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A PRIORI ERROR ANALYSIS FOR DISCRETIZATION OF SPARSE ELLIPTIC OPTIMAL CONTROL PROBLEMS IN MEASURE SPACE*

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Abstract. In this paper an optimal control problem is considered, where the control variable lies in a measure space and the state variable fulfills an elliptic equation. This formulation leads to a sparse structure of the optimal control. In this setting we prove a new regularity result for the optimal state and the optimal control. Moreover, a finite element discretization based on [E. Casas, C. Clason, and K. Kunisch, *SIAM J. Control Optim.*, 50 (2012), pp. 1735–1752] is discussed and a priori error estimates are derived, which significantly improve the estimates from that paper. Numerical examples for problems in two and three space dimensions illustrate our results.

Key words. optimal control, sparsity, finite elements, error estimates

AMS subject classifications. 35J05, 35Q93, 49M25, 49M29, 65N30, 65N15

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1. Introduction.

In this paper we consider the following optimal control problem:

$$\text{Minimize}\quad J(q,u) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \alpha\|q\|_{\mathcal{M}(\Omega)}, \quad q \in \mathcal{M}(\Omega), \tag{1.1}$$

subject to

$$\begin{cases} -\Delta u = q & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ is a convex bounded domain with a $C^{2,\beta}$-boundary $\partial\Omega$. The control variable q is searched for in the space of regular Borel measures $\mathcal{M}(\Omega)$, which is identified with the dual of the space of continuous functions vanishing on the boundary $C_0(\Omega)$. The state variable u is the solution of the state equation (1.2); see the next section for the precise weak formulation. The desired state u_d is in $L^2(\Omega)$; see also further assumptions ($u_d \in L^p(\Omega)$ or $u_d \in L^\infty(\Omega)$) below. The parameter α is assumed to be positive.

This problem setting with the control from a measure space was considered in [11], where it has been observed that this setting leads to optimal controls with sparse structure. This is important for many applications; cf., e.g., [12]. For another functional analytic concept utilizing the $L^1(\Omega)$-norm of the control combined with an L^2-regularization and/or with control constraints, we refer, e.g., to [20, 22, 10].

This paper is mainly concerned with the discretization of the problem (1.1)–(1.2). In [8] a discretization concept for this problem is presented and the following error estimates are derived:

$$J(\bar q,\bar u) - J(\bar q_h,\bar u_h) = \mathcal{O}\big(h^{2-\frac{d}{2}}\big) \quad \text{and} \quad \|\bar u-\bar u_h\|_{L^2(\Omega)} = \mathcal{O}\big(h^{1-\frac{d}{4}}\big),$$

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where $(\bar{q}, \bar{u})$ is the unique solution to (1.1)–(1.2), h is the discretization parameter, and $(\bar{q}_h, \bar{u}_h)$ is the discrete solution. Our main contribution is the improvement of these estimates using the same discretization concept as

$$(1.3)\quad J(\bar{q}, \bar{u}) - J(\bar{q}_h, \bar{u}_h) = \mathcal{O}\big(h^{4-d} |\ln h|^{\gamma}\big) \quad \text{and} \quad \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} = \mathcal{O}\big(h^{2-\frac{d}{2}} |\ln h|^{\frac{\gamma}{2}}\big)$$

with $\gamma = \frac{7}{2}$ for $d = 2$ and $\gamma = 1$ for $d = 3$. Moreover we provide an estimate for the error in the control variable. Although one can only expect $\bar{q}_h \overset{*}{\rightharpoonup} \bar{q}$ in $\mathcal{M}(\Omega)$ (see [8]), we derive the following estimate with respect to the $H^{-2}(\Omega)$-norm:

$$\|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} = \mathcal{O}\big(h^{2-\frac{d}{2}} |\ln h|^{\frac{\gamma}{2}}\big).$$

We obtain these improved estimates with similar assumptions as in [8], employing error estimates for the state solution in $L^p(\Omega)$ for $p < 2$, which are of (almost) optimal order (see Lemma 3.3), combined with a more careful study of the regularity of the state solutions for a measure valued right-hand side. However, the assumption on the desired state u_d needs to be slightly stronger than in [8]; see Remark 4.1 below.

The numerical examples (see section 8) indicate that the estimates (1.3) are sharp. However, we make the following observation: In the two-dimensional case we see the predicted order of almost $\mathcal{O}(h)$ with respect to the state variable in all examples. But for the three-dimensional case, the predicted order of (almost) $\mathcal{O}(h^{\frac{1}{2}})$ is observed only in examples where the exact optimal control contains Dirac measures. For optimal controls $\bar{q}$ with better regularity, we observe convergence rates similar to the two-dimensional case. Motivated by this observation, we show in section 2 (see Theorem 2.5) that assuming a bounded desired state $u_d \in L^\infty(\Omega)$ implies that $\bar{u}$ must be bounded as well, which immediately rules out controls containing Dirac measures. Another direct consequence is $\bar{q} \in H^{-1}(\Omega)$, which allows us to show an order of convergence of (almost) order $\mathcal{O}(h)$ for the state error $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$ independent of dimension d; see Theorem 5.1.

We remark that these improved regularity results and the improved convergence estimates strongly exploit the specific structure of the problem under consideration. In a more general setting, where the control and the observation domains do not coincide with the whole domain Ω, the optimal control may contain Dirac measures, even if the desired state u_d is bounded; see the discussion in section 6.

The paper is structured as follows. In the next section we recall the optimality conditions from [11] and [8], discuss some consequences of them, and prove that the optimal state $\bar{u}$ is bounded provided that $u_d \in L^\infty(\Omega)$. In section 3 we describe the finite element discretization and derive some error estimates for the state equation. In section 4 we prove the main estimates (1.3), and in section 5 we derive an improved estimate resulting from additional regularity. In section 6 we discuss some extensions of our results for the case where the control action is restricted to a subdomain $\Omega_c \subset \Omega$ and the observation domain is another subset $\Omega_o \subset \Omega$. In the last section we present numerical examples illustrating our results.

Throughout we will denote by $(\cdot, \cdot)$ the $L^2(\Omega)$ inner product and by $\langle \cdot, \cdot \rangle$ the duality product between $\mathcal{M}(\Omega)$ and $C_0(\Omega)$.

2. Optimality system and regularity. As the first step we recall the weak formulation of the state equation (1.2). For a given $q \in \mathcal{M}(\Omega)$ the solution $u = u(q)$ is determined by

$$u \in L^2(\Omega) \; : \; (u, -\Delta\varphi) = \langle q, \varphi \rangle \quad \text{for all} \;\; \varphi \in H^2(\Omega) \cap H^1_0(\Omega).$$

It is well-known that the above formulation possesses a unique solution, which belongs to $W_0^{1,s}(\Omega)$ for all $1 \le s < \frac{d}{d-1}$; see, e.g., [7]. Moreover, there holds the following stability estimate.

LEMMA 2.1. *For each* $0 < \varepsilon \le \frac{1}{d-1}$ *let* s_ε *be given as*

$$s_\varepsilon = \frac{d}{d-1} - \varepsilon.$$

There exists a constant c *independent of* ε *such that for all* $q \in \mathcal{M}(\Omega)$ *and the corresponding solution* u *of* (1.2) *the following estimate holds:*

$$\|u\|_{W_0^{1,s_\varepsilon}(\Omega)} \le \frac{c}{\varepsilon} \|q\|_{\mathcal{M}(\Omega)}.$$

Proof. The estimate for $\|u\|_{W_0^{1,s}(\Omega)}$ with an s-dependent constant is shown in [7]. To obtain the precise dependence of ε we use the continuous embedding of $W_0^{1,s'_\varepsilon}(\Omega)$ into $C_0(\Omega)$, where

$$\frac{1}{s'_\varepsilon} + \frac{1}{s_\varepsilon} = 1, \quad s'_\varepsilon > d.$$

From Theorem 8.10 in [2] we obtain

$$\|v\|_{C_0(\Omega)} \le \frac{c}{\varepsilon} \|v\|_{W_0^{1,s'_\varepsilon}(\Omega)}$$

for all $v \in W_0^{1,s'_\varepsilon}(\Omega)$ with the constant c independent of ε. Using the result from [1] (see also [16]), we estimate

$$\|\nabla u\|_{L^{s_\varepsilon}(\Omega)} \le c \sup_{v \in W_0^{1,s'_\varepsilon}(\Omega)} \frac{(\nabla u, \nabla v)}{\|\nabla v\|_{L^{s'_\varepsilon}(\Omega)}} = c \sup_{v \in W_0^{1,s'_\varepsilon}(\Omega)} \frac{\langle q, v\rangle}{\|\nabla v\|_{L^{s'_\varepsilon}(\Omega)}} \le \frac{c}{\varepsilon} \|q\|_{\mathcal{M}(\Omega)}.$$

This completes the proof. ∎

Due to the embedding of $W_0^{1,s}(\Omega)$ into $L^2(\Omega)$ for $\frac{2d}{d+2} \le s < \frac{d}{d-1}$ the cost functional (1.1) is well-defined. Moreover, the solution operator mapping $q \in \mathcal{M}(\Omega)$ to $u = u(q) \in L^2(\Omega)$ is injective and therefore the cost functional is strictly convex. Using this fact, the existence of a unique solution $(\bar q, \bar u)$ to (1.1)–(1.2) can be directly obtained; see [11] for details. The following optimality system is obtained in [11, 8].

