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Finite element error estimates for Neumann boundary control problems on graded meshes

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Abstract A specific elliptic linear-quadratic optimal control problem with Neumann boundary control is investigated. The control has to fulfil inequality constraints. The domain is assumed to be polygonal with reentrant corners. The asymptotic behaviour of two approaches to compute the optimal control is discussed. In the first the piecewise constant approximations of the optimal control are improved by a postprocessing step. In the second the control is not discretized; instead the first order optimality condition is used to determine an approximation of the optimal control. Although the quality of both approximations is in general affected by corner singularities a convergence order of 3/2 can be proven provided that the mesh is sufficiently graded.

Keywords: linear-quadratic elliptic optimal control problem, boundary control, a-priori error estimates, mesh grading, postprocessing, variational discretization

1 Introduction

In this paper, we consider the discretization of the optimal control problem

$$\begin{aligned} J(\bar{u}) &= \min_{u\in U_{ad}} J(u), \\ J(u) &:= F(Su, u), \\ F(y,u) &:= \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \end{aligned} \tag{1.1}$$

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where the associated state $y = Su$ to the control u is the weak solution of the state equation

$$\begin{aligned} -\Delta y + y &= 0 \qquad \text{in } \Omega, \\ \partial_n y &= u \qquad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{1.2}$$

and the control variable is constrained by

$$a \le u \le b \quad \text{for a.a. } x \in \Gamma.$$

The domain Ω is a bounded, two dimensional, polygonal domain with boundary Γ and m corner points $x^{(j)}$, $j = 1, \dots, m$, counting counterclockwise. In particular, Γ_j denotes the side on the boundary Γ which connects the corners $x^{(j)}$ and $x^{(j+1)}$ except that $x^{(1)}$ is the intersection of $\bar{\Gamma}_m$ and $\bar{\Gamma}_1$. The set of all corner points is denoted by S. We further distinguish between the sets of convex corners C^e and concave corners $C^a = S \backslash C^e$. The regularization parameter $\nu \in \mathbb{R}_+$ and the bounds $a, b \in \mathbb{R}$ are fixed numbers. The function $y_d \in W^{0,2}_\beta(\Omega)$ denotes the desired state. Furthermore we define the set of admissible controls by

$$U_{ad} := \{u \in L^2(\Gamma) : \ a \le u \le b \text{ a.e. on } \Gamma\}.$$

In that what follows we discuss the full discretization and the variational discretization of the optimal control problem. Both approaches have in common that the state and the adjoint state are discretized by a finite element method. The difference occurs when discretizing the control. The full discretization approach uses piecewise constant approximations for the control together with an additional postprocessing step. This concept was established by Meyer and Rösch in [27] for distributed control. In that case they proved a convergence order of 2 in the $L^2(\Omega)$-norm for the optimal solution. In [25] it was applied to the Neumann boundary control problem which is also considered in this paper. Mateos and Rösch proved an order of convergence of $2 - 1/p$ for linear finite elements on convex domains where p depends on the regularity of the solution of the optimal control problem. For nonconvex domains they proved a convergence rate of $1/2 + \pi/\omega$, where ω denotes the largest inner angle in the domain. They also used higher order finite elements for the approximation of the state and the adjoint state and obtained better results. The second discretization concept was introduced by Hinze in [20] for distributed control problems. Instead of discretizing the control, the first order optimality condition is used within this approach to compute an approximation of the control. In the case of distributed control the convergence order is also 2 in the $L^2(\Omega)$-norm. Using this approach Neumann boundary control problems were primarily discussed in [11], where the focus was on convex domains, semilinear partial differential equations and not necessarily quadratic functionals. The variational approach for problem (1.1) on quasiuniform meshes was considered in [25, 21]. Casas and Mateos [11] and Hinze and Matthes [21] proved that the error in the control is bounded by $ch^{3/2}$ on convex domains when using this discretization concept. Mateos and Rösch [25] obtained for the variational approach the order $2 - 1/p$ on convex domains and $1/2 + \pi/\omega$ on nonconvex domains as for the fully discrete approach. But the results of Mateos and Rösch on nonconvex domains for both, the full and the variational discretization approach, have in common that the order of convergence is lower than 3/2 and decreases further in case that the largest inner angle in the domain increases. In the sequel we will analyze the asymptotic behaviour of the discrete optimal control problems and prove that the error in the control is bounded by $ch^{3/2}$ for both approaches if one uses appropriately graded meshes in the neighborhood of reentrant corners.

