of (2.5) and let Assumption 2.2 be satisfied. We introduce the following auxiliary problem

$$\min_{q\in Q_{\mathrm{ad}}^{\varepsilon}} j_k(q), \tag{5.2}$$

where $Q_{\mathrm{ad}}^{\varepsilon}$ is defined as

$$Q_{\mathrm{ad}}^{\varepsilon} := \{q \in Q_{\mathrm{ad}}\colon\ \|\bar{q}-q\|_I \le \varepsilon\}.$$

Lemma 5.1 *Let $\varepsilon > 0$ be small enough, that Lemma 3.2 is satisfied. Then, for k sufficiently small, the auxiliary problem* (5.2) *admits a unique global solution $\bar{q}_k^{\varepsilon}$.*

Proof The existence of a solution is clear noting that $Q_{\mathrm{ad}}^{\varepsilon}$ is not empty. To show uniqueness of $\bar{q}_k^{\varepsilon}$, let us assume that there exist two global minima $\bar{q}_k^{\varepsilon}$, $\bar{r}_k^{\varepsilon}$ of (5.2) with $\bar{q}_k^{\varepsilon} \neq \bar{r}_k^{\varepsilon}$ and $j_k(\bar{q}_k^{\varepsilon}) = j_k(\bar{r}_k^{\varepsilon})$. For some $\xi \in (0,1)$, we obtain

$$\begin{aligned} j_k(\bar{r}_k^{\varepsilon}) &= j_k(\bar{q}_k^{\varepsilon}) + j_k'(\bar{q}_k^{\varepsilon})(\bar{r}_k^{\varepsilon}-\bar{q}_k^{\varepsilon}) + \frac{1}{2} j_k''(\bar{q}_k^{\varepsilon}+\xi(\bar{r}_k^{\varepsilon}-\bar{q}_k^{\varepsilon}))(\bar{r}_k^{\varepsilon}-\bar{q}_k^{\varepsilon},\bar{r}_k^{\varepsilon}-\bar{q}_k^{\varepsilon}) \\ &\ge j_k(\bar{q}_k^{\varepsilon}) + \frac{\gamma}{8}\|\bar{r}_k^{\varepsilon}-\bar{q}_k^{\varepsilon}\|_I^2 > j_k(\bar{q}_k^{\varepsilon}) \end{aligned}$$

for k sufficiently small by Taylor expansion, first order optimality conditions for $\bar{q}_k^{\varepsilon}$ and Lemma 3.2, that is applicable to $q^{\xi} = \bar{q}_k^{\varepsilon} + \xi(\bar{r}_k^{\varepsilon}-\bar{q}_k^{\varepsilon})$ due to the convexity of $Q_{\mathrm{ad}}^{\varepsilon}$. This contradiction yields uniqueness of $\bar{q}_k^{\varepsilon}$.

Remark 5.1 In view of numerical solution algorithms, note that there is even a unique stationary point in a small neighborhood of $\bar{q}$. For that, assume there exist two stationary points $\tilde{q}_k^{\varepsilon}, \tilde{r}_k^{\varepsilon} \in Q_{\mathrm{ad}}^{\varepsilon}$, $\tilde{q}_k^{\varepsilon} \neq \tilde{r}_k^{\varepsilon}$ for ε sufficiently small i. e.

$$j_k'(\tilde{q}_k^{\varepsilon})(q-\tilde{q}_k^{\varepsilon}) \ge 0, \quad j_k'(\tilde{r}_k^{\varepsilon})(q-\tilde{r}_k^{\varepsilon}) \ge 0 \quad \forall q \in Q_{\mathrm{ad}}^{\varepsilon}.$$

Then we obtain

$$0 \le j_k'(\tilde{r}_k^{\varepsilon})(\tilde{q}_k^{\varepsilon}-\tilde{r}_k^{\varepsilon}) - j_k'(\tilde{q}_k^{\varepsilon})(\tilde{q}_k^{\varepsilon}-\tilde{r}_k^{\varepsilon}) = -j_k''(q_{\xi})(\tilde{r}_k^{\varepsilon}-\tilde{q}_k^{\varepsilon},\tilde{r}_k^{\varepsilon}-\tilde{q}_k^{\varepsilon}) \tag{5.3}$$

with $q_{\xi} = \tilde{q}_k^{\varepsilon} + \xi(\tilde{r}_k^{\varepsilon}-\tilde{q}_k^{\varepsilon})$ for a $0 < \xi < 1$ by Taylor expansion of j_k'. Note further that

$$\|q_{\xi}-\bar{q}\|_I = \|(1-\xi)(\tilde{q}_k^{\varepsilon}-\bar{q}) + \xi(\tilde{r}_k^{\varepsilon}-\bar{q})\|_I \le \xi\varepsilon \le \varepsilon,$$

and therefore Lemma 3.2 is applicable for $\varepsilon > 0$ sufficiently small. Then, (5.3) yields

$$0 \le -\frac{\gamma}{4}\|\tilde{r}_k^\varepsilon - \tilde{q}_k^\varepsilon\|_I^2.$$

This contradiction implies uniqueness of the stationary points.