THEOREM 2.2. *Let* $(\bar q, \bar u)$ *be the solution to* (1.1)–(1.2). *Then there exists a unique adjoint state* $\bar z \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C_0(\Omega)$ *satisfying*

$$\begin{cases} -\Delta \bar z = \bar u - u_d & \text{in } \Omega, \\ \bar z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and

$$-\langle q - \bar q, \bar z\rangle + \alpha \|\bar q\|_{\mathcal{M}(\Omega)} \le \alpha \|q\|_{\mathcal{M}(\Omega)} \quad \text{for all } q \in \mathcal{M}(\Omega). \tag{2.2}$$

Furthermore this implies

$$\|\bar z\|_{C_0(\Omega)} \le \alpha, \tag{2.3}$$

the support of $\bar q$ *is contained in the set* $\{\, x \in \Omega \mid |\bar z(x)| = \alpha \,\}$, *and for the Jordan-decomposition* $\bar q = \bar q^+ - \bar q^-$ *we have*

$$\text{(2.4)} \qquad \operatorname{supp} \bar q^+ \subset \{x \in \Omega | \bar z(x) = -\alpha\} \quad \textit{and} \quad \operatorname{supp} \bar q^- \subset \{x \in \Omega | \bar z(x) = \alpha\}.$$

Remark 2.3. The optimality condition (2.2) can be equivalently reformulated as

$$\text{(2.5)} \qquad (u(q) - \bar u, \bar u - u_d) + \alpha\big(\|q\|_{\mathcal{M}(\Omega)} - \|\bar q\|_{\mathcal{M}(\Omega)}\big) \geq 0 \quad \text{for all} \ \ q \in \mathcal{M}(\Omega).$$

The statement of the above theorem directly implies the following corollary on the structure of the optimal control $\bar q$.

Corollary 2.4. *There exist* $\eta > 0$ *depending on the data of the problem such that*

$$\text{(2.6)} \qquad \operatorname{supp} \bar q \subset \Omega_\eta = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \eta\},$$

and additionally

$$\text{(2.7)} \qquad \operatorname{dist}(\operatorname{supp} \bar q^+, \operatorname{supp} \bar q^-) > \eta.$$

The first property implies that the support is compact.

Proof. The adjoint state $\bar z$ belongs to $W^{1,p}(\Omega)$ with $p > d$ and $W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\bar\Omega)$ with some $\beta > 0$. This implies (due to the homogeneous Dirichlet boundary conditions) the existence of $\eta > 0$ such that

$$|\bar z(x)| < \frac{\alpha}{2} \quad \text{for } x \in \Omega \setminus \Omega_\eta.$$

We complete the first part of the proof using the statement on the support of $\bar q$ from Theorem 2.2. With a similar argument we derive the second statement, since due to (2.4), the adjoint state attains the values $\pm\alpha$ respectively on the support of $\bar q^-$ and $\bar q^+$. ◻

Finally, we will derive an additional regularity for $\bar u$ if the desired state u_d is bounded. This structural property is not required for the general error estimate in section 4 and is only used for the improved estimate in section 5.

Theorem 2.5. *Assume that the desired state* u_d *is in* $L^\infty(\Omega)$. *Then the optimal state* $\bar u$ *is also in* $L^\infty(\Omega)$ *and there holds*

$$\|\bar u\|_{L^\infty(\Omega)} \leq \|u_d\|_{L^\infty(\Omega)}.$$

A direct consequence of this theorem is an additional regularity for the optimal control $\bar q$ and for the optimal state $\bar u$.

Corollary 2.6. *Assume that the desired state* u_d *is in* $L^\infty(\Omega)$. *Then the optimal state* $\bar u$ *lies in* $H^1_0(\Omega) \cap L^\infty(\Omega)$ *and the optimal control* $\bar q$ *lies in* $H^{-1}(\Omega)$. *There holds*

$$\|\nabla \bar u\|^2_{L^2(\Omega)} \leq \|\bar q\|_{\mathcal{M}(\Omega)} \|u_d\|_{L^\infty(\Omega)} \quad \textit{and} \quad \|\bar q\|_{H^{-1}(\Omega)} = \|\nabla \bar u\|_{L^2(\Omega)}.$$

In order to prove Theorem 2.5 and Corollary 2.6 we use some results from potential theory. First, we introduce the Green's function $G_\Omega \colon \Omega \times \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ as in, e.g., [3] or [15]. Then, for a positive measure $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, we define the numeric function $v^* \colon \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\text{(2.8)} \qquad v^* = S(\mu) := \int_\Omega G_\Omega(\cdot, y) \mathrm{d}\mu(y),$$

which is subharmonic and thus lower semicontinuous (see again [3]). If we normalize G_Ω by the right constant, we obtain the following simple result.

LEMMA 2.7. *For a compactly supported $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, the weak solution $v \in W_0^{1,s}(\Omega)$ with $1 \leq s < \frac{d}{d-1}$ to the problem*

$$\begin{aligned} -\Delta v &= \mu \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.9}$$

is equal to $v^* = S(\mu)$ (Lebesgue-) almost everywhere.

Proof. With [3, Theorem 4.3.8] the function v^* is a distributional solution of (2.9), and by a density argument, it is also a weak solution, unique in an almost everywhere sense. ∎

With the help of the above lemma, we obtain a pointwise representative of the optimal solution $u^* \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$, defined as

$$u^* := S(\bar{q}^+) - S(\bar{q}^-) = S(\bar{q}).$$

Due to (2.6) the measures $\bar{q}^+$ and $\bar{q}^-$ are compactly supported, and with (2.7) u^* is well-defined with values in $\mathbb{R} \cup \{-\infty, \infty\}$. With Lemma 2.7 we easily derive that $u^* = \bar{u}$ almost everywhere.

The next lemma states (roughly speaking) that if the optimal state is bounded on $\operatorname{supp} \bar{q}$, then it is bounded everywhere on Ω by the same constant. For positive measures in $\mathcal{M}(\Omega)$ this statement can be directly obtained from [15, Theorem 1.6′] in the two-dimensional case. For $d = 3$, the analogous theorem (see [15, Theorem 1.10]) is stated only for $\Omega = \mathbb{R}^d$. Therefore, we provide a direct proof.

LEMMA 2.8. *Let $\bar{q} \in \mathcal{M}(\Omega)$ be the optimal control. If $u^* = S(\bar{q})$ is bounded from above by some constant $C^+ \geq 0$ on $\operatorname{supp} q^+$, then it is bounded everywhere by C^+. Analogously, if u^* is bounded from below by some $C^- \leq 0$ on $\operatorname{supp} q^-$, then u^* is bounded from below everywhere by C^-.*

Proof. Suppose $u^* \leq C^+$ on $\operatorname{supp} q^+$. With (2.7) we estimate

$$S(\bar{q}^+) = u^* + S(\bar{q}^-) \leq C^+ + c_\eta \|\bar{q}^-\|_{\mathcal{M}(\Omega)} \quad \text{on} \quad \operatorname{supp} \bar{q}^+ ,$$

where $c_\eta = c \log(\frac{1}{\eta} \operatorname{diam} \Omega)$ for $d = 2$ and $c_\eta = \frac{c}{\eta}$ for $d = 3$ due to the growth properties of the Green's function. Thus, $S(\bar{q}^+)$ is bounded on $\operatorname{supp} \bar{q}^+$ as well. With [3, Corollary 4.5.2] we can now construct a sequence of compact sets $\{K_i\}$ with

$$\bar{q}^+(\operatorname{supp} \bar{q}^+ \setminus K_i) \to 0 \quad \text{for } i \to \infty \tag{2.10}$$

such that the functions $S(\bar{q}^+|_{K_i})$ are continuous. Now, we consider the solutions

$$u_i = S(\bar{q}^+|_{K_i}) - S(\bar{q}^-) \leq u^*.$$

Recalling that $-S(\bar{q}^-)$ is upper semicontinuous, we obtain that each u_i is upper semicontinuous as well. For each x_0 on the boundary of $\Omega \setminus \operatorname{supp} \bar{q}^+$, which is a subset of $\operatorname{supp} \bar{q}^+ \cup \partial\Omega$, we have $u_i(x_0) \leq u^*(x_0) \leq C^+$ and with upper semicontinuity

$$\limsup_{x \to x_0} u_i(x) \leq C^+. \tag{2.11}$$

Using the fact that u_i is subharmonic on $\Omega \setminus \operatorname{supp} q^+$ and the condition (2.11) we apply the maximum principle for subharmonic functions [3, Theorem 3.1.5] and obtain that u_i is bounded by C^+ everywhere on Ω for every i.

To complete the proof, it remains to show the convergence $u_i(x) \to u^*(x)$ for all $x \in \Omega \setminus \operatorname{supp} \bar{q}^+$. Let $x \in \Omega \setminus \operatorname{supp} \bar{q}^+$ be fixed. We denote by $\delta = \operatorname{dist}(x, \operatorname{supp} \bar{q}^+) > 0$. There holds

$$|u_i(x) - u^*(x)| = |S(\bar{q}^+|_{K_i})(x) - S(\bar{q}^+)(x)| \le c_\delta \bar{q}^+(\operatorname{supp} \bar{q}^+ \setminus K_i) \to 0, \quad i \to \infty,$$

where we have again used growth properties of the Green's function and (2.10).

The second statement is proved completely analogously. ∎

With these preparations we can give proofs of the claimed results.