Let us state some further results from the literature before we go into detail. Results on discretizations of optimal control problems by piecewise constant functions were already presented by Falk [17], Geveci [18] and Malanowski [24]. Recently, Arada, Casas and Tröltzsch [7], and Casas, Mateos and Tröltzsch [12] stated some new results using piecewise constant controls. Piecewise linear controls were considered in Casas [10], Casas and Mateos [11], Casas and Tröltzsch [13], Meyer and Rösch [26] and Rösch [30, 31]. In all those papers it was proven that the error in the control in the L^2-norm is bounded by ch or $ch^{3/2}$ on quasi-uniform meshes provided that the solution is smooth enough. In Apel, Rösch, Winkler [2], Apel, Winkler [6], Apel, Sirch, Winkler [5] and Apel, Rösch, Sirch [1] mesh grading and discretization techniques were successfully applied to improve the order of convergence not only in concave 2D and 3D domains.

The paper is organized as follows: In Section 2 we recall regularity results in suitable spaces for the elliptic boundary value problem. We distinguish weighted spaces with homogeneous and nonhomogeneous norms. In Section 3 the discretization concept is presented and the convergence results for the fully discrete approach are formulated. Section 4 contains results from the finite element theory and from numerical integration in order to prove the supercloseness and superconvergence properties of the fully discrete approach in Section 5. Section 6 is devoted the approach of variational discretizations. The paper ends with results from a numerical experiment using the postprocessing approach and some concluding remarks.

We will denote by c a generic constant which may take different values at each occurrence, but is always independent of the discretization.

2 Regularity results in suitable spaces and first order optimality conditions

The regularity of the solution of the elliptic boundary value problem can be described by using the classical Sobolev spaces $W^{s,p}(\Omega)$, compare e.g. [19, 15]. But in our context it is more convenient to use weighted Sobolev spaces. For this, we assume that the underlying domain coincides with an infinite angle in the neighborhood N_j of each corner. This is satisfied for polygonal domains. Let $K_j \subset \mathbb{R}^2$ be such an infinite angle with vertex at the point $x^{(j)}$ and aperture $\omega_j \in (0, 2\pi)$, i.e.

$$K_j := \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < r_j < \infty, 0 < \varphi_j < \omega_j\},$$

where r_j, φ_j denote the polar coordinates located at the point $x^{(j)}$. In particular, the distance r_j to the corner is defined by $r_j := |x - x^{(j)}|$. The sides of K_j (excluding the vertices) are denoted by Γ_j^1 ($\varphi_j = \omega_j$) and Γ_j^2 ($\varphi_j = 0$). Moreover, let ξ_j, $j = 1, \ldots, m$, be infinitely differentiable cut-off functions defined in the closure of Ω, which are equal to one in a neighborhood of the corner $x^{(j)}$ and equal to zero in $\bar{\Omega} \backslash N_j$. Moreover, we set $\xi_0 = 1 - \sum_{j=1}^m \xi_j$ and $\beta = (\beta_1, \ldots, \beta_m)$, a real-valued vector. To ensure $\xi_0 \geq 0$ the sets K_j should not overlap. In the sequel we distinguish between two types of weighted Sobolev spaces

$$V_\beta^{k,2}(\Omega) := \{v \in D'(\Omega) : \|v\|_{V_\beta^{k,2}(\Omega)} < \infty\}$$

and

$$W_\beta^{k,2}(\Omega) := \{v \in D'(\Omega) : \|v\|_{W_\beta^{k,2}(\Omega)} < \infty\}$$

with $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}^m$, equipped with the norms

$$\|v\|_{V_\beta^{k,2}(\Omega)} := \|\xi_0 v\|_{W^{k,2}(\Omega)} + \sum_{j=1}^m \|\xi_j v\|_{V_{\beta_j}^{k,2}(K_j)}$$

and

$$\|v\|_{W_\beta^{k,2}(\Omega)} := \|\xi_0 v\|_{W^{k,2}(\Omega)} + \sum_{j=1}^m \|\xi_j v\|_{W_{\beta_j}^{k,2}(K_j)},$$

where the Sobolev spaces $W^{k,p}(\Omega)(= H^k(\Omega)$ for $p = 2)$ are defined as usual. By using standard multi-index notation the weighted norms for each angle K_j are defined by

$$\|v\|_{V_{\beta_j}^{k,2}(K_j)} := \left(\int\limits_{K_j} \sum_{|\alpha| \leq k} r_j^{2(\beta_j - k + |\alpha|)} |D^\alpha v|^2 dx \right)^{\frac{1}{2}}$$

and

$$\|v\|_{W_{\beta_j}^{k,2}(K_j)} := \left(\int\limits_{K_j} \sum_{|\alpha| \leq k} r_j^{2\beta_j} |D^\alpha v|^2 dx \right)^{\frac{1}{2}}.$$

