Technische Universität München

Department of Mathematics

Chair of Optimal Control

Errata:
Optimal Control of Parabolic Obstacle Problems

Optimality Conditions and Numerical Analysis

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Die Dissertation wurde am 18.12.2019 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 07.04.2020 angenommen. Die Errata wurden am 12.02.2021 nachgereicht.

I would like to sincerely thank Boris Vexler and Constantin Christof for their support, suggestions and feedback regarding these errata.

The proofs of Theorem 8.17 and Lemma 2.26

Unfortunately the proof of Lemma 2.26 and therefore the proof of Theorem 8.17, which references it, are wrong. The proof accidentally presumed that the non-linearity f, resp. $\tilde{f}$, is independent of $(t, x) \in Q$. For those f, e.g. $f(t, x, y) = y^3$, the proof still applies and nothing changes.

We now provide the reader with correct variants of Theorem 8.17 and Lemma 2.26 that include dependencies on $(t, x) \in Q$ for the non-linearities. We first provide the new statements, shortly comment on the changes and then give a detailed breakdown of how the rest of the thesis is affected.

All the references are references to the dissertation, unless noted otherwise.

Theorem 1 (Correct Version of Theorem 8.17) *Assume Assumption 8.16 holds. Let $u \in L^{q_u}(Q)$ and $y_0 \in \mathbb{W}_{q_u}$. There exists a unique weak solution y of*

$$\begin{cases} \partial_t y + Ay + \tilde{f}(y) = u \text{ in } Q, \\ y(0) = y_0, \quad y|_{\Sigma_D} = 0. \end{cases}$$

We have the following estimate, with a constant $C > 0$ independent of $\tilde{f}$, u and y_0:

$$\|y\|_{L^\infty(Q) \cap L^2(I,V)} \le C(\|u - \tilde{f}(0)\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)}). \tag{0.1}$$

Furthermore, there are $C_{lip} > 0, \kappa^* \in (0, 1)$ such that for any $\kappa_\Omega \in [0, \kappa^*)$ and $\kappa_I \in (0, 1)$ with

$$\frac{1}{q_u}(1 + N/2) + \frac{\kappa_\Omega}{2} < 1 \text{ and } \kappa_I \in \left(0, 1 - \frac{1}{q_u}(1 + N/2) - \frac{\kappa_\Omega}{2}\right)$$

we have

$$\begin{aligned} &\|y\|_{C^{\kappa_I}(I, C^{\kappa_\Omega}(\Omega))} + \|\partial_t y\|_{L^{q_u}(Q)} + \|y\|_{L^2(I,V)} + \|Ay\|_{L^{q_u}(Q)} + \|\tilde{f}(y)\|_{L^{q_u}(Q)} \\ &\quad \le C_{lip} \left(\|u - \tilde{f}(0)\|_{L^{q_u}(Q)} + \|y_0\|_{\mathbb{W}_{q_u}}\right). \end{aligned}$$

The constant $C_{lip} > 0$ does depend on the local Lipschitz constant of $\tilde{f}$ in its third argument on a ball with radius greater or equal to the right hand side of (0.1). This constant is independent of the first two arguments of $\tilde{f}$.

The statement now explicitly states the behaviour of the constants. It is now clear that the constant C_{lip} does depend on $\|u\|_{L^{q_u}(Q)}$ or rather an upper bound to it. This will thankfully have no mathematical impact on the rest of the thesis, but some of the phrasing of the statements needs minor modifications. We give a detailed list after the proofs.

Proof. The proof will have three major steps: first we study a truncated $\tilde{f}_k$ with corresponding solutions y_k and prove an upper bound on $\|y_k\|_{L^\infty(Q)}$. We will compare y_k to solutions to linear problems without $\tilde{f}_k$. Due to its monotonicity $\tilde{f}_k$ has a "dampening" effect on y_k and considering PDEs without it yields solutions that serve as overestimators. Thus the obtained bound will be independent of the truncated nonlinearity. The second step is simply a transfer of the results from the truncated to the untruncated case. Lastly we will show all the other regularities.