Proof of Theorem 2.5. Assume the contrary, i.e., that we have $C, \varepsilon > 0$, such that $|u_d| \le C$ almost everywhere in Ω, but $|\bar{u}| > C + \varepsilon$ on some set of positive Lebesgue measure,

$$|\{\, x \in \Omega \mid \bar{u}(x) > C + \varepsilon \,\}| > 0.$$

Due to Lemma 2.8 we can find some point $x \in \operatorname{supp} q^+$ where the state $u^* = S(\bar{q})$ is larger than $C+\varepsilon$. Considering a ball $B_\eta(x)$ of radius η around this point, we have with Corollary 2.4 that $\bar{q}^-|_{B_\eta(x)} = 0$ and therefore that $S(\bar{q}|_{B_\eta(x)})$ is lower semicontinuous. We decompose

$$u^* = S(\bar{q}|_{B_\eta(x)}) + S(\bar{q}|_{\Omega \setminus B_\eta(x)})$$

and obtain that $S(\bar{q}|_{\Omega \setminus B_\eta(x)})$ is harmonic and consequently continuous on $B_\eta(x)$. This implies the lower semicontinuity of u^* on $B_\eta(x)$. Thus, the set

$$\{\, y \in B_\eta(x) \mid u^*(y) > C + \varepsilon \,\}$$

is open, and we can find a radius $r > 0$ such that $\bar{u} \ge C + \varepsilon$ almost everywhere in the ball $B_r(x)$.

Note that $x \in \operatorname{supp} \bar{q}^+$ implies $\bar{z}(x) = -\alpha$ with Theorem 2.2. We define w to be the solution to

$$\begin{aligned} -\Delta w &= \varepsilon \quad \text{in } B_r(x), \\ w &= 0 \quad \text{on } \partial B_r(x), \end{aligned}$$

which is clearly strictly positive at x. Considering the minimum principle for $\tilde{z} = \bar{z} - w$ which solves

$$\begin{aligned} -\Delta \tilde{z} &= \bar{u} - u_d - \varepsilon \ge 0 \quad \text{in } B_r(x), \\ \tilde{z} &= \bar{z} \quad \text{on } \partial B_r(x), \end{aligned}$$

we see that the minimum value $z_{\min} = \inf_{x \in B_r(x)} \tilde{z}(x)$ must be attained for some $x' \in \partial B_r(x)$. Comparing with the center x we find

$$\bar{z}(x') = \tilde{z}(x') = (\bar{z} - w)(x') \le (\bar{z} - w)(x) < \bar{z}(x) = -\alpha,$$

which is a violation of the bounds on the adjoint state (2.3) and thus a contradiction. ∎

Proof of Corollary 2.6. The result can be derived by considering a sequence of smooth approximations to $\bar{q}$, testing the corresponding state equation with the smooth solution and a subsequential weak limit argument.

However, the statement directly follows from a well-known classical result: Since u^* is Borel-measurable (as the difference of two lower semicontinuous functions) we can pair u^* with $\bar{q}$, and since, by the previous theorem, u^* is bounded, we obtain

$$\|\bar{q}\|_{\mathcal{M}(\Omega)} \|u^*\|_{L^\infty(\Omega)} \geq \langle \bar{q}, u^* \rangle = \int_\Omega u^*(x) \mathrm{d}\bar{q}(x) = \int_\Omega \int_\Omega G_\Omega(x,y) \mathrm{d}\bar{q}(x) \mathrm{d}\bar{q}(y).$$

With [15, Theorem 1.20], this implies $\nabla u^* \in L^2(\Omega)$ and

$$\int_\Omega \int_\Omega G_\Omega(x,y) \mathrm{d}\bar{q}(x) \mathrm{d}\bar{q}(y) = \|\nabla u^*\|^2_{L^2(\Omega)},$$

which implies the first part of the claim. The second assertion is evident. ◻

3. Discretization. For the discretization of the state equation we use linear finite elements on a family of shape regular quasi-uniform triangulations $\{\mathcal{T}_h\}_h$; see, e.g., [4]. The discretization parameter h denotes the maximal diameter of cells $K \in \mathcal{T}_h$. We set

$$\bar{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} \bar{K}$$

and make the usual assumption

$$|\Omega \setminus \Omega_h| \leq ch^2.$$

The finite element space associated with $\mathcal{T}_h$ is defined as usual by

$$V_h = \{\, v_h \in C_0(\Omega) \mid v_h|_K \in \mathcal{P}_1(K) \quad \text{for all } K \in \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \Omega \setminus \Omega_h \,\}.$$

For a given $q \in \mathcal{M}(\Omega)$ the discrete solution $u_h = u_h(q)$ is determined by

$$u_h \in V_h \ : \ (\nabla u_h, \nabla v_h) = \langle q, v_h \rangle \quad \text{for all } v_h \in V_h. \tag{3.1}$$

To define the approximation of the optimal control problem (1.1)–(1.2) we follow the approach from [8] and do not discretize the control space; cf. the variational approach by [14]. The discrete optimal control problem is then given as

$$\text{Minimize} \ \ J(q_h, u_h), \quad q_h \in \mathcal{M}(\Omega) \text{ and subject to (3.1).} \tag{3.2}$$

The existence of a solution can be shown as on the continuous level. The optimal state $\bar{u}_h$ is unique. The discrete solution operator mapping $q \in \mathcal{M}(\Omega)$ to $u_h(q)$ is not injective and the uniqueness of the optimal control cannot be guaranteed. However, one special solution can be identified, which is numerically accessible; see [8] and the discussion below.

By $\{x_i\}$, $i = 1, 2 \ldots, N_h$, we denote the interior nodes of Ω_h and by $\{e_i\} \subset V_h$ the corresponding node basis functions. We introduce the space $\mathcal{M}_h$ consisting of linear combination of Dirac functionals associated with the nodes x_i:

$$\mathcal{M}_h = \left\{ q_h \in \mathcal{M}(\Omega) \;\middle|\; q_h = \sum_{i=1}^{N_h} \beta_i \, \delta_{x_i}, \ \ \beta_i \in \mathbb{R}, \ i = 1, 2, \ldots, N_h \right\}$$

and an operator $\Lambda_h \colon \mathcal{M}(\Omega) \to \mathcal{M}_h$ (see [8]) by

$$\Lambda_h q = \sum_{i=1}^{N_h} \langle q, e_i \rangle \, \delta_{x_i}. \tag{3.3}$$

There holds the following theorem; see [8].

THEOREM 3.1. *Among the solutions to* (3.2) *there exists a unique solution* $\bar q_h \in \mathcal{M}_h$ *with the corresponding state* $\bar u_h = u_h(\bar q_h)$. *Any other solution* $\tilde q_h \in \mathcal{M}(\Omega)$ *satisfies* $\Lambda_h \tilde q_h = \bar q_h$. *Moreover there holds*

$$\bar q_h \overset{*}{\rightharpoonup} \bar q \ \text{ in } \mathcal{M}(\Omega) \quad \text{and} \quad \|\bar q_h\|_{\mathcal{M}(\Omega)} \to \|\bar q\|_{\mathcal{M}(\Omega)}$$

for $h \to 0$.

For the solution $(\bar q_h, \bar u_h)$ from this theorem the following discrete version of the optimality conditions holds, which can be derived as in the continuous case; cf. [8].

THEOREM 3.2. *Let* $(\bar q_h, \bar u_h) \in \mathcal{M}_h \times V_h$ *be the discrete solution; see Theorem* 3.1. *Then there exists the discrete adjoint state* $\bar z_h \in V_h$ *fulfilling*

$$(\nabla v_h, \nabla \bar z_h) = (\bar u_h - u_d, v_h) \quad \text{for all } v_h \in V_h$$

and the optimality condition

$$-\langle q - \bar q_h, \bar z_h\rangle + \alpha \|\bar q_h\|_{\mathcal{M}(\Omega)} \le \alpha \|q\|_{\mathcal{M}(\Omega)} \quad \text{for all } \ q \in \mathcal{M}(\Omega). \tag{3.4}$$

The last condition can be equivalently rewritten as

$$(u_h(q) - \bar u_h, \bar u_h - u_d) + \alpha\big(\|q\|_{\mathcal{M}(\Omega)} - \|\bar q_h\|_{\mathcal{M}(\Omega)}\big) \ge 0 \quad \text{for all } \ q \in \mathcal{M}(\Omega); \tag{3.5}$$

cf. Remark 2.3.

In order to prove our main result mentioned in the introduction, we first provide some estimates for the error $u(q) - u_h(q)$ for a fixed control $q \in \mathcal{M}(\Omega)$.

LEMMA 3.3. *Let* $q \in \mathcal{M}(\Omega)$ *with associated continuous and discrete states* $u = u(q)$ *and* $u_h = u_h(q)$ *be given. Then the following holds:*

$$\text{(i)} \quad \|u - u_h\|_{L^p(\Omega)} \le c_p h^{2-\frac{d}{p'}} \|q\|_{\mathcal{M}(\Omega)}, \quad p \in \left(1, \tfrac{d}{d-2}\right), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

$$\text{(ii)} \quad \|u - u_h\|_{L^1(\Omega)} \le c h^2 |\ln h|^r \, \|q\|_{\mathcal{M}(\Omega)}$$

with $r = 2$ *for* $d = 2$ *and* $r = \frac{11}{4}$ *for* $d = 3$.