Lemma 5.2 *Let $\bar{q}_k^\varepsilon \in Q_{ad}$ denote the solution of Problem* (5.2). *There exists a constant ε such that*

$$j_k''(q)(\delta q, \delta q) \ge \frac{\gamma}{8}\|\delta q\|_I^2$$

for all $\|q - \bar{q}_k^\varepsilon\|_I \le \varepsilon$ and all $\delta q \in Q$.

Proof This follows directly from Lemma 3.3 and Lemma 3.2 considering

$$j_k''(q)(\delta q, \delta q) = j_k''(\bar{q}_k^\varepsilon)(\delta q, \delta q) + \left(j_k''(q)(\delta q, \delta q) - j_k''(\bar{q}_k^\varepsilon)(\delta q, \delta q)\right),$$

and noting that $\|\bar{q}_k^\varepsilon - \bar{q}\|_I \le \varepsilon$.

Lemma 5.3 *Let $\varepsilon > 0$ and $k > 0$ be small enough that Lemmas 2.6 and 3.2 as well as 5.1 are satisfied. Then there exists a constant $C > 0$ such that*

$$\|\bar{q} - \bar{q}_k^\varepsilon\|_I \le Ck.$$

Proof We make use of the variational inequalities for Problems (2.5) and (5.2) and observe that $0 \le j'(\bar{q})(\bar{q}_k^\varepsilon - \bar{q})$ as well as $0 \le j_k'(\bar{q}_k^\varepsilon)(\bar{q} - \bar{q}_k^\varepsilon)$ is satisfied. Summation of both inequalities yields

$$\begin{aligned} 0 &\le (j'(\bar{q}) - j_k'(\bar{q}_k^\varepsilon))(\bar{q}_k^\varepsilon - \bar{q}) \\ &= (j'(\bar{q}) - j_k'(\bar{q}))(\bar{q}_k^\varepsilon - \bar{q}) + (j_k'(\bar{q}) - j_k'(\bar{q}_k^\varepsilon))(\bar{q}_k^\varepsilon - \bar{q}) \\ &= (z(\bar{q}) - z_k(\bar{q}), \bar{q}_k^\varepsilon - \bar{q})_I - j_k''(q^\xi)(\bar{q}_k^\varepsilon - \bar{q}, \bar{q}_k^\varepsilon - \bar{q}), \end{aligned}$$

with $q^\xi = \bar{q} + \xi(\bar{q}_k^\varepsilon - \bar{q})$ for some $0 < \xi < 1$ after Taylor expansion. With Lemma 5.2, this yields

$$\frac{\gamma}{8}\|\bar{q}_k^\varepsilon - \bar{q}\|_I^2 \le \|z(\bar{q}) - z_k(\bar{q})\|_I \|\bar{q}_k^\varepsilon - \bar{q}\|_I$$

from which we obtain

$$\|\bar{q}_k^\varepsilon - \bar{q}\|_I \le ck(\|\partial_t z(\bar{q})\|_I + \|\partial_t u(\bar{q})\|_I)$$

with Proposition 3.3.

The main result of this section is now a direct consequence of Lemma 5.3. We state it in the next theorem.

Theorem 5.1 *Let $\bar{q}$ be an optimal control of Problem* (2.5) *satisfying the first order optimality conditions* (2.11) *as well as the second order sufficient conditions from Assumption 2.2. Then there exists a sequence of locally optimal controls $\bar{q}_k$ to problem* (3.15) *converging strongly in $L^2(I \times \Omega)$ to $\bar{q}$ as k tends to zero, and the discretization error estimate*

$$\|\bar{q} - \bar{q}_k\|_I \le Ck$$

is satisfied for a constant $C > 0$, which is independent of k.

Proof It is sufficient to mention that by Lemma 5.3 $\bar{q}_k^\varepsilon$ converges to $\bar{q}$ in $L^2(I\times\Omega)$ as k tends to zero. Hence, for k sufficiently small, $\bar{q}_k^\varepsilon =: \bar{q}_k$ is a local solution to Problem 3.15, since $\|\bar{q}_k^\varepsilon - \bar{q}_k\|_I < \varepsilon$ and therefore the additional constraints are inactive. Then, the error estimate from Lemma 5.3 is applicable.

5.2 Error estimates for the discrete optimal control problem

Let us now develop the spatial error estimate in the control. We focus in detail on a cellwise constant control discretization with the same discretization parameter h as is used for the PDE. Results for further control discretization concepts are briefly outlined in Section 5.3. We make the following general assumption that is reasonable due to the semidiscrete convergence result from Theorem 5.1.