We start like in the original proof. By considering $u - \tilde{f}(0)$ instead of u we may assume $\tilde{f}(0) = 0$. For $k \geq 0$ consider

$$\tilde{f}_k(t,x,y) := \begin{cases} \tilde{f}(t,x,k) & \text{if } k \leq y, \\ \tilde{f}(t,x,y) & \text{if } -k < y < k, \\ \tilde{f}(t,x,-k) & \text{if } y \leq -k. \end{cases} \tag{0.2}$$

Note that $\tilde{f}_k$ is bounded by the local Lipschitz continuity assumption and $\tilde{f}(0) = 0$. By Proposition 8.18 the problem

$$\begin{cases} \partial_t y_k + A y_k + \tilde{f}_k(y_k) = u \text{ in } Q, \\ y_k(0) = y_0, \quad y_k|_{\Sigma_D} = 0 \end{cases} \tag{0.3}$$

has a unique solution $y_k \in W(I)$ satisfying

$$\|y_k\|_{C(\bar{I},H)} + \|y_k\|_{L^2(I,V)} \leq C_{y\!\!\!/,\tilde{f}_k} \left(\|u\|_{L^{q_u}(Q)} + \|y_0\|_{L^2(\Omega)} \right).$$

We now show that y_k can be estimated in $L^\infty(Q)$ with an estimate independent of $\tilde{f}_k$. Rearranging (0.3) to $\partial_t y_k + A y_k = u - \tilde{f}_k(y_k)$, we see that y_k satisfies a linear PDE with a right hand side in $L^{q_u}(Q)$ as $\tilde{f}_k(y_k)$ is bounded. By Theorem 8.20 this yields that $y_k \in C(\bar{Q})$.

Using the local Lipschitz continuity of $\tilde{f}$ in its third argument, which translates to $\tilde{f}_k$, and $\tilde{f}(0) = 0$ we realize that

$$b_k(t,x) := \begin{cases} y_k^{-1} \tilde{f}_k(y_k) & \text{if } y_k \neq 0, \\ 0 & \text{if } y_k = 0, \end{cases}$$

is bounded, as y_k is bounded, and non-negative by the monotonicity of $\tilde{f}_k$. Recall that the local Lipschitz continuity is uniform in (t,x) and note that the bound on b_k does depend on y_k. We will never explicitly need this bound. Note that Proposition 8.18 implies that y_k is the unique solution to the linear PDE

$$\begin{cases} \partial_t y_k + A y_k + b_k y_k = u \text{ in } Q, \\ y_k(0) = y_0, \quad y_k|_{\Sigma_D} = 0 \end{cases} \tag{0.4}$$

Now we see that the solutions to the following PDEs exist by Proposition 8.18

$$\begin{cases} \partial_t y_k^p + A y_k^p + b_k y_k^p = u^+ \text{ in } Q, \\ y_k^p(0) = y_0^+, \qquad y_k^p|_{\Sigma_D} = 0, \end{cases} \quad \begin{cases} \partial_t y_k^m + A y_k^m + b_k y_k^m = u^- \text{ in } Q, \\ y_k^m(0) = y_0^-, \qquad y_k^m|_{\Sigma_D} = 0, \end{cases}$$

and satisfy $y_k^p + y_k^m = y_k$ as y_k was the unique solution to (0.4). Here, $u^+, y_0^+ \geq 0$ and $u^-, y_0^- \leq 0$ are the positive and negative parts of $u = u^+ + u^-$ and $y_0 = y_0^+ + y_0^-$. By Theorem 8.15, the maximum principle, we have $y_k^p \geq 0$ and $y_k^m \leq 0$. We see that

$$y_k^m = y_k - y_k^p \leq y_k \leq y_k - y_k^m = y_k^p. \tag{0.5}$$

We will thus provide an overestimate of y_k^p and an underestimate of y_k^m. We will prove the bound on $\|y_k^p\|_{L^\infty(Q)}$. The bound on $\|y_k^m\|_{L^\infty(Q)}$ can be obtained completely analogously.

We now define $\hat{y}^p$ as the solution to the linear PDE

$$\begin{cases} \partial_t \hat{y}^p + A \hat{y}^p = u^+ \text{ in } Q, \\ \hat{y}^p(0) = y_0^+, \qquad \hat{y}^p|_{\Sigma_D} = 0. \end{cases}$$

Taking the difference for the equations for y_k^p and $\hat{y}^p$ we see that

$$\begin{cases} \partial_t (\hat{y}^p - y_k^p) + A(\hat{y}^p - y_k^p) = b_k y_k^p \text{ in } Q, \\ (\hat{y}^p - y_k^p)(0) = 0, \qquad (\hat{y}^p - y_k^p)|_{\Sigma_D} = 0. \end{cases}$$