Proof. (i) For the first estimate in case $p = 2$ we refer, e.g., to [6]. For a general case, $p \in (1, \frac{d}{d-2})$, we set $e = u - u_h$ and

$$g_p(x) = |e(x)|^{p-1} \operatorname{sgn}(e(x)).$$

By a direct calculation it follows that $g_p \in L^{p'}(\Omega)$ and

$$\|g_p\|_{L^{p'}(\Omega)} = \|e\|^{p-1}_{L^p(\Omega)}.$$

We consider a dual problem

$$w \in H^1_0(\Omega) \ : \ (\nabla w, \nabla v) = (g_p, v) \quad \text{for all } v \in H^1_0(\Omega)$$

and its Ritz projection

$$w_h \in V_h \ : \ (\nabla w_h, \nabla v_h) = (g_p, v_h) \quad \text{for all } v_h \in V_h.$$

With the help of this we can write

$$\begin{aligned} \|e\|^p_{L^p(\Omega)} &= (e, g_p) = (\nabla e, \nabla w) \\ &= (\nabla e, \nabla (w - w_h)) = (\nabla u, \nabla (w - w_h)) \\ &= \langle q, w - w_h\rangle \le \|q\|_{\mathcal{M}(\Omega)} \, \|w - w_h\|_{C_0(\Omega)} \end{aligned}$$

using the Galerkin orthogonality for both errors $u - u_h$ and $w - w_h$. By the elliptic regularity we obtain $w \in W^{2,p'}(\Omega)$ with

$$\|\nabla^2 w\|_{L^{p'}(\Omega)} \leq c\|g_p\|_{L^{p'}(\Omega)}$$

and since $p' > \frac{2}{d}$, a corresponding L^∞-estimate can be obtained. With an inverse estimate we get

$$\|w - w_h\|_{C_0(\Omega)} \leq \|w - i_h w\|_{C_0(\Omega)} + ch^{-\frac{d}{p'}}\|i_h w - w_h\|_{L^{p'}(\Omega)},$$

where i_h is the nodal interpolation. With well-known interpolation estimates for the nodal interpolant in L^∞ and $L^{p'}$ and a further application of the triangle inequality, we arrive at

$$\|w - w_h\|_{C_0(\Omega)} \leq ch^{2-\frac{d}{p'}}\|\nabla^2 w\|_{L^{p'}(\Omega)} + ch^{-\frac{d}{p'}}\|w - w_h\|_{L^{p'}(\Omega)}.$$

The optimal estimate

$$\|w - w_h\|_{L^{p'}(\Omega)} \leq c_p h^2 \|\nabla^2 w\|_{L^{p'}(\Omega)}$$

was first given in [18], albeit only for $d = 2$. However, the stability of the Ritz-projection in $W^{1,p'}$, which is the central ingredient of the proof, is also known to hold for $d = 3$ (see [4, Theorem 8.5.3]), so the proof can be repeated one for one.

Putting everything together, we obtain for the error $\|e\|_{L^p(\Omega)}$ the estimate

$$\begin{aligned}\|e\|^p_{L^p(\Omega)} &\leq \|q\|_{\mathcal{M}(\Omega)}\,\|w - w_h\|_{C_0(\Omega)}\\ &\leq c_p h^{2-\frac{d}{p'}}\,\|q\|_{\mathcal{M}(\Omega)}\|e\|^{p-1}_{L^p(\Omega)},\end{aligned}$$

which gives the desired result.

(ii) To obtain the second estimate, we set $g_1 = \operatorname{sgn}(e) \in L^\infty(\Omega)$. There holds

$$\|e\|_{L^1(\Omega)} = (e, g_1).$$

We consider a dual problem

$$w \in H^1_0(\Omega) \ : \ (\nabla w, \nabla v) = (g_1, v) \quad \text{for all } v \in H^1_0(\Omega)$$

and its Ritz projection

$$w_h \in V_h \ : \ (\nabla w_h, \nabla v_h) = (g_1, v_h) \quad \text{for all } v_h \in V_h.$$

Then we obtain using the Galerkin orthogonality for both errors $u - u_h$ and $w - w_h$

$$\begin{aligned}\|e\|_{L^1(\Omega)} &= (e, g_1) = (\nabla e, \nabla w)\\ &= (\nabla e, \nabla(w - w_h)) = (\nabla u, \nabla(w - w_h))\\ &= \langle q, w - w_h\rangle \leq \|q\|_{\mathcal{M}(\Omega)}\,\|w - w_h\|_{C_0(\Omega)}.\end{aligned}$$

For the pointwise error in w we use the result from Frehse and Rannacher [13] for $d = 2$ and Rannacher [17] for $d = 3$ and obtain

$$\|w - w_h\|_{C_0(\Omega)} \leq ch^2|\ln h|^r\,\|g_1\|_{L^\infty(\Omega)}.$$

This completes the proof. $\square$

Via the Sobolev embedding theorem we can easily derive an estimate of the form

$$\|u\|_{L^t(\Omega)} \leq c_t \|q\|_{\mathcal{M}(\Omega)} \quad \text{for all } t < \frac{d}{d-2}$$

for the continuous solutions. For the discrete solutions we can also give a result in the limiting case for t.

LEMMA 3.4. *Let $q \in \mathcal{M}(\Omega)$ with the discrete solution $u_h = u_h(q)$ as above. Then we have*

$$\|u_h\|_{L^\infty(\Omega)} \leq c|\ln h|^{\frac{3}{2}} \|q\|_{\mathcal{M}(\Omega)} \quad \textit{for } d = 2,$$
$$\|u_h\|_{L^3(\Omega)} \leq c|\ln h| \|q\|_{\mathcal{M}(\Omega)} \quad \textit{for } d = 3.$$

Proof. In the first step we estimate

$$\|u_h\|_{L^\infty(\Omega)} \leq c|\ln h|^{\frac{1}{2}} \|\nabla u_h\|_{L^2(\Omega)} \quad \text{for } d = 2$$

by the discrete Sobolev inequality (see [4, Lemma 4.9.1]) and

$$\|u_h\|_{L^3(\Omega)} \leq c\|\nabla u_h\|_{L^{\frac{3}{2}}(\Omega)} \quad \text{for } d = 3$$

by the Sobolev embedding. Defining $\sigma = \frac{d}{d-1}$ ($\sigma = 2$ and $\sigma = \frac{3}{2}$ for two and three dimensions, respectively), we proceed in a common way with an inverse estimate and the stability of the Ritz projection with respect to the $W^{1,s}$-seminorm (see [4, Theorem 8.5.3]),

$$\begin{aligned} \|\nabla u_h\|_{L^\sigma(\Omega)} &\leq c\, h^{\frac{d}{\sigma} - \frac{d}{s}} \|\nabla u_h\|_{L^s(\Omega)} \\ &\leq c h^{\frac{d}{\sigma} - \frac{d}{s}} \|\nabla u\|_{L^s(\Omega)} \end{aligned}$$

for any $1 < s < \sigma$, where the constant c is independent of s. Then we choose $s = s_\varepsilon = \sigma - \varepsilon$ for $0 < \varepsilon < \sigma - 1$, which implies that

$$\frac{d}{\sigma} - \frac{d}{s_\varepsilon} = -\frac{d\varepsilon}{\sigma(\sigma - \varepsilon)} > -\varepsilon\, d\sigma^{-1} = -\varepsilon(d-1).$$

We obtain by Lemma 2.1

$$\|\nabla u_h\|_{L^\sigma(\Omega)} \leq \frac{c}{\varepsilon} h^{-\varepsilon(d-1)} \|q\|_{\mathcal{M}(\Omega)}.$$

Choosing now $\varepsilon = \frac{1}{|\ln h|}$ we obtain

$$\|\nabla u_h\|_{L^\sigma(\Omega)} \leq c|\ln h| \|q\|_{\mathcal{M}(\Omega)},$$

which, together with the first estimate, completes the proof. ∎

4. General error estimates. In the next theorem we provide an error estimate for the error with respect to the cost functional. To state this theorem we need an assumption on the desired state u_d.

Assumption 1. We assume

$$u_d \in \begin{cases} L^\infty(\Omega) & \text{for } d = 2, \\ L^3(\Omega) & \text{for } d = 3. \end{cases}$$

Remark 4.1. Assumption 1 is only slightly stronger than the corresponding assumption in [8], where $u_d \in L^4(\Omega)$ in two dimensions and $u_d \in L^{\frac{8}{3}}(\Omega)$ in three dimensions is assumed.

THEOREM 4.2. *Let Assumption* 1 *be fulfilled. Moreover let* $(\bar{q}, \bar{u})$ *be the solution to* (1.1)–(1.2) *and* $(\bar{q}_h, \bar{u}_h) \in \mathcal{M}_h \times V_h$ *be the discrete solution; see Theorem* 3.1. *Then there holds*

$$|J(\bar{q}, \bar{u}) - J(\bar{q}_h, \bar{u}_h)| \le c\, h^{4-d} |\ln h|^\gamma$$

with $\gamma = \frac{7}{2}$ *for* $d = 2$ *and* $\gamma = 1$ *for* $d = 3$.

Proof. By the optimality we obtain

$$J(\bar{q}, \bar{u}) \le J(\bar{q}_h, u(\bar{q}_h)) \quad \text{and} \quad J(\bar{q}_h, \bar{u}_h) \le J(\bar{q}, u_h(\bar{q})).$$

Consequently we have

$$J(\bar{q}, \bar{u}) - J(\bar{q}, u_h(\bar{q})) \le J(\bar{q}, \bar{u}) - J(\bar{q}_h, \bar{u}_h) \le J(\bar{q}_h, u(\bar{q}_h)) - J(\bar{q}_h, \bar{u}_h).$$

Therefore, it remains to estimate the error with respect to the cost functional for a fixed $q \in \mathcal{M}(\Omega)$, i.e.,

$$|J(q, u(q)) - J(q, u_h(q))| = \left| \frac{1}{2} \|u(q) - u_d\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_h(q) - u_d\|^2_{L^2(\Omega)} \right|$$

and then to apply this estimate for both $q = \bar{q}$ and $q = \bar{q}_h$.