Assumption 5.1 *For the remainder of this section, let $k>0$ be small enough that $\bar{q}_k \in B_\varepsilon(\bar{q})$ with $\varepsilon > 0$ small enough that Lemmas 3.2 and 5.2 are satisfied.*

We define the discrete admissible set

$$Q_{\mathrm{ad}}^\delta := \{q \in Q_{\mathrm{ad}} \colon\ q_{|_{I_m\times K}} \in \mathscr{P}_0(I_m \times K),\ m=1,2,\dots,M,\ K\in\mathscr{T}_h\}$$

and consider a completely discretized problem

$$\min_{q\in Q_{\mathrm{ad}}^\delta} j_{kh}(q). \tag{5.4}$$

A local solution to (5.4) will be denoted by $\bar{q}_\sigma$, and we will now estimate the error between corresponding semidiscrete solutions $\bar{q}_k$ and the fully discrete solution $\bar{q}_\sigma$. We again introduce an auxiliary problem to discuss local solutions. Let therefore the auxiliary set $Q_{\mathrm{ad}}^{\delta,\varepsilon}$ be defined by

$$Q_{\mathrm{ad}}^{\delta,\varepsilon} := \{q \in Q_{\mathrm{ad}} \colon\ q\in L^2(I,V_h^s) \text{ and } q_{|_{I_m}} \in \mathscr{P}_0(I_m, V_h^s), \text{ and } \|\bar{q}_k - q\|_I \le \varepsilon\}$$

and consider a fully discrete auxiliary problem

$$\min_{q\in Q_{\mathrm{ad}}^{\delta,\varepsilon}} j_{kh}(q). \tag{5.5}$$

We first prove a coercivity result for the second derivative of j_{kh}.

Lemma 5.4 *Let k and ε be small enough that Assumption 5.1 holds. Then, for all $q\in Q_{ad}$ with $\|q-\bar{q}_k\|_I \le \varepsilon$ and all $p\in Q$, the inequality*

$$j_{kh}''(q)(p,p) \ge \frac{\gamma}{16}\|p\|_I^2$$

is satisfied for all h sufficiently small.

Proof With Lemma 5.2, the proof follows as the one for Lemma 3.2, making use of Theorems 4.1 and 4.2, Propositions 4.1 and 4.2, as well as the stability estimates (4.3)-(4.5).

As an analogue to the Lemmas 5.1 and 5.3 we obtain existence of a unique global solution to (5.5) as well as an estimate for $\|\bar{q}_k - \bar{q}_\sigma^\varepsilon\|$.

Lemma 5.5 *Let $\varepsilon > 0$ be so small, that Lemma 5.2 is satisfied. For all $\varepsilon > 0$ and $\delta > 0$ sufficiently small, the auxiliary problem* (5.5) *admits a unique global solution $\bar{q}_\sigma^\varepsilon$.*

Proof The proof of existence follows by standard arguments noting that $\bar{q}_k$ is piecewise constant in time and hence $\pi_h \bar{q}_k \in Q_{\text{ad}}^{\delta,\varepsilon}$ in case of cellwise constant discretization for all h sufficiently small. The proof of uniqueness then follows similarly to Lemma 5.1.

Again, it is possible to prove uniqueness of stationary points, cf. Remark 5.1. Before proving an error estimate, we show an auxiliary result for j_k' and j_{kh}'.

Lemma 5.6 *Let $q \in Q_{ad}$ and $\delta q \in Q$ be given. Then the estimate*

$$|j_k'(q)\delta q - j_{kh}'(q)\delta q| \le Ch^2 \|\delta q\|_I$$

holds with a constant $C > 0$ independent of h. Moreover, j_{kh}' fulfills a Lipschitz condition, i. e. there exists a constant $C > 0$ such that for all $p, q \in Q_{ad}$ and all $\delta q \in Q$

$$|j_{kh}'(q)\delta q - j_{kh}'(p)\delta q| \le C\|q - p\|_I \|\delta q\|_I$$

is satisfied.

Proof With $u_k := G_k(q)$, $u_{kh} := G_{kh}(q)$, $\tilde{u}_k := G_k'(q)\delta q$, and $\tilde{u}_{kh} := G_{kh}'(q)\delta q$, direct calculations imply

$$\begin{aligned}
|j_k'(q)\delta q - j_{kh}'(q)\delta q| &= |(u_{kh} - \hat{u}, \tilde{u}_{kh})_I - (u_k - \hat{u}, \tilde{u}_k)_I| \\
&= |(u_{kh} - \hat{u}, \tilde{u}_{kh} - \tilde{u}_k)_I + (u_{kh} - u_k, \tilde{u}_k)_I| \\
&\le c\|\tilde{u}_{kh} - \tilde{u}_k\|_I + c\|u_{kh} - u_k\|_I \|\tilde{u}_k\|_I \\
&\le Ch^2 \|\delta q\|_I
\end{aligned}$$

by the boundedness of Q_{ad}, the boundedness results of Theorem 4.1 and estimates (3.12) and (3.13), and the error estimates from Theorem 4.2 and Proposition 4.1. The second estimate follows in a similar manner, replacing u_k by $v_{kh} := G_{kh}(p)$, as well as $\tilde{u}_k$ by $\tilde{v}_{kh} = G_{kh}'(p)\delta q$, and making use of Lipschitz results for the state and linearized state equation from Lemma 4.1.

