A dual singular complement method for the numerical solution of the Poisson equation with L^2 boundary data in non-convex domains*

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Abstract The very weak solution of the Poisson equation with L^2 boundary data is defined by the method of transposition. The finite element solution with regularized boundary data converges with order 1/2 in convex domains but has a reduced convergence order in non-convex domains. As a remedy, a dual variant of the singular complement method is proposed. The error order of the convex case is retained. Numerical experiments confirm the theoretical results.

Key Words Elliptic boundary value problem, very weak formulation, finite element method, singular complement method, discretization error estimate

AMS subject classification 65N30; 65N15

1 Introduction

In this paper we consider the boundary value problem

$$-\Delta y = f \quad \text{in } \Omega, \qquad\qquad y = u \quad \text{on } \Gamma = \partial\Omega, \tag{1.1}$$

with right hand side $f \in H^{-1}(\Omega)$ and boundary data $u \in L^2(\Gamma)$. We assume $\Omega \subset \mathbb{R}^2$ to be a bounded polygonal domain with boundary Γ. Such problems arise in optimal control when the Dirichlet boundary control is considered in $L^2(\Gamma)$ only, see for example the papers by Deckelnick, Günther, and Hinze, [7], French and King, [8], and May, Rannacher, and Vexler, [11].

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For boundary data $u \in L^2(\Gamma)$ we cannot expect a weak solution $y \in H^1(\Omega)$. Therefore we define a very weak solution by the method of transposition which goes back at least to Lions and Magenes [10]: Find

$$y \in L^2(\Omega): \quad (y, \Delta v)_\Omega = (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in V \tag{1.2}$$

with $(w, v)_G := \int_G wv$ denoting the $L^2(G)$ scalar product or an appropriate duality product. In our previous paper [1] we showed that the appropriate space V for the test functions is

$$V := H^1_\Delta(\Omega) \cap H^1_0(\Omega) \quad \text{with} \quad H^1_\Delta(\Omega) := \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}. \tag{1.3}$$

In particular it ensures $\partial_n v \in L^2(\Gamma)$ for $v \in V$ such that the formulation (1.2) is well defined. We proved the existence of a unique solution $y \in L^2(\Omega)$ for $u \in L^2(\Gamma)$ and $f \in H^{-1}(\Omega)$, and that the solution is even in $H^{1/2}(\Omega)$. The method of transposition is used in different variants also in [8, 2, 4, 3, 7, 11].

Consider now the discretization of the boundary value problem. Let $\mathcal{T}_h$ be a family of quasi-uniform, conforming finite element meshes, and introduce the finite element spaces

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} = Y_h \cap H^1_0(\Omega), \quad Y_h^\partial = Y_h|_{\partial\Omega}.$$

Since the boundary datum u is in general not contained in Y_h^∂ we have to approximate it by $L^2(\Gamma)$-projection or by quasi-interpolation. We showed in [1] that we can construct in this way a function u^h with

$$\|u - u^h\|_{H^{-1/2}(\Gamma)} \le C h^{1/2} \|u\|_{L^2(\Gamma)}.$$

As a side effect, the boundary datum is regularized since $u^h \in H^{1/2}(\Gamma)$. Hence we can consider a regularized (weak) solution in $Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$,

$$y^h \in Y_*^h: \quad (\nabla y^h, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H^1_0(\Omega). \tag{1.4}$$

The finite element solution y_h is now searched in $Y_{*h} := Y_*^h \cap Y_h$ and is defined in the classical way: find

$$y_h \in Y_{*h}: \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \tag{1.5}$$

The same discretization was derived previously by Berggren [2] from a different point of view. In [1] we showed that the discretization error estimate

$$\|y - y_h\|_{L^2(\Omega)} \le C h^s \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

holds for $s = 1/2$ if the domain is convex; this is a slight improvement of the result of Berggren.

Let us now consider non-convex domains. Although the very weak solution y is also in $H^{1/2}(\Omega)$ the convergence order is reduced; the finite element method does not lead to the best approximation in $L^2(\Omega)$. In order to describe the result we assume for simplicity that Ω has only one corner with interior angle $\omega \in (\pi, 2\pi)$. We proved in [1] the convergence order $s \in (0, \lambda - \frac{1}{2})$, where $\lambda := \frac{\pi}{\omega}$, and showed by numerical experiments that the order of almost $\lambda - \frac{1}{2}$ is sharp.