By our earlier arguments we have $b_k y_k^p \geq 0$ and thus we find by the maximum principle, Theorem 8.15,

$$\hat{y}^p - y_k^p \geq 0 \text{ a.e. in } Q. \tag{0.6}$$

We will now show $\|\hat{y}^p\|_{L^\infty(Q)} \leq C(\|u\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)})$. We decompose $\hat{y}^p = v + w$ with v and w being the solutions to

$$\begin{cases} \partial_t v + Av = 0 \text{ in } Q, \\ v(0) = y_0^+, \quad v|_{\Sigma_D} = 0, \end{cases} \qquad \begin{cases} \partial_t w + Aw = u^+ \text{ in } Q, \\ w(0) = 0, \quad w|_{\Sigma_D} = 0. \end{cases}$$

By Theorem 8.20 (note that y_0^+ does not necessarily lie in $\mathbb{W}_{q_u}$) we have $\|w\|_{L^\infty(Q)} \leq C\|u^+\|_{L^{q_u}(Q)}$. The constant does not depend on $\tilde{f}_k$, k or u. By the following Proposition 2 we also get $\|v\|_{L^\infty(Q)} \leq \|y_0^+\|_{L^\infty(\Omega)}$.

Together we find

$$\|\hat{y}^p\|_{L^\infty(Q)} \leq C(\|u\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)}).$$

The constant does not depend on $\tilde{f}_k$, k, u or y_0. Thus, by (0.5) and (0.6) we have

$$y_k \leq C(\|u\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)})$$

in Q. Doing the analogous arguments for some $\hat{y}^m$ defined by u^- and y_0^- we get the analogous lower bound. Therefore we have, with a constant independent of $\tilde{f}_k$, k, u or y_0,

$$\|y_k\|_{L^\infty(Q)} \leq C(\|u\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)}).$$

Thus, for k larger than the right hand side, we find that $\tilde{f}_k(y_k) = \tilde{f}(y_k)$. This means y_k solves the original PDE. If the solution to the original PDE is unique we immediately get $y_k = y$ and the bound (0.1) as we included $\tilde{f}(0)$ into u.

To see the uniqueness let $y' \in W(I)$ be a second solution to the PDE. Taking the difference of their equations and testing that difference with $(y - y') \cdot 1_{(0,\hat{t})}$ for $\hat{t} \in I$ yields

$$\begin{aligned} 0 &= (\partial_t(y - y'), y - y')_{L^2((0,\hat{t}),V^*,V)} + a_{(0,\hat{t})}(y - y', y - y') + (f(y) - f(y'), y - y')_{L^2((0,\hat{t})\times\Omega)} \\ &\geq \frac{1}{2}\|(y - y')(t)\|_H^2. \end{aligned}$$

Here we used integration by parts, the ellipticity of the coefficient matrix of the operator A and the monotonicity of f. Since $t \in I$ was arbitrary this shows that $y = y'$ is the unique solution.

Now we prove the estimates for the higher regularity. We realize that by the local Lipschitz continuity we have

$$\|\tilde{f}(y)\|_{L^\infty(Q)} \leq C_{Lip}\|y\|_{L^\infty(Q)} \leq C_{Lip}(\|u\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)}).$$

Here, C_{Lip} depends on the local Lipschitz constant of $\tilde{f}$ in its third argument on a ball with radius proportional to the right hand side of (0.1). Now the rest of the proof can be done as before, continuing after (8.22).

□

Proposition 2 *Let* $0 \leq y_0 \in L^\infty(\Omega)$ *be given. Then there exists a unique weak solution* $v \in W(I)$ *to*

$$\begin{cases} \partial_t v + Av = 0 \textit{ in } Q, \\ v(0) = y_0, \quad v|_{\Sigma_D} = 0. \end{cases}$$

It satisfies $\|v\|_{L^\infty(Q)} \leq \|y_0\|_{L^\infty(\Omega)}$.

Proof. By Proposition 8.18 there exists a unique solution $v \in W(I)$ to the given PDE. To see the bound we make arguments similar to those in [CV21, Lemma A.3]. By Proposition 8.19 we get for $l := \|y_0\|_{L^\infty(\Omega)}$ that $v^l := \max(v - l, 0)$ lies in $L^2(I, V) \cap C(\bar{I}, H)$. Thus testing the equation of v with $v^l \cdot 1_{(0,\hat{t})}$ for some $\hat{t} \in I$ yields

$$0 = (\partial_t v, v^l)_{L^2((0,\hat{t}),V^*,V)} + \int_0^{\hat{t}} \int_\Omega \nabla v^T(t,x) A(x) \nabla v^l(t,x)\, d(t,x).$$

