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„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations“: Sheet 2

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 2.1: Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\Gamma = \partial\Omega$ denote its boundary. Consider the boundary value problem

$$\begin{cases} -\Delta u + \beta u = f & \text{in } \Omega, \\ \partial_n u + \alpha u = g & \text{on } \Gamma \end{cases} \tag{BVP}$$

with $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $\beta \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma)$ and $\beta(x) \geq 0$ for almost all $x \in \Omega$, $\alpha(s) \geq 0$ for almost all $s \in \Gamma$. This type of boundary condition is also called *Robin boundary condition* or *third type boundary condition.*

Prove the following statement: if the condition

$$\|\beta\|_{L^2(\Omega)} + \|\alpha\|_{L^2(\Gamma)} > 0 \tag{B}$$

is satisfied, then the boundary value problem (BVP) has an unique weak solution $u \in H^1(\Omega)$ and there is a constant c_R independent of f and g, such that the inequality

$$\|u\|_{H^1(\Omega)} \leq c_R \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} \right). \tag{A}$$

holds true.

Hint: Use the following two inequalities.

Generalized Poincaré inequality:

Let $E \subset \Omega$ be a set of positive measure, i. e., $|E| > 0$. Then there is a constant c_E independent of $u \in H^1(\Omega)$ such that for any $u \in H^1(\Omega)$ we have the estimate

$$\|u\|^2_{H^1(\Omega)} \leq c_E \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(E)} \right).$$

Generalized Friedrich's inequality:

Let $\tilde{\Gamma} \subset \Gamma$ be a set with positive $d-1$ dimensional Hausdorff measure denoted by $\mu(\tilde{\Gamma}) > 0$. Then there is a constant $c_{\tilde{\Gamma}}$ independent of $u \in H^1(\Omega)$ such that the estimate

$$\|u\|^2_{H^1(\Omega)} \leq c_{\tilde{\Gamma}} \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\tilde{\Gamma})} \right)$$

holds true for any $u \in H^1(\Omega)$.

Exercise 2.2 (A weak comparison principle): Let $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain. We consider the problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

with $f \in L^2(\Omega)$ and g the trace of some function $u_g \in H^1(\Omega)$.

Given two sets of data f_1, g_1 and f_2, g_2 with $f_1 \geq f_2$ almost everywhere in Ω and $g_1 \geq g_2$ almost everywhere on the boundary $\partial\Omega$, show that the corresponding weak solutions $u_1, u_2 \in H^1(\Omega)$ satisfy $u_1 \geq u_2$ almost everywhere. You can use the following auxiliary result without proof:

Lemma: On a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, let $u \in H^1(\Omega)$. Then the positive part of u denoted by $u^+ := \max(u, 0)$ is also in $H^1(\Omega)$ and the identities

$$\begin{aligned} \partial_{x_i} u^+ &= \chi(u > 0)\partial_{x_i} u \text{ for } i = 1, \dots, d, \text{ and} \\ \tau u^+ &= (\tau u)^+ \end{aligned}$$

hold true where $\chi(u > 0)$ denotes the characteristic function of the set $\{x \in \Omega \,|\, u(x) > 0\}$.

Exercise 2.3: Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. We consider the equation

$$\begin{cases} -\Delta u + qu = f & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f, q \in L^2(\Omega)$ and $q \geq 0$ almost everywhere in Ω.

(a) Prove the existence of a solution $u \in H_0^1(\Omega)$ and show the a priori estimate

$$\|\nabla u\|_{L^2(\Omega)} \leq c_L \|f\|_{L^2(\Omega)} \tag{*}$$

with a constant $c_L > 0$ independent of f and q.

(b) Assume additionally that Ω is a convex polyhedron and that $q \in L^3(\Omega)$. Show that in this setting u is in $H^2(\Omega)$ and derive a corresponding estimate for $\|u\|_{H^2(\Omega)}$.