In this paper, we modify the discrete solution y_h from (1.5) in order to retain the convergence order $s = \frac{1}{2}$. In particular, we suggest to compute a function

$$z_h \in Y_h \oplus \operatorname{Span}\{r^{-\lambda} \sin(\lambda\theta)\},$$

where r, θ are polar coordinates at the concave corner, such that the error estimate

$$\|y - z_h\|_{L^2(\Omega)} \leq C h^{1/2} \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

can be shown. This method is a dual variant of the singular complement method introduced by Ciarlet and He [5]. Numerical experiments confirm the theoretical results.

2 Analytical background and regularization

As in the introduction, let Ω be a domain with exactly one concave corner, and denote this interior angle by $\omega \in (\pi, 2\pi)$. This corner is located at the origin of the coordinate system, and one boundary edge is contained in the positive x_1-axis. It is well known that the weak solution of the boundary value problem

$$-\Delta v = g \quad \text{in } \Omega, \qquad\qquad v = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{2.1}$$

with $g \in L^2(\Omega)$ is not contained in $H^2(\Omega)$ but in

$$H^1_\Delta(\Omega) \cap H^1_0(\Omega) = \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \oplus \operatorname{Span}\{\xi(r)\, r^{\lambda} \sin(\lambda\theta)\},$$

ξ being a cut-off function, see for example the monograph of Grisvard [9]. This means that

$$R := \{\Delta v : v \in H^2(\Omega) \cap H^1_0(\Omega)\},$$

is a closed subspace of $L^2(\Omega)$. It is shown in [9, Sect. 2.3] that

$$L^2(\Omega) = R \overset{\perp}{\oplus} \operatorname{Span}\{p_s\}, \tag{2.2}$$

with the *dual singular function*

$$p_s = r^{-\lambda} \sin(\lambda\theta) + \tilde{p}_s \tag{2.3}$$

where $\tilde{p}_s \in H^1(\Omega)$ is chosen such that the decomposition (2.2) is orthogonal for the $L^2(\Omega)$ inner product. Therefore, the dual singular function p_s is a solution of

$$w \in L^2(\Omega): \quad (\Delta v, w) = 0 \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega), \tag{2.4}$$

which proves the non-uniqueness of the solution of (2.4). This is the dual property to the non-existence of a solution of (2.1) in $H^2(\Omega) \cap H^1_0(\Omega)$, see [9, Introduction].

Due to (2.2) we can split any $L^2(\Omega)$-function into $L^2(\Omega)$-orthogonal parts. To this end denote by Π_R and Π_{p_s} the orthogonal projections on R and on $\mathrm{Span}\{p_s\}$, respectively, i.e., for $g \in L^2(\Omega)$, it is $g = \Pi_R g + \Pi_{p_s} g$ where

$$\Pi_{p_s} g = \alpha(g)\, p_s \quad \text{with} \quad \alpha(g) = \frac{(g, p_s)_\Omega}{\|p_s\|^2_{L^2(\Omega)}},$$
$$\Pi_R g = g - \Pi_{p_s} g.$$

Since $p_s \in L^2(\Omega)$ there exists

$$\phi_s \in H^1_\Delta(\Omega) \cap H^1_0(\Omega) : \quad -\Delta\phi_s = p_s, \tag{2.5}$$

see also Section 4 for more details on ϕ_s. For the moment we assume that p_s and ϕ_s are explicitly known; hence the decomposition $g = \Pi_R g + \alpha(g)\, p_s$ can be computed once g is given. Computable approximations of p_s and ϕ_s are discussed in Section 4.

Now we come back to problem (1.2) and decompose its solution y in the form

$$y = \Pi_R y + \alpha(y)\, p_s. \tag{2.6}$$

From the decomposition (2.2) we see that problem (1.2) is equivalent to

$$(y, p_s)_\Omega = -(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega,$$
$$(y, \Delta v)_\Omega = (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega)$$

and with the orthogonal splitting (2.6) to

$$\alpha(y)\,(p_s, p_s)_\Omega = -(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega,$$
$$(\Pi_R y, \Delta v)_\Omega = (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega).$$

The first equation directly yields $\alpha(y)$, namely

$$\alpha(y) = \frac{-(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega}{(p_s, p_s)_\Omega}, \tag{2.7}$$

hence the projection of y on p_s is known. It remains to find an approximation of $\Pi_R y$.