Lemma 5.7 *Let $\varepsilon > 0$ and $h > 0$ be small enough that Assumption 5.1 and Lemma 5.4 are satisfied. Then there exists a constant $C > 0$ such that*

$$\|\bar{q}_k - \bar{q}_\sigma^\varepsilon\|_I \le Ch.$$

Proof Note first that $\pi_h \bar{q}_k \in Q_{\text{ad}}^{\delta,\varepsilon}$ if h is sufficiently small. Then, by Taylor expansion and the help of Lemma 5.5 we obtain that

$$\begin{aligned}
j_{kh}'(\pi_h \bar{q}_k)(\pi_h \bar{q}_k - \bar{q}_\sigma^\varepsilon) - j_{kh}'(\bar{q}_\sigma^\varepsilon)(\pi_h \bar{q}_k - \bar{q}_\sigma^\varepsilon) &= j_{kh}''(q^\xi)(\pi_h \bar{q}_k - \bar{q}_\sigma^\varepsilon, \pi_h \bar{q}_k - \bar{q}_\sigma^\varepsilon) \\
&\ge \frac{\gamma}{16} \|\pi_h \bar{q}_k - \bar{q}_\sigma^\varepsilon\|_I^2,
\end{aligned}$$

where $q^\xi = \pi_h\bar q_k + \xi(\bar q^\varepsilon_\sigma - \pi_h\bar q_k) \in Q^{\delta,\varepsilon}_{\mathrm{ad}}$ for a $\xi \in (0,1)$. Moreover, from the optimality conditions for problems (3.15) and (5.5) we obtain

$$-j'_{kh}(\bar q^\varepsilon_\sigma)(\pi_h\bar q_k - \bar q^\varepsilon_\sigma) \le 0 \le -j'_k(\bar q_k)(\bar q_k - \bar q^\varepsilon_\sigma) = -j'_k(\bar q_k)(\pi_h\bar q_k - \bar q^\varepsilon_\sigma) - j'_k(\bar q_k)(\bar q_k - \pi_h\bar q_k),$$

and hence

$$\begin{aligned}\tfrac{\gamma}{16}\|\pi_h\bar q_k - \bar q^\varepsilon_\sigma\|_I^2 &\le j'_{kh}(\pi_h\bar q_k)(\pi_h\bar q_k - \bar q^\varepsilon_\sigma) - j'_k(\bar q_k)(\pi_h\bar q_k - \bar q^\varepsilon_\sigma) - j'_k(\bar q_k)(\bar q_k - \pi_h\bar q_k)\\ &= (j'_{kh}(\pi_h\bar q_k) - j'_{kh}(\bar q_k))(\pi_h\bar q_k - \bar q^\varepsilon_\sigma)\\ &\quad + (j'_{kh}(\bar q_k) - j'_k(\bar q_k))(\pi_h\bar q_k - \bar q^\varepsilon_\sigma) - j'_k(\bar q_k)(\bar q_k - \pi_h\bar q_k)\\ &\le c\|\bar q_k - \pi_h\bar q_k\|_I\|\pi_h\bar q_k - \bar q^\varepsilon_\sigma\|_I + ch^2\|\pi_h\bar q_k - \bar q^\varepsilon_\sigma\|_I\\ &\quad - (\nu\bar q_k + z_k(\bar q_k), \bar q_k - \pi_h\bar q_k)_I,\end{aligned} \tag{5.6}$$

where we used the Lipschitz stability result and error estimate from Lemma 5.6. By Corollary 3.2 and the properties of π_h, the last term of (5.6) can be estimated by

$$\begin{aligned}(\nu\bar q_k + z_k(\bar q_k), \bar q_k - \pi_h\bar q_k) &= (\nu\bar q_k + z_k(\bar q_k) - \pi_h(\nu\bar q_k - z_k(\bar q_k)), \bar q_k - \pi_h\bar q_k)_I\\ &\le ch^2(\nu\|\nabla\bar q_k\|_I + \|\nabla z_k(\bar q_k)\|_I)\|\nabla\bar q_k\|_I\\ &\le ch^2.\end{aligned}$$