For fixed $q \in \mathcal{M}(\Omega)$ we now use the notation $u = u(q)$ and $u_h = u_h(q)$. There holds

$$\begin{aligned} J(q, u) - J(q, u_h) &= \frac{1}{2} \|u - u_d\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_h - u_d\|^2_{L^2(\Omega)} \\ &= \frac{1}{2} (u - u_h, u + u_h - 2u_d) \\ &= -(u - u_h, u_d) + \frac{1}{2} \|u - u_h\|^2_{L^2(\Omega)} + (u - u_h, u_h). \end{aligned} \tag{4.1}$$

For the second term in (4.1) we obtain by the estimate (i) for $p = 2$ from Lemma 3.3

$$\|u - u_h\|^2_{L^2(\Omega)} \le c h^{4-d} \|q\|^2_{\mathcal{M}(\Omega)}.$$

The other terms are estimated separately in two dimensions and in three dimensions.

The case $d = 2$. The first and last terms in (4.1) are estimated using (ii) from Lemma 3.3:

$$\begin{aligned} (u - u_h, u_d) &\le \|u - u_h\|_{L^1(\Omega)} \|u_d\|_{L^\infty(\Omega)} \le c h^2 |\ln h|^2 \|q\|_{\mathcal{M}(\Omega)}, \\ (u - u_h, u_h) &\le \|u - u_h\|_{L^1(\Omega)} \|u_h\|_{L^\infty(\Omega)} \le c h^2 |\ln h|^2 \|q\|_{\mathcal{M}(\Omega)} \|u_h\|_{L^\infty(\Omega)}. \end{aligned}$$

Additionally, by Lemma 3.4 we have $\|u_h\|_{L^\infty(\Omega)} \le |\ln h|^{\frac{3}{2}} \|q\|_{\mathcal{M}(\Omega)}$.

The case $d = 3$. Now, we use (i) for $p = \frac{3}{2}$ from Lemma 3.3 for the remaining terms in (4.1) to obtain

$$\begin{aligned} (u - u_h, u_d) &\le \|u - u_h\|_{L^{\frac{3}{2}}(\Omega)} \|u_d\|_{L^3(\Omega)} \le c h \|q\|_{\mathcal{M}(\Omega)}, \\ (u - u_h, u_h) &\le \|u - u_h\|_{L^{\frac{3}{2}}(\Omega)} \|u_h\|_{L^3(\Omega)} \le c h \|q\|_{\mathcal{M}(\Omega)} \|u_h\|_{L^3(\Omega)}. \end{aligned}$$

We apply Lemma 3.4 again and complete the proof. ∎

Remark 4.3. Assumption 1 excludes the case where the desired state u_d is given as a Green's function. However, for construction of irregular examples with known exact solutions (see section 8), it is desirable to choose u_d to be the solution of

$$\begin{aligned} -\Delta u_d &= \delta_{x_0} && \text{in } \Omega, \\ u_d &= 0 && \text{on } \partial\Omega \end{aligned}$$

with some $x_0 \in \Omega$. For this choice of u_d there holds

$$u_d \in L^p(\Omega) \quad \text{for all } p \in (1, \infty) \quad \text{for } d = 2$$

and

$$u_d \in L^{3-\varepsilon}(\Omega) \quad \text{for all } \varepsilon \in (0, 1) \quad \text{for } d = 3.$$

The result of Theorem 4.2 can be directly extended to this situation. In this case an additional logarithmic term $|\ln h|$ will appear.

In the next theorem we prove the main estimate for the error in the state variable, as announced in (1.3).

THEOREM 4.4. *Let the conditions of Theorem* 4.2 *be fulfilled. Then there holds*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le ch^{2-\frac{d}{2}} |\ln h|^{\frac{\gamma}{2}}.$$

Proof. We use the optimality condition (2.5), choose $q = \bar{q}_h$, and obtain

$$(u(\bar{q}_h) - \bar{u}, \bar{u} - u_d) + \alpha\big(\|\bar{q}_h\|_{\mathcal{M}(\Omega)} - \|\bar{q}\|_{\mathcal{M}(\Omega)}\big) \ge 0.$$

For the corresponding discrete optimality condition (3.5) we choose $q = \bar{q}$, resulting in

$$(u_h(\bar{q}) - \bar{u}_h, \bar{u}_h - u_d) + \alpha\big(\|\bar{q}\|_{\mathcal{M}(\Omega)} - \|\bar{q}_h\|_{\mathcal{M}(\Omega)}\big) \ge 0.$$

Adding these two inequalities we arrive at

$$(u(\bar{q}_h) - \bar{u}, \bar{u} - u_d) + (u_h(\bar{q}) - \bar{u}_h, \bar{u}_h - u_d) \ge 0.$$

Rearranging the terms we obtain

$$(\bar{u}_h - \bar{u}, \bar{u} - u_d) + (u(\bar{q}_h) - \bar{u}_h, \bar{u} - u_d) + (\bar{u} - \bar{u}_h, \bar{u}_h - u_d) + (u_h(\bar{q}) - \bar{u}, \bar{u}_h - u_d) \ge 0,$$

resulting in

(4.2)
$$\begin{aligned} \|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} &\le (u(\bar{q}_h) - \bar{u}_h, \bar{u} - u_d) + (u_h(\bar{q}) - \bar{u}, \bar{u}_h - u_d) \\ &= (u(\bar{q}_h) - \bar{u}_h, \bar{u} - u_h(\bar{q})) + (u(\bar{q}_h) - \bar{u}_h, u_h(\bar{q}) - u_d) + (u_h(\bar{q}) - \bar{u}, \bar{u}_h - u_d). \end{aligned}$$

For the first term in (4.2) we obtain by the estimate (i) for $p = 2$ from Lemma 3.3

$$(u(\bar{q}_h) - \bar{u}_h, \bar{u} - u_h(\bar{q})) \le \|u(\bar{q}_h) - \bar{u}_h\|_{L^2(\Omega)} \|\bar{u} - u_h(\bar{q})\|_{L^2(\Omega)} \le ch^{4-d} \|\bar{q}\|_{\mathcal{M}(\Omega)} \|\bar{q}_h\|_{\mathcal{M}(\Omega)}.$$

The second and third terms in (4.2) are estimated with the same procedure as in the proof of Theorem 4.2, resulting in

$$\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \le ch^{4-d} |\ln h|^{\gamma}.$$

This completes the proof. ∎

With help of this result, we can also provide an estimate for the error of the control in $H^{-2}(\Omega)$.

COROLLARY 4.5. *Let the conditions of Theorem* 4.2 *be fulfilled. Then there holds*

$$\|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} \le ch^{2-\frac{d}{2}}|\ln h|^{\frac{\gamma}{2}}.$$

Proof. For each $q \in \mathcal{M}(\Omega)$ we have

$$\|q\|_{H^{-2}(\Omega)} = \sup_{\varphi \in H^2(\Omega)} \frac{(q,\varphi)}{\|\varphi\|_{H^2(\Omega)}} = \sup_{\varphi \in H^2(\Omega)} \frac{(u(q), -\Delta\varphi)}{\|\varphi\|_{H^2(\Omega)}} \le c\|u(q)\|_{L^2(\Omega)}.$$

Thus we obtain (recall that $\bar{u} = u(\bar{q})$, $\bar{u}_h = u_h(\bar{q}_h)$)

$$\|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} \le c\|u(\bar{q}) - u(\bar{q}_h)\|_{L^2(\Omega)} \le c(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|u_h(\bar{q}_h) - u(\bar{q}_h)\|_{L^2(\Omega)}).$$

The first term is covered by Theorem 4.4, and for the second term we can apply the a priori estimate from Lemma 3.3(i) with $p = 2$. ∎

5. Improved error estimates. In the following we exploit the additional regularity derived in section 2 to provide an improved estimate under the assumption that u_d is bounded.

THEOREM 5.1. *For both $d = 2$ and $d = 3$, let $(\bar{q}, \bar{u})$ be the solution to* (1.1)–(1.2) *and let $(\bar{q}_h, \bar{u}_h) \in \mathcal{M}_h \times V_h$ be the discrete solution; see Theorem* 3.1. *Moreover let $u_d \in L^\infty(\Omega)$, which implies $\bar{u} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\bar{q} \in H^{-1}(\Omega)$ with Theorems* 2.5 *and* 2.6. *Then there holds*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le c\, h|\ln h|^{\frac{r}{2}}$$

with the constant r as in Lemma 3.3.