At this point we recall the regularization approach from [1] which we summarized already in the introduction. Let $u^h \in H^{1/2}(\Gamma)$ be a regularized boundary datum such that we can define the regularized (weak) solution in $Y^h_* := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$,

$$y^h \in Y^h_* : \quad (\nabla y^h, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H^1_0(\Omega). \tag{2.8}$$

In [1] we showed that the regularization error can be estimated by

$$\|y - y^h\|_{L^2(\Omega)} \le c\|u - u^h\|_{H^{-s}(\Gamma)}$$

where $s = \frac{1}{2}$ if Ω is convex and $s \in [0, \lambda - \frac{1}{2})$ if Ω is non-convex, that means the regularization error is in general bigger in the non-convex case. With the next lemma we show that $\Pi_R(y - y^h)$ is not affected by non-convex corners.

Lemma 1. *If the domain Ω is non-convex, the estimate*

$$\|\Pi_R(y - y^h)\|_{L^2(\Omega)} \le C\|u - u^h\|_{H^{-1/2}(\Gamma)}$$

holds.

Proof. Recall $V = H^1_\Delta(\Omega) \cap H^1_0(\Omega)$ from (1.3). From (2.8) and the Green formula, we have for any $v \in V$

$$(f, v)_\Omega = (\nabla y^h, \nabla v)_\Omega = -(y^h, \Delta v)_\Omega + (y^h, \partial_n v)_\Gamma.$$

Note that $v \in V$ is sufficient, see [6, Lemma 3.4]. Subtracting this expression from the very weak formulation (1.2), we get

$$(y - y^h, \Delta v)_\Omega = (u - u^h, \partial_n v)_\Gamma \quad \forall v \in V.$$

Restricting this identity to $v \in H^2(\Omega) \cap H^1_0(\Omega)$, we have

$$(\Pi_R(y - y^h), \Delta v)_\Omega = (u - u^h, \partial_n v)_\Gamma \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega). \tag{2.9}$$

Now for any $z \in R$, we let $v_z \in H^2(\Omega) \cap H^1_0(\Omega)$ be the unique solution of

$$\Delta v_z = z, \tag{2.10}$$

that satisfies

$$\|\partial_n v_z\|_{H^{1/2}(\Gamma)} \le c\|v_z\|_{H^2(\Omega)} \le c\|z\|_{L^2(\Omega)}. \tag{2.11}$$

Since for any $g \in L^2(\Omega)$ the equality

$$(\Pi_R(y - y^h), g)_\Omega = (\Pi_R(y - y^h), \Pi_R g)_\Omega = (y - y^h, \Pi_R g)_\Omega$$

holds we get with (2.9)–(2.11)

$$\begin{aligned}\|\Pi_R(y - y^h)\|_{L^2(\Omega)} &= \sup_{z\in R, z\neq 0} \frac{(y - y^h, z)_\Omega}{\|z\|_{L^2(\Omega)}} = \sup_{z\in R, z\neq 0} \frac{(u - u^h, \partial_n v_z)_\Gamma}{\|z\|_{L^2(\Omega)}} \\ &\le \|u - u^h\|_{H^{-1/2}(\Gamma)} \sup_{z\in R, z\neq 0} \frac{\|\partial_n v_z\|_{H^{1/2}(\Gamma)}}{\|z\|_{L^2(\Omega)}} \le c\|u - u^h\|_{H^{-1/2}(\Gamma)}\end{aligned}$$

which is the estimate to be proved. □

3 Discretization by standard finite elements

Recall from the introduction the finite element spaces

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} = Y_h \cap H^1_0(\Omega), \quad Y_h^\partial = Y_h|_{\partial\Omega},$$

defined on a family $\mathcal{T}_h$ of quasi-uniform, conforming finite element meshes. Assume that the regularized boundary datum u^h is contained in Y_h^∂ such that the estimates

$$\|u^h\|_{L^2(\Gamma)} \leq c\|u\|_{L^2(\Gamma)}, \tag{3.1}$$

$$\|u - u^h\|_{H^{-1/2}(\Gamma)} \leq Ch^{1/2}\|u\|_{L^2(\Gamma)}, \tag{3.2}$$

hold. It is proved in [1] that this can be accomplished by using the $L^2(\Gamma)$-projection or by quasi-interpolation. A consequence of Lemma 1 is the estimate

$$\|\Pi_R(y - y^h)\|_{L^2(\Omega)} \leq Ch^{1/2}\|u\|_{L^2(\Gamma)} \tag{3.3}$$

in the case of a non-convex domain Ω. (In the case of a convex domain the operator Π_R is the identity, and the corresponding error estimates were already proven in [1].)