Inserting this into (5.6) and applying Young's inequality yields

$$\|\pi_h\bar q_k - \bar q^\varepsilon_\sigma\|_I \le ch$$

by known projection error estimates. Then, the assertion is obtained by known projection error estimates noting that

$$\|\bar q_k - \bar q^\varepsilon_\sigma\|_I \le \|\bar q_k - \pi_h\bar q_k\|_I + \|\pi_h\bar q_k - \bar q^\varepsilon_\sigma\|_I.$$

Again, we obtain that $\bar q^\varepsilon_\sigma$ converges to $\bar q_k$ in $L^2(I\times\Omega)$ as h tends to zero. Therefore, $\bar q^\varepsilon_\sigma =: \bar q_\sigma$ is a local solution of Problem (5.4) for all h sufficiently small, and we directly obtain

Theorem 5.2 *Let $\bar q$ be an optimal solution of Problem* (2.5) *satisfying Assumption 2.2, and let $\bar q_k \in B_\varepsilon(\bar q)$ be an optimal control of Problem* (3.15) *satisfying the first order optimality conditions* (3.16)*, with k small enough such that Lemma 5.2 holds. Then there exists a sequence of locally optimal controls to problem* (4.6) *converging strongly in $L^2(I\times\Omega)$ to $\bar q_k$ as h tends to zero, and the discretization error estimate*

$$\|\bar q_k - \bar q_\sigma\|_I \le Ch$$

is satisfied for a constant $C > 0$, which is independent of h.

Combining the Theorems 5.1 and 5.2, we arrive at the main result of this article, an a priori error estimate between a completely discrete solution $\bar q_\sigma$ and the corresponding continuous solution $\bar q$.

Theorem 5.3 *Let $\bar q$ be an optimal control of Problem* (2.5) *satisfying the first order optimality conditions* (2.11) *as well as second order sufficient condition from Assumption 2.2. Then there exists a sequence of locally optimal controls $\bar q_\sigma$ to problem* (4.6) *converging strongly in $L^2(I\times\Omega)$ to $\bar q$ as k,h tend to zero, and the discretization error estimate*

$$\|\bar q - \bar q_\sigma\|_I \le C(k+h)$$

is fulfilled for a constant $C > 0$ independent of k and h.

5.3 Extensions and additional results

In this section, we will give an overview over some direct consequences and extensions of the previously shown results. That includes the discussion of further types of control discretization as well as the analysis of a class of problems with finitely many time-dependent controls that appear more often in practice.

5.3.1 Cellwise bilinear control discretization

For the discretization of the control, we now choose the same discretization as for the state variable. With

$$Q_h = \{v \in C(\bar{\Omega}) | v_{|_K} \in \mathscr{Q}_1(K) \text{ for } K \in \mathscr{T}_h\}$$

we define $Q^\delta \supset X_{k,h}^{0,1}$ as

$$Q^\delta = \{q \in Q | q_{|_{I_m}} \in \mathscr{P}_0(I_m, Q_h)\},$$

as well as $Q_{\text{ad}}^\delta = Q^\delta \cap Q_{\text{ad}}$. We consider again the formal splitting of the error into

$$\|\bar{q} - \bar{q}_\sigma\|_I \leq \|\bar{q} - \bar{q}_k\|_I + \|\bar{q}_k - \bar{q}_\sigma\|_I, \tag{5.7}$$

where $\bar{q}_\sigma$ denotes a local solution of the completely discretized problem

$$\min_{q \in Q_{\text{ad}}^\delta} j_{kh}(q). \tag{5.8}$$

The discussion of the term $\|\bar{q} - \bar{q}_k\|_I$ with the help of corresponding auxiliary problems from the last section remains valid here. For the second term in (5.7), the arguments from [24, Section 5.2] can be combined with our argumentation for local solutions. Therefore, for each time interval I_m we group the cells K of the mesh $\mathscr{T}_h$ into three pairwise disjoint sets $\mathscr{T}_h = \mathscr{T}_{h,m}^1 \cup \mathscr{T}_{h,m}^2 \cup \mathscr{T}_{h,m}^3$, with

$$\begin{aligned}
\mathscr{T}_{h,m}^1 &:= \{K \in \mathscr{T}_h \,|\, \bar{q}_k(t_m, x) = q_a \text{ or } \bar{q}_k(t_m, x) = q_b \quad \forall x \in K\}, \\
\mathscr{T}_{h,m}^2 &:= \{K \in \mathscr{T}_h \,|\, q_a < \bar{q}_k(t_m, x) < q_b \quad \forall x \in K\}, \\
\mathscr{T}_{h,m}^3 &:= \mathscr{T}_h \setminus (\mathscr{T}_{h,m}^1 \cup \mathscr{T}_{h,m}^2),
\end{aligned}$$

and rely on the following

Assumption 5.2 *There exists a positive constant C independent of k, h, and m, such that*

$$\sum_{K \in \mathscr{T}_{h,m}^3} |K| \leq Ch$$

is satisfied separately for all $m = 1, \ldots, M$.