Proof. First, we obtain an $L^2(\Omega)$ estimate for $\bar{u}_h - \bar{u}$ in terms of an $L^\infty(\Omega)$-error for the adjoint state. For that, we use the optimality condition (2.2), choosing $q = \bar{q}_h$,

$$-\langle \bar{q}_h - \bar{q}, \bar{z}\rangle + \alpha\|\bar{q}\|_{\mathcal{M}(\Omega)} \le \alpha\|\bar{q}_h\|_{\mathcal{M}(\Omega)},$$

and the optimality condition (3.4), choosing $q = \bar{q}$,

$$-\langle \bar{q} - \bar{q}_h, \bar{z}_h\rangle + \alpha\|\bar{q}_h\|_{\mathcal{M}(\Omega)} \le \alpha\|\bar{q}\|_{\mathcal{M}(\Omega)}.$$

Adding these two inequalities results in

$$\langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h\rangle \ge 0.$$

We introduce a discrete adjoint state $\hat{z}_h \in V_h$ for the continuous optimal solution defined by

$$(\nabla v_h, \nabla \hat{z}_h) = (\bar{u} - u_d, v_h) \quad \text{for all } v_h \in V_h$$

and $\hat{u}_h = u_h(\bar{q})$, the discrete solution for the continuous optimal control. There holds

$$\begin{aligned} 0 \le \langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h\rangle &= \langle \bar{q}_h - \bar{q}, \bar{z} - \hat{z}_h\rangle + \langle \bar{q}_h - \bar{q}, \hat{z}_h - \bar{z}_h\rangle \\ &= \langle \bar{q}_h - \bar{q}, \bar{z} - \hat{z}_h\rangle + (\nabla(\bar{u}_h - \hat{u}_h), \nabla(\hat{z}_h - \bar{z}_h)) \\ &= \langle \bar{q}_h - \bar{q}, \bar{z} - \hat{z}_h\rangle + (\bar{u}_h - \hat{u}_h, \bar{u} - \bar{u}_h) \\ &= \langle \bar{q}_h - \bar{q}, \bar{z} - \hat{z}_h\rangle + (\bar{u} - \hat{u}_h, \bar{u} - \bar{u}_h) - \|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)}. \end{aligned}$$

Rearranging terms and using Young's inequality we obtain

$$\|\bar u - \bar u_h\|^2_{L^2(\Omega)} \le \|\bar q_h - \bar q\|_{\mathcal{M}(\Omega)}\|\bar z - \hat z_h\|_{L^\infty(\Omega)} + \frac{1}{2}\|\bar u - \hat u_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\bar u - \bar u_h\|^2_{L^2(\Omega)},$$

which results in

$$\|\bar u - \bar u_h\|^2_{L^2(\Omega)} \le c\|\bar z - \hat z_h\|_{L^\infty(\Omega)} + \|\bar u - \hat u_h\|^2_{L^2(\Omega)}, \tag{5.1}$$

since $\|\bar q\|_{\mathcal{M}(\Omega)}$ and $\|\bar q_h\|_{\mathcal{M}(\Omega)}$ are bounded. For the first term we obtain with an L^∞-estimate as in the proof of Lemma 3.3(ii)

$$\|\bar z - \hat z_h\|_{L^\infty(\Omega)} \le ch^2|\ln h|^r\|\bar u - u_d\|_{L^\infty(\Omega)}.$$

The square root of the second term in (5.1) can be estimated by

$$\|\bar u - \hat u_h\|_{L^2(\Omega)} \le ch\|\bar q\|_{H^{-1}(\Omega)},$$

which can be obtained from standard estimates with a simple duality argument. Together with the improved regularity for $\bar u$ and $\bar q$ this completes the proof. ∎

6. Extensions. Let us consider some possible extensions of our results to more general problem settings. We remark that all the results will transfer with only minor modifications to the case of more general elliptic operators with smooth coefficients instead of the Laplacian in (1.2).

In the following, we will briefly consider a problem where the control is allowed to act only on a subset $\Omega_c \subset \Omega$ of the domain and the observation is restricted to another subset $\Omega_o \subset \Omega$:

$$\text{Minimize } J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega_o)} + \alpha\|q\|_{\mathcal{M}(\Omega_c)}, \quad q \in \mathcal{M}(\Omega_c), \tag{6.1}$$

subject to the state equation (1.2). For well-definedness, we require Ω_o to be open and Ω_c to be relatively closed in Ω, i.e.,

$$\Omega_c = \overline{\Omega}_c \setminus \partial\Omega.$$

See [12] for a detailed exposition. Denote by χ_{Ω_o} the characteristic function of Ω_o. Note that the solutions to (6.1) are not unique in general, since the strict convexity of the first term of J only guarantees uniqueness for the expression $\chi_{\Omega_o}\bar u$ for any optimal state solutions $\bar u$. The corresponding optimality system, as obtained in [12], is given in the following theorem.

THEOREM 6.1. *Let $(\bar q, \bar u)$ be a (not necessarily unique) solution to* (6.1). *The corresponding unique adjoint state is given by*

$$\begin{cases} -\Delta \bar z = \chi_{\Omega_o}(\bar u - u_d) & \text{in } \Omega, \\ \bar z = 0 & \text{on } \partial\Omega \end{cases} \tag{6.2}$$

and satisfies the inequality

$$|\bar z| \le \alpha \text{ on } \Omega_c. \tag{6.3}$$

Furthermore, the support of $\bar q$ is contained in the set $\{ x \in \Omega_c \mid |\bar z(x)| = \alpha \}$, and for the Jordan-decomposition $\bar q = \bar q^+ - \bar q^-$ we have

$$\operatorname{supp}\bar q^+ \subset \{ x \in \Omega_c \mid \bar z(x) = -\alpha \} \quad \text{and} \quad \operatorname{supp}\bar q^- \subset \{ x \in \Omega_c \mid \bar z(x) = \alpha \}. \tag{6.4}$$

The optimality of $\bar{u}$ can also be characterized by the following variational inequality:

$$(6.5)\quad (u(q)-\bar{u},\chi_{\Omega_o}(\bar{u}-u_d))+\alpha\|q\|_{\mathcal{M}(\Omega_c)}-\alpha\|\bar{q}\|_{\mathcal{M}(\Omega_c)}\geq 0 \quad \textit{for all}\ \ q\in\mathcal{M}(\Omega_c).$$

In the discrete setting, we consider the optimal control problem

$$(6.6)\qquad \text{Minimize}\ \ J(q_h,u_h),\quad q_h\in\mathcal{M}(\Omega_c)\text{ and subject to (3.1).}$$

As before, the control is not discretized at first. However, for practical computations it should be replaced with a discrete control $\bar{q}_h\in\mathcal{M}_h$ as in Theorem 3.1. Therefore, to ensure that the operator Λ_h defined in (3.3) maps from $\mathcal{M}(\Omega_c)$ to $\mathcal{M}(\Omega_c)\cap\mathcal{M}_h$, we require that for each h we have

$$(6.7)\qquad \overline{\Omega}_c\cap\overline{\Omega}_h=\bigcup_{K\in\mathcal{T}_h^c}\overline{K},$$

where $\mathcal{T}_h^c\subset\mathcal{T}_h$ is the collection of all the cells of the triangulation which make up the control region. Then, we can verify that

$$\Lambda_h(q)\in\mathcal{M}(\Omega_c)\cap\mathcal{M}_h\text{ for all }q\in\mathcal{M}(\Omega_c),$$

and any optimal solution $\tilde{q}_h$ of (6.6) can be replaced by a discrete optimal solution $\bar{q}_h=\Lambda_h(\tilde{q}_h)$ with the same objective value as in Theorem 3.1. There may still be more than one discrete solution for the same reasons as in the continuous case.

Most of our results are valid as well for problem (6.1) without any additional assumptions, since Lemmas 3.3 and 3.4 are applicable to this case without modification. In fact, if we repeat the steps of Theorems 4.2 and 4.4 line by line we obtain the following result.

THEOREM 6.2. *Let Assumption* 1 *be fulfilled. Moreover let $(\bar{q},\bar{u})$ be any solution to* (6.1) *and let $(\bar{q}_h,\bar{u}_h)\in(\mathcal{M}(\Omega_c)\cap\mathcal{M}_h)\times V_h$ be any discrete solution. Then there holds*

$$|J(\bar{q},\bar{u})-J(\bar{q}_h,\bar{u}_h)|\leq c\,h^{4-d}|\ln h|^{\gamma},$$
$$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega_o)}\leq c\,h^{2-\frac{d}{2}}|\ln h|^{\frac{\gamma}{2}}$$

with γ as in 4.2.

In comparison, Theorem 6.2 is not as strong as the version for $\Omega_o=\Omega_c=\Omega$, since we only get an estimate for the state on the observation domain. However, since $\bar{u}$ is not even unique (there are counterexamples), an estimate on the whole domain or any estimate for the controls cannot be expected in general.

In this setting we can also construct examples where the optimal control contains Dirac measures, even if u_d is bounded, by making sure that the singularities of $\bar{u}$ are located outside of Ω_o. Thus, the higher regularity of Theorem 2.5 and Corollary 2.6 does not hold in general. If Ω_c and $\overline{\Omega}_o$ are disjoint, we can typically expect the optimal control to be a linear combination of Dirac delta functions; cf. [5] for an application, where this is explicitly desired.

However, we can extend the estimate for the state error from Theorem 6.2 to the whole domain if the control domain is contained in the observation domain, i.e., if we have $\Omega_c\subset\Omega_o$. Then, the optimal control is uniquely determined by the values of $\bar{u}$ on the observation domain: Since $\chi_{\Omega_o}u(q)=0$ implies $u(q)=0$ with Lemma 2.8 and thus $q=0$ for any $q\in\mathcal{M}(\Omega_c)$, the control to observation operator mapping

$q \in \mathcal{M}(\Omega_c)$ to $\chi_{\Omega_o} u(q) \in L^2(\Omega_o)$ is injective. Therefore, the optimal solution $(\bar{q}, \bar{u})$ is unique. Under this condition we can additionally prove the following lemma.