As already done in the introduction, define further the finite element solution $y_h \in Y_{*h} := Y_*^h \cap Y_h$ via

$$y_h \in Y_{*h}: \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \tag{3.4}$$

We proved in [1] that

$$\|y - y_h\|_{L^2(\Omega)} \leq Ch^s \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right) \tag{3.5}$$

holds for $s = \frac{1}{2}$ if the domain is convex but only $s \in (0, \lambda - \frac{1}{2})$ in the non-convex case. In the next lemma we show that $\Pi_R(y - y_h)$ is not affected by the non-convex corners.

Lemma 2. *For non-convex domains Ω the discretization error estimate*

$$\|\Pi_R(y - y_h)\|_{L^2(\Omega)} \leq Ch^{1/2} \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

holds.

Proof. By the triangle inequality we have

$$\|\Pi_R(y - y_h)\|_{L^2(\Omega)} \leq \|\Pi_R(y - y^h)\|_{L^2(\Omega)} + \|\Pi_R(y^h - y_h)\|_{L^2(\Omega)}. \tag{3.6}$$

The first term is estimated in (3.3). For the second term we first notice that $y^h - y_h \in H_0^1(\Omega)$ satisfies the Galerkin orthogonality

$$(\nabla(y^h - y_h), \nabla v_h)_\Omega = 0 \quad \forall v_h \in Y_{0h}, \tag{3.7}$$

see (1.4) and (1.5). With that, we estimate $\|\Pi_R(y^h - y_h)\|_{L^2(\Omega)}$ by a similar arguments as $\|\Pi_R(y - y^h)\|_{L^2(\Omega)}$ in the proof of Lemma 1. Recall from (2.10) and (2.11) that $v_z \in H^2(\Omega) \cap H_0^1(\Omega)$ is the weak solution of $\Delta v_z = z \in R$. It can be approximated by the Lagrange interpolant $I_h v_z$ satisfying

$$\|\nabla(v_z - I_h v_z)\|_{L^2(\Omega)} \leq ch\|v_z\|_{H^2(\Omega)} \leq ch\|z\|_{L^2(\Omega)}.$$

We get

$$\begin{aligned}\|\Pi_R(y^h - y_h)\|_{L^2(\Omega)} &= \sup_{z\in R, z\neq 0} \frac{(y^h - y_h, z)_\Omega}{\|z\|_{L^2(\Omega)}} = \sup_{z\in R, z\neq 0} \frac{(\nabla(y^h - y_h), \nabla v_z)_\Omega}{\|z\|_{L^2(\Omega)}} \\ &= \sup_{z\in R, z\neq 0} \frac{(\nabla(y^h - y_h), \nabla(v_z - I_h v_z))_\Omega}{\|z\|_{L^2(\Omega)}} \\ &\leq ch\|\nabla(y^h - y_h)\|_{L^2(\Omega)}. \end{aligned} \tag{3.8}$$

In order to bound $\|\nabla(y^h - y_h)\|_{L^2(\Omega)}$ by the data we consider a lifting $B_h u^h \in Y_{*h}$ defined by the nodal values as follows:

$$(B_h u^h)(x) = \begin{cases} u^h(x), & \text{for all nodes } x \in \Gamma, \\ 0 & \text{for all nodes } x \in \Omega. \end{cases} \tag{3.9}$$

The homogenized solution $y_0^h = y^h - B_h u^h \in H_0^1(\Omega)$ satisfies

$$(\nabla y_0^h, \nabla v)_\Omega = (f, v)_\Omega - (\nabla(B_h u^h), \nabla v)_\Omega \quad \forall v \in H_0^1(\Omega).$$

By taking $v = y_0^h$ we see that

$$\|\nabla y_0^h\|^2_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \|y_0^h\|_{H^1(\Omega)} + \|\nabla(B_h u^h)\|_{L^2(\Omega)} \|\nabla y_0^h\|_{L^2(\Omega)}.$$

Using the Poincaré inequality we obtain

$$\|\nabla y_0^h\|_{L^2(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)} + \|\nabla(B_h u^h)\|_{L^2(\Omega)}, \tag{3.10}$$

and with the Céa lemma

$$\begin{aligned}\|\nabla(y^h - y_h)\|_{L^2(\Omega)} &\leq \|\nabla(y^h - B_h u^h)\|_{L^2(\Omega)} = \|\nabla y_0^h\|_{L^2(\Omega)} \\ &\leq c\|f\|_{H^{-1}(\Omega)} + \|\nabla(B_h u^h)\|_{L^2(\Omega)}.\end{aligned}$$