Retrieving to the analysis of auxiliary problems in the neighborhood of a semidiscrete local solution, we only need to ensure coercivity of j''_{kh} in the neighborhood of $\bar{q}_k$ in order to apply the argumentation from the linear-quadratic setting. Since the desired coercivity is guaranteed in Lemma 5.4, we formulate the following result without detailed proof.

Proposition 5.1 *Let $\bar{q}$ be an optimal control of Problem* (2.5) *satisfying the first order optimality conditions* (2.11) *as well as second order sufficient condition from Assumption 2.2. Moreover, let Assumption 5.2 hold. Then there exists a sequence of locally optimal controls to problem* (5.8) *converging strongly in $L^2(I \times \Omega)$ to $\bar{q}$ as k,h tend to zero, and the discretization error estimate*

$$\|\bar{q} - \bar{q}_\sigma\|_I \leq C(k + h^{\frac{3}{2}-\varepsilon})$$

is fulfilled for every $\varepsilon > 0$ and a constant $C > 0$ independent of k and h.

5.3.2 Variational approach

In Section 5.3 in [24], an error estimate for the variational discretization approach from, e. g., [17], without discretization of the control extended to parabolic problems is analyzed for the linear-quadratic problem. Here, $Q^\delta = Q$ is chosen, which implies $Q^\delta_{\text{ad}} = Q_{\text{ad}}$. Then, formally $\bar{q}_\sigma = \bar{q}_{kh}$ holds, where again associated local solutions have to be considered. Again, the key point when adapting the argumention for the linear-quadratic setting to our model problem is the coercivity of j''_{kh}, which is used in the proof of Theorem 5.10 from [24] and guaranteed for our model problem by Lemma 5.4.

Proposition 5.2 *Let $\bar{q}$ be an optimal control of Problem* (2.5) *satisfying the first order optimality conditions* (2.11) *as well as second order sufficient condition from Assumption 2.2. Then there exists a sequence of locally optimal controls to problem* (4.6) *converging strongly in $L^2(I \times \Omega)$ to $\bar{q}$ as k,h tend to zero, and the discretization error estimate*

$$\|\bar{q} - \bar{q}_{kh}\|_I \leq C(k + h^2)$$

is fulfilled with a constant $C > 0$ independent of k and h.

We omit the proof.

5.3.3 Postprocessing strategy

The discussion of a post processing strategy in Section 5.4 in [24] is quite involved, but most results can be applied directly to our setting. Again, the crucial point is the coercivity of j''_{kh} used in the proof of Theorem 5.16, [24], which is guaranteed for our model problem. The control is discretized piecewise constant in time and cellwise constant in space. Then, after the computation of a solution $\bar{q}_\sigma$ of Problem (5.4) the approximation is improved by constructing

$$\tilde{q}_\sigma = P_{Q_{\text{ad}}}\left(-\frac{1}{\nu} z_{kh}(\bar{q}_\sigma)\right)$$

As an analogue to [24, Corollary 5.17], we obtain, again omitting the proof:

Proposition 5.3 *Let $\bar{q}$ be an optimal control of Problem* (2.5) *satisfying the first order optimality conditions* (2.11) *as well as second order sufficient condition from Assumption 2.2. Moreover, let Assumption 5.2 hold. Then there exists a sequence controls $\tilde{q}_\sigma := P_{Q_{ad}}\left(-\frac{1}{\nu} z_{kh}(\bar{q}_\sigma)\right)$, where $\bar{q}_\sigma$ is a solution to problem* (5.8) *converging strongly in $L^2(I \times \Omega)$ to $\bar{q}$ as k,h tend to zero, and for the error estimate*

$$\|\bar{q} - \tilde{q}_\sigma\|_I \le C(k + h^{2-\varepsilon})$$

is fulfilled for every $\varepsilon > 0$ and a constant $C > 0$ independent of k and h.

5.3.4 Model problems with finitely many time-dependent controls

We conclude this section by briefly discussing a setting with control

$$\tilde{q}(t,x) = \sum_{i=1}^{N} q_i(t) e_i(x),$$

with fixed functions $e_i \in L^\infty(\Omega)$, $i = 1, \dots, N$, i. e. we consider a problem with finitely many time-dependent controls. Introducing the control space

$$Q = L^\infty(I, \mathbb{R}^N),$$

it is clear that our theory is valid for this specific setting. By minor modifications, it is possible to consider bounds $q_a, q_b \in \mathbb{R}^N$ for each control component, i. e.

$$q_{a,i} \le q_i(t) \le q_{b,i}$$

with $q_{a,i} < q_{b,i}$ for all $i = 1, \dots, N$. Then, the admissible set has to be defined as

$$Q_{\text{ad}} = \{ q \in Q \mid q_{a,i}(t) \le q_i(t) \le q_{b,i}(t) \quad \forall i = 1, \dots, N \}.$$