LEMMA 6.3. *Assume* $\Omega_c \subset \Omega_o$. *Then*

$$\operatorname{supp} \bar{q}_h \,,\, \operatorname{supp} \bar{q} \subset \{\, x \in \Omega_o \mid \operatorname{dist}(x, \partial\Omega_o) > \eta \,\}$$

for some $\eta > 0$ *depending only on the data.*

Proof. We use that the adjoint state $\bar{z}$ is Hölder continuous as in Corollary 2.4. For the discrete adjoint states we can obtain the uniform bound

$$\|\bar{z}_h\|_{C^{0,\beta}(\Omega)} \le c\|\bar{z}_h\|_{W^{1,p}(\Omega)} \le c\|z(\bar{u}_h)\|_{W^{1,p}(\Omega)} \le c\|\bar{u}_h - u_d\|_{L^2(\Omega_o)} \le c\|u_d\|_{L^2(\Omega_o)}$$

using the stability of the Ritz projection in $W^{1,p}$ for $p > d$, where $z(\bar{u}_h)$ solves the continuous adjoint equation (6.3) with the discrete $\bar{u}_h$ instead of $\bar{u}$ on the right-hand side. Together with the Dirichlet boundary conditions and the conditions on the support of the optimal controls we therefore get

$$\operatorname{supp} \bar{q}_h \,,\, \operatorname{supp} \bar{q} \subset \{\, x \in \Omega_c \mid \operatorname{dist}(x, \partial\Omega) \ge \eta_1 \,\} = A_{\eta_1}$$

for some $\eta_1 > 0$ depending on the constant in the estimate before. The set A_{η_1} is compact since Ω_c is relatively closed. With $A_{\eta_1} \subset \Omega_o$ and Ω_o open, we find a suitable $\eta \le \eta_1$ by considering that $\operatorname{dist}(\cdot, \partial\Omega_o) > 0$ must assume a minimum on A_{η_1}. ∎

Under these conditions the higher regularity results from Theorem 2.5 and Corollary 2.6 can be transferred without modification and we can also derive an error estimate for the state on the whole domain.

THEOREM 6.4. *Assume* $\Omega_c \subset \Omega_o$ *and the conditions of Theorem* 6.2. *Then we have the estimate*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} \le c_\eta\, h^{2-\frac{d}{2}} |\ln h|^{\frac{\gamma}{2}}$$

with γ *as in* Theorem 4.2 *and* η *from Lemma* 6.3.

Proof. With the elliptic regularity and Lemma 3.3(i) we obtain

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le \|u(\bar{q} - \bar{q}_h)\|_{L^2(\Omega)} + \|u(\bar{q}_h) - \bar{u}_h\|_{L^2(\Omega)} \le \|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} + c\, h^{2-\frac{d}{2}}.$$

For the estimate of the control we choose a smooth function $\kappa_\eta \in C_0^\infty(\Omega)$ which is zero on $\Omega \setminus \Omega_o$ and equal to one on $\{\, x \in \Omega_o \mid \operatorname{dist}(x, \partial\Omega_o) > \eta \,\} \subseteq \Omega_c$. This is possible due to Lemma 6.3. Then we have for any $\psi \in H^2(\Omega)$ that

$$\langle \bar{q} - \bar{q}_h, \psi \rangle = \langle \bar{q} - \bar{q}_h, \kappa_\eta \psi \rangle = (\nabla u(\bar{q} - \bar{q}_h), \nabla(\kappa_\eta \psi)) = -(\bar{u} - u(\bar{q}_h), \Delta(\kappa_\eta \psi)).$$

For the expression in the last term we obtain

$$\Delta(\kappa_\eta \psi) = \Delta\kappa_\eta \psi + 2\nabla\kappa_\eta \nabla\psi + \kappa_\eta \Delta\psi,$$

and since the derivatives of κ_η are bounded and depend only on η, we can estimate

$$\|\Delta(\kappa_\eta \psi)\|_{L^2(\Omega)} \le c_\eta \|\psi\|_{H^2(\Omega)}.$$

Moreover $\Delta(\kappa_\eta \psi) = 0$ on $\Omega \setminus \Omega_o$ and thus we have

$$\langle \bar{q} - \bar{q}_h, \psi \rangle \le c_\eta \|\psi\|_{H^2(\Omega)} \|\bar{u} - u(\bar{q}_h)\|_{L^2(\Omega_o)}.$$

Dividing by $\|\psi\|_{H^2(\Omega)}$ and taking the supremum, we obtain

$$\begin{aligned} \|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} &\le c_\eta \|\bar{u} - u(\bar{q}_h)\|_{L^2(\Omega_o)} \\ &\le c_\eta (\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_o)} + \|\bar{u}_h - u(\bar{q}_h)\|_{L^2(\Omega_o)}) \le c_\eta h^{2-\frac{d}{2}} |\ln h|^{\frac{\gamma}{2}}, \end{aligned}$$

where we applied with Theorem 6.2 the result of Lemma 3.3(i). This concludes the proof. ∎

7. Computational aspects. For the numerical computation of optimal controls we are going to consider a Tikhonov regularized version of the optimal control problem. Then, the Tichonov-parameter is driven to zero with a continuation method. The regularized problem is given in the continuous setting by

$$\begin{aligned} \min_{q\in L^2(\Omega)} \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \alpha\|q\|_{L^1(\Omega)} + \frac{\varepsilon}{2}\|q\|^2_{L^2(\Omega)} \\ \text{s.t. } (\nabla u, \nabla v) = (q, v) \text{ for all } v \in V, \end{aligned} \tag{7.1}$$

where $\varepsilon \geq 0$ is the regularization parameter. See [11] for a detailed analysis of the connection of (7.1) and the original problem. Specifically, it is shown there that the optimal controls $\bar{q}_\varepsilon$ converge to $\bar{q}$ weakly in $H^{-2}(\Omega)$ for $\varepsilon \to 0$. For analysis of the problem (7.1) for a fixed ε see also [20] and [10].

The optimality condition for (7.1) with $\varepsilon > 0$ is known to be given by the projection formula

$$\bar{q}_\varepsilon = \tfrac{1}{\varepsilon}\operatorname{sh}_\alpha(-\bar{z}_\varepsilon),$$

where the Nemizkij-operator sh_α (soft-shrinkage) can be written as

$$\operatorname{sh}_\alpha(y) = \max(0, y-\alpha) - \min(0, y+\alpha),$$

and $\bar{z}_\varepsilon$ fulfills the adjoint equation (2.1) with a corresponding state solution $\bar{u}_\varepsilon$ solving (1.2). Thus, the control variable can be eliminated to obtain the system

$$G(z,u) = \begin{pmatrix} -u + u_d - \Delta z \\ -\Delta u - \frac{1}{\varepsilon}\operatorname{sh}_\alpha(-z) \end{pmatrix} = 0,$$

which can be solved with a semismooth Newton method; see, e.g., [21].

We proceed completely analogously for the discrete problem. However, since the controls are discretized as nodal Diracs measures, it is not immediately clear how to interpret the regularization term in the discrete setting. For simplicity, we implement the regularization term as

$$\frac{\varepsilon}{2}\|q_h\|^2_{L^2_h} = \frac{\varepsilon}{2}\sum_{i=1}^{N} d_i^{-1} q_i^2, \tag{7.2}$$

where q_i is the coefficient of the control $q_h \in \mathcal{M}_h$ at the nodal Dirac measure δ_{x_i} and $(d_i)_{i=1\ldots N}$ is the diagonal of the lumped mass matrix. The discrete regularized problem is then given by

$$\begin{aligned} \min_{q_h\in \mathcal{M}_h} \frac{1}{2}\|u_h-u_d\|^2_{L^2(\Omega)} + \alpha\|q_h\|_{\mathcal{M}(\Omega)} + \frac{\varepsilon}{2}\|q_h\|^2_{L^2_h} \\ \text{s.t. } (\nabla u_h, \nabla v_h) = \langle q_h, v_h\rangle \text{ for all } v_h \in V_h. \end{aligned} \tag{7.3}$$

A related mass lumping for discretization of L^1-control costs is also employed in [9].

The optimality system for (7.3) can then be derived as in the continuous setting. We only point out that here we obtain the optimality condition

$$d_i^{-1} q_i = \frac{1}{\varepsilon}\operatorname{sh}_\alpha(-\bar{z}_{h,\varepsilon}(x_i)) \quad \text{for } i = 1\ldots N,$$

where q_i is the coefficient of the optimal control $q_{h,\varepsilon} \in \mathcal{M}_h$ at the nodal Dirac δ_{x_i}. The corresponding algorithm for the discrete regularized problem (7.3) was implemented with [19], and the arising linear systems were solved with a Schur-complement method and conjugate gradients.

8. Numerical examples. We present some examples to verify the rates of convergence established in sections 4 and 5.

8.1. Example for $d = 2$. We take $\Omega = B_1(0)$ as the unit ball and construct a radially symmetric example with the optimal state given as

$$\bar{u}(x) = -\frac{1}{2\pi} \ln(\max\{\rho, |x|\})$$

with a kink in the radial direction at $\rho \in [0, 1)$. See Figure 8.1 for the representative cases $\rho = \frac{1}{2}$ and $\rho = 0$. For $\rho = 0$ the state $\bar{u}$ is simply a Green's function, and the optimal control is then given by $\bar{q} = \delta_0$. For $\rho > 0$ we obtain the surface measure (given in terms of the one-dimensional Hausdorff measure $\mathcal{H}$)

$$\bar{q} = \frac{1}{2\pi\rho} \mathcal{H}^1|_{\partial B_\rho(0)},$$

which, due to the choice of scaling, has a norm of $\|\bar{q}\|_{\mathcal{M}(\Omega)} = 1$. The optimal dual state can then be chosen as any element in $H^2(\Omega) \cap H_0^1(\Omega)$ such that $|\bar{z}| \leq \alpha$ and $\bar{z}|_{\partial B_\rho(0)} = -\alpha$. We make the specific choice

$$\bar{z}(x) = h(|x|),$$

where $h \in C^1([0, 1])$ is a piecewise cubic polynomial interpolating $h(0) = h(1) = 0$, $h(\rho) = -\alpha$ with the choices $h'(\rho) = h'(0) = h'(1) = 0$. (For $\rho = 0$, the conditions $h(0) = h'(0) = 0$ are dropped.) This yields $\bar{z} \in C^1(\Omega)$, which is piecewise twice