The remaining term $\|\nabla(B_h u^h)\|_{L^2(\Omega)}$ is estimated by using the inverse inequality

$$\|\nabla(B_h u^h)\|_{L^2(T)} \leq ch^{-1/2}\|u^h\|_{L^2(E)}.$$

for $E \subset T \cap \Gamma$, $T \in \mathcal{T}_h$, which can be proved by standard scaling arguments, to get

$$\|\nabla(B_h u^h)\|_{L^2(\Omega)} \leq ch^{-1/2}\|u^h\|_{L^2(\Gamma)}. \tag{3.11}$$

Hence we proved

$$\|\nabla(y^h - y_h)\|_{L^2(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)} + ch^{-1/2}\|u^h\|_{L^2(\Gamma)}.$$

With (3.6), (3.3), (3.8), the previous inequality, and (3.1) we finish the proof. □

With (2.6) we can immediately conclude the following result.

Corollary 3. *Let Ω be a non-convex domain and let $y_h \in Y_{*h}$ be the solution of* (3.4)*, then the discretization error estimate*

$$\|y - (\Pi_R y_h + \alpha(y) p_s)\|_{L^2(\Omega)} \le C h^{1/2} \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

holds, reminding that p_s and $\alpha(y)$ are given by (2.3) *and* (2.7)*, respectively.*

Hence the positive result is that $\Pi_R y_h + \alpha(y) p_s$ is a better approximation of y than y_h. The problem is that p_s and ϕ_s are used explicitly, and in practice they are not known. A remedy of this drawback is the aim of the next section.

4 Approximate singular functions

Following [5], we approximate p_s from (2.3) by

$$\begin{aligned} p_s^h &= p_h^* - r_h + r^{-\lambda} \sin(\lambda\theta), \quad r_h = B_h \left(r^{-\lambda} \sin(\lambda\theta) \right), \\ p_h^* &\in Y_{0h} : \quad (\nabla p_h^*, \nabla v_h)_\Omega = (\nabla r_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}, \end{aligned} \tag{4.1}$$

with B_h from (3.9). The function ϕ_s from (2.5) admits the splitting

$$\phi_s = \tilde{\phi} + \beta r^\lambda \sin(\lambda\theta), \tag{4.2}$$

with $\tilde{\phi} \in H^2(\Omega)$ and $\beta = \pi^{-1} \|p_s\|^2_{L^2(\Omega)}$, see again [5]. It is approximated by

$$\begin{aligned} \phi_s^h &= \phi_h^* - \beta_h s_h + \beta_h r^\lambda \sin(\lambda\theta), \quad s_h = B_h \left(r^\lambda \sin(\lambda\theta) \right), \quad \beta_h = \frac{1}{\pi} \|p_s^h\|^2_{L^2(\Omega)}, \\ \phi_h^* &\in Y_{0h} : \quad (\nabla \phi_h^*, \nabla v_h)_\Omega = (p_s^h, v_h)_\Omega + \beta_h (\nabla s_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}, \end{aligned} \tag{4.3}$$

that means, $\tilde{\phi}$ is approximated by $\tilde{\phi}_h = \phi_h^* - \beta_h s_h \in Y_h$. The approximation errors are bounded by

$$\|p_s - p_s^h\|_{L^2(\Omega)} \le c h^{2\lambda - \epsilon} \le ch, \tag{4.4}$$

$$|\beta - \beta_h| \le c h^{2\lambda - \varepsilon} \le ch, \tag{4.5}$$

$$\|\phi_s - \phi_s^h\|_{1,\Omega} \le ch, \tag{4.6}$$

see [5, Lemmas 3.1–3.3], where (4.5) and (4.6) imply

$$\|\tilde{\phi} - \tilde{\phi}_h\|_{1,\Omega} \le ch. \tag{4.7}$$

At the end of Section 3 we saw that $\Pi_R y_h + \alpha(y) p_s$ is a better approximation of y than y_h. Since this function is not computable we approximate it by

$$z_h = \Pi_R^h y_h + \alpha_h p_s^h, \tag{4.8}$$

with

$$\Pi_R^h y_h = y_h - \gamma_h p_s^h, \quad \gamma_h = \frac{(y_h, p_s^h)_\Omega}{\|p_s^h\|^2_{L^2(\Omega)}} \tag{4.9}$$

and a suitable approximation α_h of

$$\alpha(y) = \frac{-(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega}{(p_s, p_s)_\Omega}$$