With $s(q) := \sum_{i=1}^{N} q_i e_i$ we obtain the problem formulation

$$\min_{q \in Q_{\text{ad}}} j(s(q)). \tag{5.9}$$

All differentiability, Lipschitz, coercivity, and boundedness results remain valid in appropriate norms on, e. g., $L^\infty(I, \mathbb{R}^N)$ or $L^2(I, \mathbb{R}^N)$. The projection formula on the continuous level changes to

$$\bar{q}_i(t) = P_{[q_{a,i}, q_{b,i}]}\left(-\frac{1}{\nu} \int_\Omega z(s(\bar{q}))(t,x) e_i(x)\, dx \right), \quad i = 1, \dots, N. \tag{5.10}$$

On the semidiscrete level, we then obtain that each $\bar{q}_{k,i}$ is piecewise constant in time. Since no control discretization in space is necessary, it is clear that formally $\bar{q}_\sigma = \bar{q}_{kh}$, where $\bar{q}_\sigma$ solves

$$\min_{q \in Q_{\text{ad}}} j_{kh}(s(q)), \tag{5.11}$$

similarly to the variationally discrete setting in case of control functions depending arbitrarily on space and time. We eventually obtain:

Proposition 5.4 *Let $\bar{q}$ be an optimal control of Problem* (5.9) *satisfying* (5.10) *as well as second order sufficient condition from Assumption 2.2. Then there exists a sequence of locally optimal controls to problem* (5.11) *converging strongly in $L^2(I,\mathbb{R}^N)$ to $\bar{q}$ as k,h tend to zero, and the discretization error estimate*

$$\|\bar{q}-\bar{q}_{kh}\|_I \leq C(k+h^2)$$

is fulfilled with a constant $C>0$ independent of k and h.

6 Numerical Experiments

In this section, we validate the proven a priori error estimates numerically. We consider a numerical example

$$\text{Minimize } J(q,u) := \frac{1}{2}\int_0^T\int_\Omega (u(t,x)-\hat{u}(t,x))^2 dx\,dt + \frac{\nu}{2}\int_0^T\int_\Omega q(t,x)^2 dx\,dt$$

$$\begin{aligned} \partial_t u - \Delta u + u^3 &= q + f \text{ in } (0,T)\times\Omega \\ u(0,\cdot) &= u_0 \quad \text{ in } \Omega \end{aligned}$$

$$\begin{aligned} u &= 0 \qquad \text{on } (0,T)\times\partial\Omega. \\ q_a \leq q(t,x) &\leq q_b \quad \text{a.e. in } (0,T)\times\Omega. \end{aligned}$$

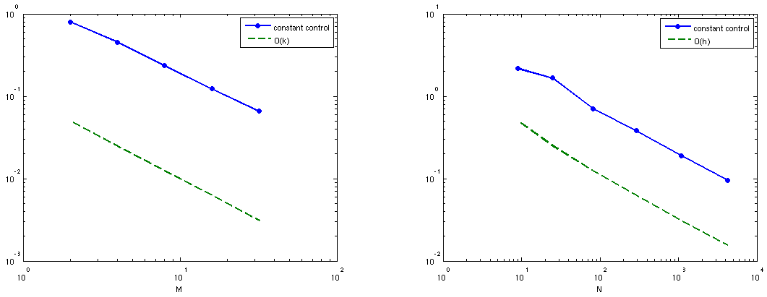
on $I\times\Omega = (0,0.1)\times(0,1)^2$, with a regularization parameter $\nu = 0.1$, and bounds $q_a = 10$ as well as $q_b = 30$, and the following data:

$$\begin{aligned} u_0(x_1,x_2) &= \sin(\pi x_1)\sin(\pi x_2) \\ \hat{u}(t,x_1,x_2) &= \left(e^{\pi^2 t} + 2\pi^2 e^{2\pi^2 T}\right)\sin(\pi x_1)\sin(\pi x_2) \\ &\quad -3e^{2\pi^2 t}\left(e^{2\pi^2 t} - e^{2\pi^2 T}\right)\sin^3(\pi x_1)\sin^3(\pi x_2) \\ f(t,x_1,x_2) &= 3\pi^2 e^{\pi^2 t}\sin(\pi x_1)\sin(\pi x_2) + e^{3(\pi^2 t)}\sin^3(\pi x_1)\sin^3(\pi x_2) \\ &\quad -P_{Q_{\text{ad}}}\left(-\frac{1}{\nu}\left(e^{2\pi^2 t} - e^{2\pi^2 T}\right)\sin(\pi x_1)\sin(\pi x_2)\right) \end{aligned}$$

An optimal solution triple $(\bar{q},\bar{u},\bar{z})$ of problem (2.5) is given by