By [Wac16, Lemma 3.3], integration by parts for positive parts of Bochner functions, we arrive at

$$0 = \frac{1}{2}\|v^l(\hat{t})\|_H^2 - \frac{1}{2}\|v^l(0)\|_H^2 + \int_0^{\hat{t}} \int_\Omega (\nabla v^l)^T(t,x) A(x) \nabla v^l(t,x)\, d(t,x) \geq \frac{1}{2}\|v^l(\hat{t})\|_H^2 - \frac{1}{2}\|v^l(0)\|_H^2.$$

As $v^l(0) = \max(y_0 - \|y_0\|_{L^\infty(\Omega)}, 0) = 0$ we get that $v^l(\hat{t})$ has to vanish and since $\hat{t}$ was arbitrary v^l has to vanish. Since $v \geq 0$ by the maximum principle, Theorem 8.15, this proves $\|v\|_{L^\infty(Q)} \leq \|y_0\|_{L^\infty(\Omega)}$. ☐

Now we can provide a correct version of Lemma 2.26.

Lemma 3 (Correct Version of Lemma 2.26) *Let $u \in L^{q_u}(Q)$ and $y_\gamma = S_\gamma(u)$. The sequence $\left(\|f(y_\gamma)\|_{L^{q_u}(Q)}\right)_{\gamma>0}$ is bounded independently of γ. In particular:*

$$\begin{aligned}\|f(y_\gamma)\|_{L^{q_u}(Q)} \leq C(&\|y_0\|_{L^\infty(\Omega)} + \|u\|_{L^{q_u}(Q)} + \|f(0)\|_{L^{q_u}(Q)} \\ &+ \|\partial_t \Psi\|_{L^{q_u}(Q)} + \|A\Psi\|_{L^{q_u}(Q)} + \|f(\Psi)\|_{L^{q_u}(Q)}).\end{aligned}$$

The constant depends on the local Lipschitz constant of f in its third argument, which is uniform in (t,x), on a ball whose radius depends on upper bounds to $\|y_0\|_{L^\infty(\Omega)}$, $\|u\|_{L^{q_u}(Q)}$, $\|f(0)\|_{L^{q_u}(Q)}$, $\|\partial_t \Psi\|_{L^{q_u}(Q)}$, $\|A\Psi\|_{L^{q_u}(Q)}$ and $\|f(\Psi)\|_{L^{q_u}(Q)}$.

Essentially the same terms as in the original Lemma 2.26 appear in the estimate. Only the behaviour of the constant has to be amended. We comment on the consequences in the following section.

Proof. We rearrange the defining semilinear PDE for y_γ, see Definition 2.19, as

$$\begin{cases} \partial_t y_\gamma + A y_\gamma + f(y_\gamma) = u - \beta_\gamma(y_\gamma - \Psi) \text{ in } Q, \\ y_\gamma|_{\Sigma_D} = 0, \qquad y_\gamma(0) = y_0. \end{cases}$$

By the new version of Theorem 8.17, i.e. Theorem 1, we find

$$\|y_\gamma\|_{L^\infty(Q)} \leq C(\|u - \beta_\gamma(y_\gamma - \Psi) - f(0)\|_{L^{q_u}(Q)} + \|y_0\|_{L^\infty(\Omega)}).$$

Using Lemma 2.25, the bound on $\|\beta_\gamma(y_\gamma - \Psi)\|_{L^{q_u}(Q)}$, we find

$$\begin{aligned}\|y_\gamma\|_{L^\infty(Q)} \leq C(&\|y_0\|_{L^\infty(\Omega)} + \|u\|_{L^{q_u}(Q)} + \|f(0)\|_{L^{q_u}(Q)} \\ &+ \|\partial_t \Psi\|_{L^{q_u}(Q)} + \|A\Psi\|_{L^{q_u}(Q)} + \|f(\Psi)\|_{L^{q_u}(Q)}).\end{aligned} \tag{0.7}$$

The constant $C > 0$ does not depend on f, γ or quantities on the right. By the local Lipschitz continuity of f in its third argument, which is uniform in its first two arguments, we get

$$\|f(y_\gamma)\|_{L^{q_u}(Q)} \leq \|f(y_\gamma) - f(0)\|_{L^{q_u}(Q)} + \|f(0)\|_{L^{q_u}(Q)} \leq C_{lip}\|y_\gamma\|_{L^{q_u}(Q)} + \|f(0)\|_{L^{q_u}(Q)}. \tag{0.8}$$

Here C_{lip} is the Lipschitz constant of f in its third component on a ball with radius equal to $\|y_\gamma\|_{L^\infty(Q)}$ which is bounded via (0.7). Thus inserting (0.7) into (0.8) yields the claim. ☐

Consequences for the text

Theorem 8.17 and Lemma 2.26 are cited in various places, mainly Chapter 2. We now chronologically mention places that are affected by the new changes. Places where Theorem 8.17 is used for linear PDEs or where the phrasing automatically encompasses the new situation are not specifically mentioned again as they were not affected by the changes.