from (2.7). To this end we write the problematic term by using (4.2) as

$$(u, \partial_n \phi_s)_\Gamma = (u, \partial_n \tilde{\phi})_\Gamma + \beta(u, \partial_n (r^\lambda \sin(\lambda\theta)))_\Gamma.$$

and replace the term $(u, \partial_n \tilde{\phi})_\Gamma$ by $(u^h, \partial_n \tilde{\phi})_\Gamma$. Since $\tilde{\phi}$ belongs to $H^2(\Omega)$ and u^h is the trace of $B_h u^h$, we get by using the Green formula

$$\begin{aligned}(u^h, \partial_n \tilde{\phi})_\Gamma &= (B_h u^h, \Delta\tilde{\phi})_\Omega + (\nabla B_h u^h, \nabla\tilde{\phi})_\Omega \\ &= -(B_h u^h, p_s)_\Omega + (\nabla B_h u^h, \nabla\tilde{\phi})_\Omega \end{aligned} \tag{4.10}$$

as $\Delta\tilde{\phi} = \Delta\phi_s = -p_s$. With all these notations and results, we define

$$\alpha_h = \frac{(B_h u^h, p_s^h)_\Omega - (\nabla B_h u^h, \nabla\tilde{\phi}_h)_\Omega - \beta_h(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma + (f, \phi_s^h)_\Omega}{(p_s^h, p_s^h)^2_\Omega}. \tag{4.11}$$

Note that α_h can be computed explicitly and therefore z_h as well.

Let us estimate the approximation errors made.

Lemma 4. *Let Ω be a non-convex domain and let $y_h \in Y_{*h}$ be the solution of (3.4). Then the error estimates*

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} \le ch \left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right), \tag{4.12}$$

$$|\alpha(y) - \alpha_h| \le ch^{1/2} \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right) \tag{4.13}$$

hold.

Proof. With the definitions of Π_R and Π_R^h, with $\gamma := (y_h, p_s)_\Omega / \|p_s\|^2_{L^2(\Omega)}$, and by using the triangle inequality we have

$$\begin{aligned}\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} &= \|\gamma p_s - \gamma_h p_s^h\|_{L^2(\Omega)} \\ &\le |\gamma - \gamma_h| \, \|p_s^h\|_{L^2(\Omega)} + |\gamma| \, \|p_s - p_s^h\|_{L^2(\Omega)}\end{aligned}$$

We write

$$\begin{aligned}\gamma - \gamma_h &= \frac{(y_h, p_s)_\Omega}{\|p_s\|^2_{L^2(\Omega)}} - \frac{(y_h, p_s^h)_\Omega}{\|p_s^h\|^2_{L^2(\Omega)}} \\ &= \frac{(y_h, p_s - p_s^h)_\Omega}{\|p_s\|^2_{L^2(\Omega)}} + (y_h, p_s^h)_\Omega \left(\frac{1}{\|p_s\|^2_{L^2(\Omega)}} - \frac{1}{\|p_s^h\|^2_{L^2(\Omega)}}\right) \\ &= \frac{(y_h, p_s - p_s^h)_\Omega}{\|p_s\|^2_{L^2(\Omega)}} + (y_h, p_s^h)_\Omega \frac{(p_s^h + p_s, p_s^h - p_s)_\Omega}{\|p_s\|^2_{L^2(\Omega)} \|p_s^h\|^2_{L^2(\Omega)}},\end{aligned}$$

and by the Cauchy-Schwarz inequality and (4.4) we get

$$|\gamma - \gamma_h| \le ch\|y_h\|_{L^2(\Omega)}.$$

We have used that $\|p_s\|_{L^2(\Omega)}$ and $\|p_s^h\|_{L^2(\Omega)}$ can be treated as constants due to the definition of p_s and due to (4.4). We conclude with $|\gamma| \le c\|y_h\|_{L^2(\Omega)}$, and (4.4) that

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} \le ch\|y_h\|_{L^2(\Omega)}. \tag{4.14}$$

In view of the finite element error estimate (3.5) and the standard a priori estimate for the very weak solution,

$$\|y\|_{L^2(\Omega)} \le c\left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right),$$

see Lemma 2.3 of [1], we have

$$\|y_h\|_{L^2(\Omega)} \le \|y\|_{L^2(\Omega)} + \|y - y_h\|_{L^2(\Omega)} \le c\left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right).$$

This estimate together with (4.14) proves (4.12).