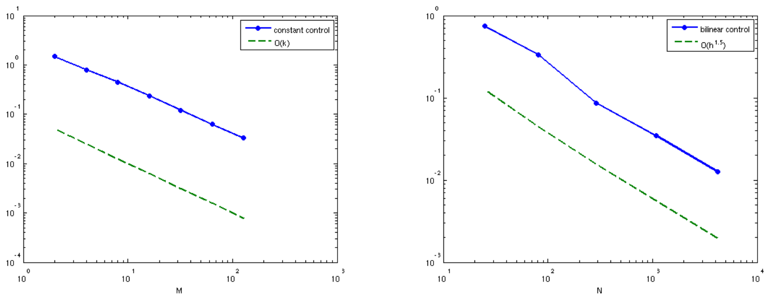
$$\begin{aligned} \bar{u} &= e^{\pi^2 t}\sin(\pi x_1)\sin(\pi x_2) \\ \bar{z} &= \left(e^{2\pi^2 t} - e^{2\pi^2 T}\right)\sin(\pi x_1)\sin(\pi x_2) \\ \bar{q} &= P_{Q_{\text{ad}}}\left(-\frac{1}{\nu}\left(e^{2\pi^2 t} - e^{2\pi^2 T}\right)\sin(\pi x_1)\sin(\pi x_2)\right). \end{aligned}$$

The optimal control problem is solved by the optimization library RoDoBo [2], and the finite element toolkit GASCOIGNE, [1]. We first discretize the controls cellwise constant in time and space, to validate Theorem 5.3. We consider first the behavior of the error for a sequence of discretizations with decreasing size of the time steps and



(a) Refinement of the time steps for $N = 66049$ spatial nodes (b) Refinement of the spatial triangulation for $M = 4000$ time steps

Fig. 1 Discretization error $\|\bar{q} - \bar{q}_\sigma\|_I$, piecewise constant control discretization in space and time.



(a) Refinement of the time steps for $N = 4096$ spatial nodes (b) Refinement of the spatial triangulation for $M = 2000$ time steps

Fig. 2 Discretization error $\|\bar{q} - \bar{q}_\sigma\|_I$, piecewise constant control discretization in time and cellwise bilinear control discretization in space

a fixed spatial discretization and $N = 66049$ nodes. Then, we analyze the behavior due to spatial discretization with $M = 4000$ time steps. In Table 1 and Figure 1, we present these computational results. Figure 1(a) shows the development of the error for decreasing time discretization parameter k, whereas Figure 1(b) depicts the error due to spatial discretization for cellwise constant controls.

We solve the same example also discretizing the controls piecewise constant in time but cellwise bilinear in space and validate the order of convergence stated in Proposition 5.1. We use first a fixed spatial discretization with $N = 4225$ nodes and analyze the error due to time discretization. Then, we repeat the calculations for $M = 2000$ time steps and decreasing spatial mesh size. Table 2 and Figure 2 show the results.

Acknowledgements The authors are grateful for interesting discussions with Klaus Krumbiegel about second order sufficient conditions and Dominik Meidner about regularity of semi-discrete solutions.

Table 1 Discretization error $\|\bar{q}-\bar{q}_\sigma\|_I$, piecewise constant control discretization in space and time

k	$\|\bar{q}-\bar{q}_\sigma\|_I$	h	$\|\bar{q}-\bar{q}_\sigma\|_I$
$0.1\cdot 2^{-1}$	7.9869e-01	2^{-1}	2.1715e+00
$0.1\cdot 2^{-2}$	4.5057e-01	2^{-4}	1.6573e+00
$0.1\cdot 2^{-3}$	2.3534e-01	2^{-5}	7.0937e-01
$0.1\cdot 2^{-4}$	1.2329e-01	2^{-6}	3.7983e-01
$0.1\cdot 2^{-5}$	6.5905e-02	2^{-7}	1.8959e-01
		2^{-8}	9.5471e-02

Table 2 Discretization error $\|\bar{q}-\bar{q}_\sigma\|_I$, piecewise constant control discretization in time and cellwise bilinear discretization in space

k	$\|\bar{q}-\bar{q}_\sigma\|_I$	h	$\|\bar{q}-\bar{q}_\sigma\|_I$
$0.1\cdot 2^{-1}$	1.4749e-00	2^{-2}	7.4278e-01
$0.1\cdot 2^{-2}$	7.9857e-01	2^{-3}	3.3482e-01
$0.1\cdot 2^{-3}$	4.5013e-01	2^{-4}	8.6501e-02
$0.1\cdot 2^{-4}$	2.3440e-01	2^{-5}	3.4536e-02
$0.1\cdot 2^{-5}$	1.2149e-01	2^{-6}	1.2524e-02
$0.1\cdot 2^{-6}$	6.2539e-02		
$0.1\cdot 2^{-7}$	3.3209e-02		

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