The actual mathematical substance changes very little. The reason for this is the fact that throughout most of the thesis the control u stems from a bounded set, causing the constant in the new Lemma 2.26, Lemma 3, to stay bounded independently of the specific control.

The first affected theorem is Theorem 2.22, where now additionally the constant $C > 0$ does depend on $\|u\|_{L^{q_u}(Q)}$ or rather an upper bound to it. The next few statements are not affected, as they only use the regularity provided by Theorem 2.22, resp. Theorem 8.17, but not the explicit estimates. This makes sense as of the goals of Chapter 2 is to reprove those estimates with stronger control over the constant.

Corollary 2.27, which is a consequence of Lemma 2.26, now has to slightly change. The constant is now dependent on an upper bound to $\|u\|_{L^{q_u}(Q)}$.

We also often explicitly refer to Corollary 2.27 for the behaviour of various constants so that the rest of the text does not need to be updated. Everywhere bounded sequences of controls are considered, so that the new constants in the estimates stay bounded.

Therefore no further changes in Section 2 are necessary up until Proposition 2.36, where the estimate for $f(y)$ is wrong, but the proof of Lemma 3 immediately gives the new bound

$$\begin{aligned}\|f(y_\gamma)\|_{L^\infty(Q)} \le C(&\|y_0\|_{L^\infty(\Omega)} + \|u\|_{L^{q_u}(Q)} + \|f(0)\|_{L^\infty(Q)} \\ &+ \|\partial_t\Psi\|_{L^{q_u}(Q)} + \|A\Psi\|_{L^{q_u}(Q)} + \|f(\Psi)\|_{L^{q_u}(Q)})\end{aligned}$$

with the same constant behaviour as in Lemma 3. The rest of Chapter 2 is unaffected.

Remark 4.62 also needs to be considered. It is phrased in such a way, such that it still applies with the modified Proposition 2.36. Only the sentence "Note that here C_∞ additionally depends on the Lipschitz constant of f on a ball with radius $\|y_0\|_{L^\infty(\Omega)}$." Needs to be corrected to "Note that here C_∞ additionally depends on the Lipschitz behaviour of f in accordance to Lemma 3." Mathematically the remark works with upper bounds on the controls, so that no other changes are necessary.

In the appendix, Chapter 8, only Corollary 8.24 is affected. The sentence "Here $C_{q_u} > 0$ does not depend on y, u or q_u, but only on the same quantities as in Theorem 8.17" has to change to"Here $C_{q_u} > 0$ does not depend on y or q_u, but only on the same quantities as in Theorem 8.17". Corollary 8.24 is only used once in the whole thesis in the proof of Theorem 4.55, where no non-linearity appears, not making any change necessary.

Minor errors and clarifications

- Notice that Remark 1.32 does not follow from Lemma 1.31 as stated. The remark is never cited throughout the rest of the thesis, so it is easy to dismiss.
- In the proof of Lemma 2.25, below (2.12), we claim $b(y - \Psi) \in L^2(I, V)$ (mistyped as $\in V$) by the trace characterization of V. While this is true and intuitive, it is also too short. The proof is not difficult and presented in the following.

  Let $k \ge 0$ and b_k be the truncation of b defined in analogy to (0.2). By the assumptions on β_γ, Definition 2.16, b_k is now Lipschitz continuous and bounded. Further, let $t \in I$ be fixed.

We abbreviate $v := (y-\Psi)(t) \in V$. We have $v|_{\Gamma_D} \geq 0$ by arguments given in the proof. Here, $\cdot|_{\Gamma_D}$ denotes the trace operator on that specific part of the spatial boundary.