The proof of the estimate (4.13) is based on writing the problematic term in the definition of $\alpha(y)$ without approximation as

$$\begin{aligned}(u, \partial_n \phi_s)_\Gamma &= (u, \partial_n \tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \\ &= (u - u^h, \partial_n \tilde{\phi})_\Gamma + (u^h, \partial_n \tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \\ &= (u - u^h, \partial_n \tilde{\phi})_\Gamma - (B_h u^h, p_s)_\Omega + (\nabla B_h u^h, \nabla \tilde{\phi})_\Omega + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma\end{aligned}$$

where we used (4.10) in the last step. Consequently, we showed that

$$\begin{aligned}\alpha(y) - \alpha_h = \frac{1}{\|p_s\|^2_{L^2(\Omega)}}\Big(&- (u - u^h, \partial_n \tilde{\phi})_\Gamma + (B_h u^h, p_s - p_s^h)_\Omega - (\nabla B_h u^h, \nabla(\tilde{\phi} - \tilde{\phi}_h))_\Omega \\ &- (\beta - \beta_h)\,(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma + (f, \phi_s - \phi_s^h)_\Omega\Big).\end{aligned}$$

To prove (4.13), in view of (4.4), (4.5), and (4.6) it remains to show that

$$\begin{aligned}\left|(u - u^h, \partial_n \tilde{\phi})_\Gamma\right| &\le ch^{1/2}\|u\|_{L^2(\Gamma)}, \\ \left|(B_h u^h, p_s - p_s^h)_\Omega\right| &\le ch^{1/2}\|u\|_{L^2(\Gamma)}, \\ \left|(\nabla B_h u^h, \nabla(\tilde{\phi} - \tilde{\phi}^h))_\Omega\right| &\le ch^{1/2}\|u\|_{L^2(\Gamma)}.\end{aligned}$$

The first estimate follows from the estimate (3.2) and the fact that $\tilde{\phi}$ belongs to $H^2(\Omega)$. The second one follows from the Cauchy-Schwarz inequality and the estimates (3.11) and (4.4). Similarly, the third estimate follows from the Cauchy-Schwarz inequality and the estimates (3.11) and (4.7). □

Corollary 5. *Let Ω be a non-convex domain and let $y_h \in Y_{*h}$ be the solution of (3.4) and let z_h be derived by (4.8), (4.9), and (4.11), then the discretization error estimate*

$$\|y - z_h\|_{L^2(\Omega)} \leq Ch^{1/2}\left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

holds.

Proof. The main ingredients of the proof were already derived. Indeed, it is

$$\begin{aligned}\|y - z_h\|_{L^2(\Omega)} &= \|\Pi_R y + \alpha(y)p_s - \Pi_R^h y_h - \alpha_h p_s^h\|_{L^2(\Omega)} \\ &\leq \|\Pi_R y - \Pi_R y_h\|_{L^2(\Omega)} + \|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} + \\ &\qquad |\alpha(y) - \alpha_h|\,\|p_s\|_{L^2(\Omega)} + |\alpha_h|\,\|p_s - p_s^h\|_{L^2(\Omega)}.\end{aligned}$$

The first three terms can be estimated by using Lemmas 2 and 4. So it remains to treat the fourth term. To bound $|\alpha_h|$ we use the triangle inequality

$$|\alpha_h| \leq |\alpha_h - \alpha(y)| + |\alpha(y)|.$$

For the first term we use (4.13), while for the second term we use (2.7) reminding that ϕ_s belongs to $H^{3/2+\epsilon}(\Omega)$ with some $\epsilon > 0$. Altogether we have

$$|\alpha_h| \leq C\left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

and conclude by using (4.4). □

Before we describe the numerical experiments, let us summarize the algorithm.

1. Compute the finite element solution

$$y_h \in Y_{*h}: \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}$$

   where $Y_{*h} = \{v_h \in Y_h : v_h|_\Gamma = u^h\}$, compare (1.5), with $u^h \in Y_h^\partial$ being an approximation of the boundary datum u satisfying (3.1) and (3.2).

2. Compute the approximate singular functions:

$$\begin{aligned}&r_h = B_h\left(r^{-\lambda}\sin(\lambda\theta)\right), \\ &p_h^* \in Y_{0h}: \quad (\nabla p_h^*, \nabla v_h)_\Omega = (\nabla r_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}, \\ &\tilde{p}_h = p_h^* - r_h, \\ &\beta_h = \frac{1}{\pi}\|\tilde{p}_h + r^{-\lambda}\sin(\lambda\theta)\|^2_{L^2(\Omega)}, \\ &s_h = B_h\left(r^{\lambda}\sin(\lambda\theta)\right), \\ &\phi_h^* \in Y_{0h}: \quad (\nabla\phi_h^*, \nabla v_h)_\Omega = (\tilde{p}_h + r^{-\lambda}\sin(\lambda\theta), v_h)_\Omega + \beta_h(\nabla s_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}, \\ &\tilde{\phi}_h = \phi_h^* - \beta_h s_h,\end{aligned}$$

   compare (4.1) and (4.3).