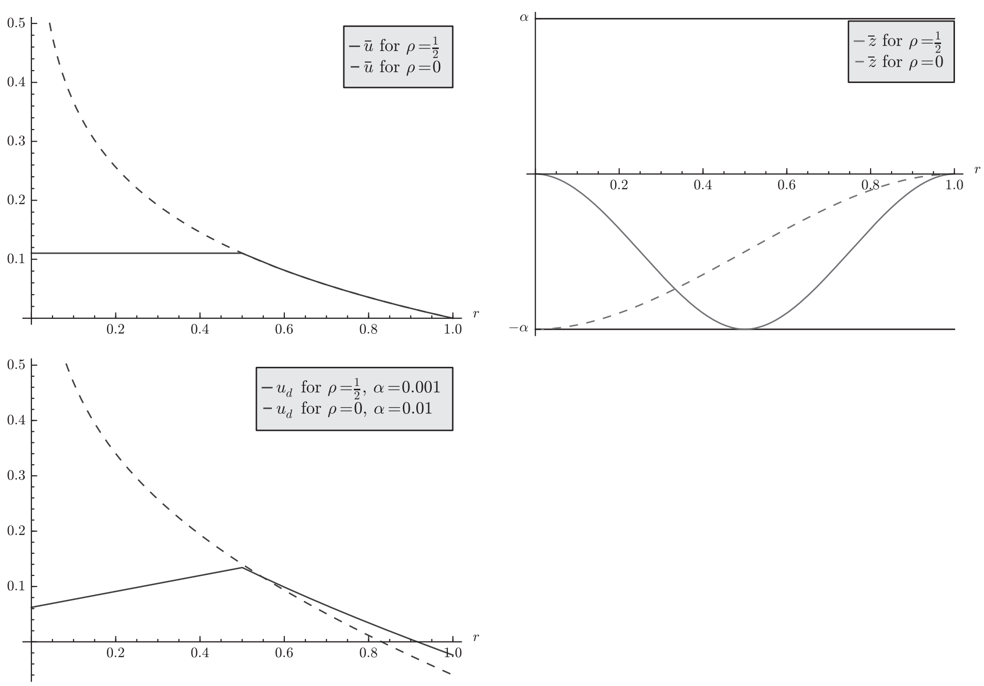
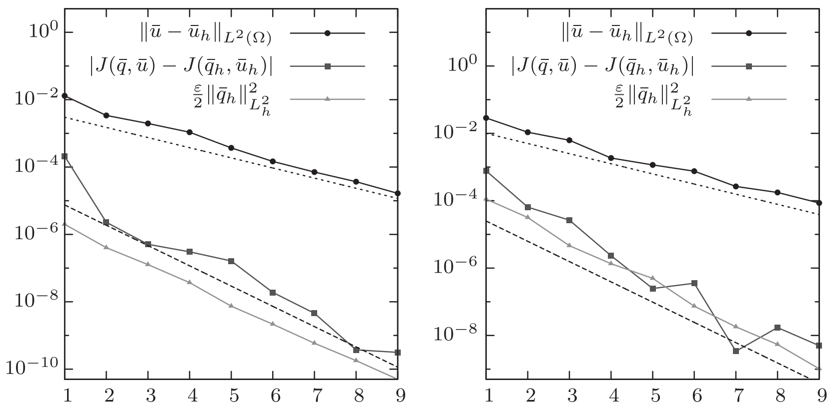


FIG. 8.1. *Radially symmetric example for the unit circle in $\mathbb{R}^2$ in radial direction r.*



(a) Errors for $\rho = \frac{1}{2}$, $\alpha = 0.001$ (b) Errors for $\rho = 0$, $\alpha = 0.01$

FIG. 8.2. *Convergence rates for the two-dimensional example at different refinement levels.*

continuously differentiable with bounded second derivatives, and a matching desired state $u_d \in L^\infty(\Omega)$ can be computed in strong formulation as

$$u_d = \Delta \bar{z} + \bar{u},$$

as depicted in Figure 8.1 for $\rho \in \{0, \frac{1}{2}\}$. For the convenience of the reader, the exact formula for u_d is given by

$$u_d(r) = \begin{cases} \alpha \, \frac{6\,(3\,r - 2\,\rho)}{\rho^3} - \frac{1}{2\pi} \ln(\rho) & \text{for } r < \rho, \\ \alpha \, \frac{6\left(3\,r^2 - 2\,r\rho - 2\,r + \rho\right)}{(\rho-1)^3 r} - \frac{1}{2\pi} \ln(r) & \text{for } r \geq \rho, \end{cases}$$

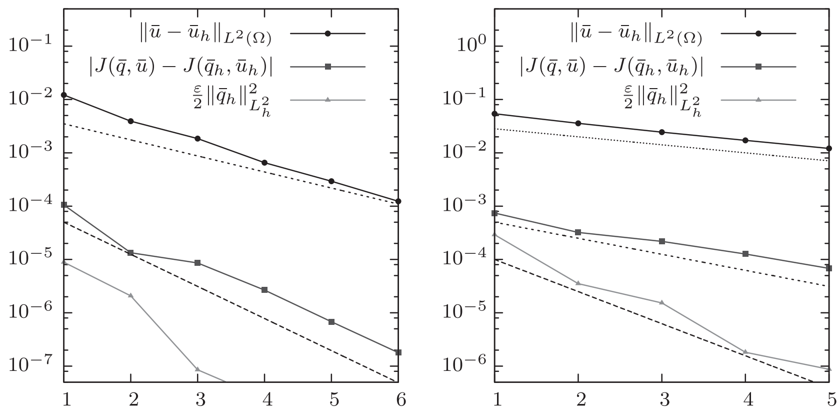
where $r = |x|$.

The convergence rates for a choice of $\rho = \frac{1}{2}$ and $\rho = 0$ are given in Figure 8.2. The inital grid (refinement level 0) consists of five cells, a small square in the middle and four additional trapezoids at each edge, glued together at the corners. For both examples we plot the error in the cost functional $J(\bar{q}, \bar{u}) - J(\bar{q}_h, \bar{u}_h)$ and the L^2-error in the state variable. The dashed lines indicate the orders of convergence $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$, which are what theory predicts for the respective quantities (up to logarithmic contributions). Since the regularization is present in the numerical computations, we also report the size of the term $\frac{\varepsilon}{2}\|\bar{q}_h\|^2_{L^2_h}$. As a parameter choice rule, at each refinement level the regularization parameter ε is decreased until

$$\frac{\varepsilon}{2}\|\bar{q}_h\|^2_{L^2_h} \leq c_{\text{reg}} h^2$$

is fulfilled, where $c_{\text{reg}} > 0$ is a constant chosen heuristically in advance. This is done to ensure that at least the asymptotic best case convergence behavior of the functional $\mathcal{O}(|\ln h|^\gamma h^2)$ is not altered by the regularization. In Figure 8.2(a), e.g., we observe that the regularization term is an order of a magnitude smaller than the exact functional error, such that the reported error in the functional should be at least accurate in the first significant digit.

We see that the observed rates agree with the rates predicted by theory. In Figure 8.2(a) the rates seem to be even slightly better; however, this is far from



(a) Errors for $\rho = \frac{1}{2}$, $\alpha = 0.001$ (b) Errors for $\rho = 0$, $\alpha = 0.01$

FIG. 8.3. *Convergence rates for the three-dimensional example at different refinement levels.*

conclusive. In Figure 8.2(b), even though the rate for the functional is somewhat wiggly, we observe the expected rates. The wiggles could be caused by the fact that the initial mesh was perturbed slightly, and thus the approximation quality depends for a large part on the smallest distance of a grid-point to the origin, where the optimal control $\bar{q} = \delta_0$ is located. If we choose a mesh which has a point at the origin, the exact control is representable at each level, and the wiggles disappear. In the Dirac case, due to the low regularity of u_d, it is also clear that the rate of almost $\mathcal{O}(h)$ for the state error is the best theoretically possible.

8.2. Example for $d = 3$. The construction of an example in three dimensions is completely analogous, except for the different Green's function

$$\bar{u}(x) = \frac{1}{4\pi}\left(\frac{1}{\max\{\rho, |x|\}} - 1\right);$$

thus we omit a detailed description. The final formula for u_d in this case is given by

$$u_d(r) = \begin{cases} \alpha\,\frac{6\,(4\,r - 3\,\rho)}{\rho^3} + \frac{1}{4\pi}\left(\frac{1}{\rho} - 1\right) & \text{for } r < \rho, \\ \alpha\,\frac{6\left(4\,r^2 - 3\,r\rho - 3\,r + 2\,\rho\right)}{(\rho-1)^3 r} + \frac{1}{4\pi}\left(\frac{1}{r} - 1\right) & \text{for } r \geq \rho, \end{cases}$$

where $r = |x|$. The computational results can be seen in Figure 8.3. Note that the parameter choice rule for ε is simply the same as before. In this case, the general theory predicts an order of convergence close to $\mathcal{O}(h)$ for the functional and close to $\mathcal{O}(h^{\frac{1}{2}})$ for the L^2-error of the state. This is clearly observed in the case $\rho = 0$, where the optimal control $\bar{q}$ is a single Dirac delta function; see Figure 8.3(b). In this case the rate for the state error is again the theoretically best possible. However, in the case $\rho = \frac{1}{2}$, depicted in 8.3(a), where u_d is bounded and the optimal control is a surface measure, the rates are clearly better. For visual comparison we plot the rates $\mathcal{O}(h)$ for the state in accordance with Theorem 5.1 and $\mathcal{O}(h^2)$ for the functional, which seems to be the closest match. Here, the order of convergence is the same as in the case $d = 2$.

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