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„Modern Methods in Nonlinear Optimization: Optimization with partial differential equations“: Sheet 4

http://www-m17.ma.tum.de/Lehrstuhl/LehreSoSe14OptPDEEn

Exercise 4.1 (Finite-dimensional Neumann boundary control): Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $Q = \mathbb{R}^l$, $l \in \mathbb{N}$ and $V = H^1(\Omega)$. We consider the optimal control problem

$$\min_{(q,u)\in Q\times V} J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)}$$

subject to

$$\begin{aligned} -\Delta u + u &= f \text{ in } \Omega, \\ \partial_n u &= Bq \text{ on } \partial\Omega, \end{aligned}$$

where the linear operator $B : Q \to L^2(\partial\Omega)$ can be written as

$$B(q) = \sum_{i=1}^{l} q_i g_i$$

with functions $g_i \in L^2(\partial\Omega)$. The functions g_i are assumed to be linearly independent. Show that this problem admits a unique solution.

Exercise 4.2 (Another formulation of Dirichlet boundary control): Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $Q = H^{\frac{1}{2}}(\partial\Omega)$, $V = H^1(\Omega)$. We consider the following variant of Dirichlet boundary control:

$$\min_{(q,u)\in Q\times V} J(q,u) = \frac{1}{2}\|u - u_d\|^2_{H^1(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\partial\Omega)}$$

subject to

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= q \text{ on } \partial\Omega, \\ a &\le q \le b \text{ almost everywhere on } \partial\Omega. \end{aligned}$$

The bounds are given as $a, b \in \bar{\mathbb{R}}$, $a \le b$ and we assume $\alpha \ge 0$, $u_d \in H^1(\Omega)$, and $f \in L^2(\Omega)$. Show existence and uniqueness of a solution.

Exercise 4.3: Derive an alternative proof for Theorem 4.7 from the lecture:

Let $\Omega \subset \mathbb{R}^n$ be polygonal and convex, $f \in L^2(\Omega)$ and $q \in L^2(\partial\Omega)$. Then there is a unique very weak solution $u \in L^2(\Omega)$ of the Poisson equation with right hand side f and Dirichlet data q, i. e.,

$$(u, -\Delta v) = (f, v) - \langle q, \partial_n v \rangle \qquad \text{for any } v \in H_0^1(\Omega) \cap H^2(\Omega) \tag{1}$$

and the a priori estimate

$$\|u\|_{L^2(\Omega)} \leq c \left(\|f\|_{L^2(\Omega)} + \|q\|_{L^2(\partial\Omega)} \right). \tag{2}$$

holds true. Proceed by showing successively:

(a) For $q \in H^{\frac{1}{2}}(\partial\Omega)$ there is a very weak solution $u \in L^2(\Omega)$ such that (2) holds.

(b) For any $q \in L^2(\partial\Omega)$ there is a sequence (q_n) in $H^{\frac{1}{2}}(\partial\Omega)$ with

$$q_n \xrightarrow{n\to\infty} q \quad \text{in } L^2(\partial\Omega).$$

(c) The sequence (u_n) in $L^2(\Omega)$ (where u_n solves (1) with $q = q_n$ for $n \in \mathbb{N}$) possesses a weakly convergent subsequence $(u_k) \subset (u_n)$, i. e., there is a $u \in L^2(\Omega)$, such that

$$u_k \rightharpoonup u \quad \text{in } L^2(\Omega) \quad \text{for } k \to \infty.$$

(d) u is a very weak solution for $q \in L^2(\partial\Omega)$ and the estimate (2) holds.

(e) The very weak solution is unique.