3. Compute

$$\gamma_h = \frac{(y_h, p_s^h)_\Omega}{(p_s^h, p_s^h)_\Omega} \quad \text{with } p_s^h = \tilde{p}_h + r^{-\lambda}\sin(\lambda\theta),$$

$$\alpha_h = \frac{(B_h u^h, p_s^h)_\Omega - (\nabla B_h u^h, \nabla\tilde{\phi}_h)_\Omega - \beta_h(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma + (f, \phi_s^h)_\Omega}{(p_s^h, p_s^h)_\Omega^2},$$

$$\delta_h = \alpha_h - \gamma_h,$$

$$\tilde{z}_h = y_h + \delta_h \tilde{p}_h,$$

compare (4.9) and (4.11). According to (4.8), the numerical solution is

$$z_h = \tilde{z}_h + \delta_h r^{-\lambda}\sin(\lambda\theta).$$

Note that all integrals with r^λ and $r^{-\lambda}$ must be computed with care.

5 Numerical experiments

This section is devoted to the numerical verification of our theoretical results. For that purpose we present examples with known solution. Furthermore, to examine the influence of the corner singularities, we consider several polygonal domain Ω_ω depending on an interior angle $\omega \in (0, 2\pi)$. The computational domains are defined by

$$\Omega_\omega := (-1,1)^2 \cap \{x \in \mathbb{R}^2 : (r(x), \theta(x)) \in (0, \sqrt{2}] \times [0, \omega]\}, \tag{5.1}$$

where r and θ stand for the polar coordinates located at the origin. The boundary of Ω_ω is denoted by Γ_ω. We solve the problem

$$-\Delta y = 0 \quad \text{in } \Omega_\omega, \qquad\qquad y = u \quad \text{on } \Gamma, \tag{5.2}$$

numerically by using the proposed dual singular function method. The boundary datum u is chosen as follows

$$u := r^{-0.4999}\sin(-0.4999\theta) \quad \text{on } \Gamma_\omega.$$

This function belongs to $L^p(\Gamma)$ for every $p < 2.0004$. The exact solution of our problem is simply

$$y = r^{-0.4999}\sin(-0.4999\theta),$$

since y is harmonic.

The quasi-uniform finite element meshes for the calculations are generated by using a newest vertex bisection algorithm. The discretization errors for different mesh sizes and the corresponding experimental orders of convergence are given in Table 1 for different interior angles $\omega = 270°$ and $\omega = 355°$. We see that the numerical results confirm the expected convergence rate 1/2.

We emphasize that the quadrature formula for the numerical integration of the integral

$$(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma$$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc
0.25000	0.58725	
0.12500	0.42338	0.47201
0.06250	0.30318	0.48177
0.03125	0.21606	0.48870
0.01562	0.15352	0.49302
0.00781	0.10888	0.49572
0.00390	0.07712	0.49742

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc
0.25000	1.02069	
0.12500	0.83402	0.29139
0.06250	0.58964	0.50025
0.03125	0.41696	0.49991
0.01562	0.29506	0.49890
0.00781	0.20903	0.49725
0.00390	0.14836	0.49462

Table 1: Discretization errors $e_h = y - z_h$ for $\omega = 3\pi/2$ (left) and $\omega = 355\pi/180$ (right)

has to be adapted in order to get a sufficiently good approximation. Otherwise, the error due to the quadrature formula dominates the overall error. In our implementation, we chose for the numerical integration a graded mesh on the boundary ($h_E \sim hr_E^{1-\mu}$ if the distance r_E of the boundary edge E satisfies $0 < r_E < R$ with R being the radius of the refinement zone and μ being the refinement parameter, and $h_T = h^{1/\mu}$ for $r_E = 0$) combined with a one-point Gauss quadrature rule on each element. Furthermore, the grading parameter μ is chosen such that

$$\mu \leq 2\pi/\omega - 1,$$

which seems to be the correct grading to achieve a convergence order of 1/2. For the results presented in Table 1 we used $R = 0.1$ and $\mu = 2\pi/\omega - 1$.

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