Using the theorem of dominated convergence, the compactness of the embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ and pointwise a.e. convergence of subsequences of L^2-convergent sequences, it is straightforward to check that the truncation $\min(\cdot, 0) : H^1(\Omega) \to H^1(\Omega)$, cf. [KS80, Section II, Theorem A.1], is continuous. Let $(v_n)_{n\in\mathbb{N}} \subset C^\infty(\bar{\Omega}) \cap H^1(\Omega)$ be a sequence approximating v in $H^1(\Omega)$. By the continuity of v_n we have that $\min(0, v_n)$ is continuous and thus $\min(0, v_n)|_{\Gamma_D} = \min(0, v_n|_{\Gamma_D})$. We conclude

$$\begin{aligned}
&\|\min(v,0)|_{\Gamma_D}\|_{L^2(\partial\Omega)} = \|\min(v,0)|_{\Gamma_D} - \min(v|_{\Gamma_D}, 0)\|_{L^2(\partial\Omega)} \\
&\leq \|\min(v,0)|_{\Gamma_D} - \min(v_n,0)|_{\Gamma_D}\|_{L^2(\partial\Omega)} + \|\min(v_n|_{\Gamma_D}, 0) - \min(v|_{\Gamma_D}, 0)\|_{L^2(\partial\Omega)} \\
&\leq C\|\min(v,0) - \min(v_n,0)\|_{H^1(\Omega)} + \|v_n|_{\Gamma_D} - v|_{\Gamma_D}\|_{L^2(\partial\Omega)} \\
&\leq C\|\min(v,0) - \min(v_n,0)\|_{H^1(\Omega)} + C\|v_n - v\|_{H^1(\Omega)} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Thus $\min(v, 0) \in V$. Therefore, by definition of V, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C^\infty_{\Gamma_D}(\Omega)$ converging to $\min(v, 0)$ in $H^1(\Omega)$.

By the chainrule for weak derivatives, Rademacher's theorem and our Lipschitz assumption we have

$$\|b_k(\varphi_n)\|_{H^1(\Omega)} \leq \|b_k(\varphi_n)\|_{L^2(\Omega)} + \|b_k'(\varphi_n)\nabla\varphi_n\|_{L^2(\Omega)} \leq C < \infty. \tag{0.9}$$

Because $b_k(\varphi_n) \in C(\bar{\Omega})$ we have $b_k(\varphi_n)|_{\Gamma_D} = 0$, i.e. $b_k(\varphi_n) \in V$.

By (0.9) and the reflexivity of the Hilbert space V we have, along a subsequence denoted by the same indices, $b_k(\varphi_n) \xrightarrow{n\to\infty} w$ weakly for some $w \in V$. We also have, by the fact that $b_k(r) = 0$ for $r \geq 0$, that

$$\|b_k(v) - b_k(\varphi_n)\|_{L^2(\Omega)} = \|b_k(\min(v,0)) - b_k(\varphi_n)\|_{L^2(\Omega)} \leq \mathrm{Lip}_{b_k}\|\min(v,0) - \varphi_n\|_{L^2(\Omega)} \xrightarrow{n\to\infty} 0.$$

We therefore finally see $b_k(v) = w \in V$. Since y and Ψ are bounded choosing $k = \|y-\Psi\|_{L^\infty(Q)}$ yields $b((y-\Psi)(t)) = b_k(v) \in V$.

- In the proof of Lemma 2.25 we also accidentally did not give the source for the weak chain rule of Bochner derivatives. (One can still see the accidental space.) The appropriate source is given by [Bar84, Lemma 1.2].
- Lemma 2.29 claims that $\partial_t y, Ay \in L^{q_u}(Q)$ without the proof. These regularities, together with the convergence $\partial_t S_{\gamma_{n_k}}(u) \xrightarrow{k\to\infty} \partial_t y$ weakly in $L^{q_u}(Q)$ (and the analogous convergence for Ay), are an almost immediate consequence of the weak $W(I)$ convergence, the bounds from Corollary 2.27 and the reflexivity of $L^{q_u}(Q)$.
- In Remark 4.53, Theorem 4.61, Theorem 4.65 we consequently wrote $|\ln h|^2$ instead of $|\ln h|^4$, likely due to an copy-paste mistake.
- At the beginning of Section 6.1.1 we have the sentence "It is well-known, or at least easily provable, that a discontinuous Galerkin method [...] is equivalent to an implicit Euler scheme in the finite dimensional space V_h [...]". This is only true if the right hand side is piecewise constant and the non-linearities do not depend on t. Both things are the case for our experiments, so that the thesis is essentially unaffected.
- In the appendix, in Assumption 8.16, the condition in the second bullet point is clearly supposed to hold $\forall y_1, y_2 \in B_M(0), (t, x) \in Q$.

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