The corresponding trace spaces are $V_\beta^{k-1/2,2}(\Gamma)$ and $W_\beta^{k-1/2,2}(\Gamma)$ for $k > 0$. The norm in $W_\beta^{k-1/2,2}(\Gamma)$ is defined by

$$\|v\|_{W_\beta^{k-1/2,2}(\Gamma)} := \inf\{\|u\|_{W_\beta^{k,2}(\Omega)} : u \in W_\beta^{k,2}(\Omega), u_{|\Gamma \backslash S} = v\}.$$

Recall that the set S is the set of all corner points. We could analogously define the norm in $V_\beta^{k-1/2,2}(\Gamma)$, but it will be more suitable to have a norm which is given by integral expressions. Hence we define an equivalent norm in the trace space of $V_\beta^{k,2}(\Omega)$ by

$$\|v\|_{V_\beta^{k-1/2,2}(\Gamma)} := \sum_{j=1}^m \|\xi_0 v\|_{W^{k-1/2,2}(\Gamma_j)} + \sum_{j=1}^m \sum_{i=1}^2 \|\xi_j v\|_{V_{\beta_j}^{k-1/2,2}(\Gamma_j^i)},$$

with

$$\|v\|_{V^{k-1/2,2}_{\beta_j}(\Gamma^i_j)} := \left(\int\limits_{\Gamma^i_j} \sum_{|\alpha|\le k-1} r_j^{2(\beta_j-k+1/2+|\alpha|)} |D^\alpha v|^2 ds_x \right.$$
$$\left. + \sum_{|\alpha|=k-1} \int_{\Gamma^i_j} \int_{\Gamma^i_j} \frac{\left| r_j(x)^{\beta_j}(D^\alpha v)(x) - r_j(y)^{\beta_j}(D^\alpha v)(y)\right|^2}{|x-y|^2} ds_x ds_y \right)^{\frac{1}{2}},$$

compare [32, Remark 1.1 together with Theorem 1.1]. At first glance both norms only differ in the exponent of the weighting function. But a closer examination reveals, that the space $W^{1,2}(\Omega)$ does not belong automatically to the scale of the first kind of weighted spaces. The space $W^{1,2}(\Omega) \cap V_0^{1,2}(\Omega)$ only contains functions which vanish at the corner. But the solution of the Neumann problem does not satisfy this condition in general. By contrast the second kind of weighted spaces contains the classical Sobolev spaces for $\beta = 0$. Thus we will use the spaces $W_\beta^{k,2}(\Omega)$ to describe the regularity of the Neumann problem. But as we will see in the sequel the space $V_\beta^{k,2}(\Omega)$ and its trace space are more convenient for the numerical analysis of problem (1.1) having regard to the following lemmas.

Lemma 2.1. *For $\beta \in (0,1)^m$ one has*

$$W_\beta^{2,2}(\Omega) = V_\beta^{2,2}(\Omega) \oplus \xi_1\Pi_0 \oplus \cdots \oplus \xi_m\Pi_0,$$

where $\xi_j\Pi_0$ is the set of all functions, which have the form $\xi_j p$ and p is a constant. In particular, for any $v \in W_\beta^{2,2}(\Omega)$ one can write $v = v_s + \sum_{j=1}^m \xi_j v_j$ with $v_s \in V_\beta^{2,2}(\Omega)$ and $v_j = v(x^{(j)})$, $j = 1,\dots,m$. Moreover, the norm equivalence

$$\|v\|_{W_\beta^{2,2}(\Omega)} \sim \|v_s\|_{V_\beta^{2,2}(\Omega)} + \sum_{j=1}^m |v(x^{(j)})|$$

is valid.

Proof. One can set $n = 2$, $l = 2$ and $s = 1$ in Theorem 7.1.1 of [23] to get the result for domains with one corner. The extension to general polygonal domains is obvious by writing $y = \sum_{j=0}^m \xi_j y$, see also [23, p. 273]. □

Lemma 2.2. *Let $0 < \beta_j, \beta_{j+1} < 1$. Then*

$$W_\beta^{3/2,2}(\Gamma_j) = V_\beta^{3/2,2}(\Gamma_j) \oplus \xi_j\Pi_0 \oplus \xi_{j+1}\Pi_0,$$

i.e. for any $v \in W_\beta^{3/2,2}(\Gamma_j)$ one can write $v = v_s + \xi_j v_j + \xi_{j+1}v_{j+1}$ with $v_s \in V_\beta^{3/2,2}(\Gamma_j)$ and $v_i = v(x^{(i)})$, $i = j, j+1$. Furthermore, the norm equivalence

$$\|v\|_{W_\beta^{3/2,2}(\Gamma_j)} \sim \|v_s\|_{V_\beta^{3/2,2}(\Gamma_j)} + |v(x^{(j)})| + |v(x^{(j+1)})|$$

holds true.

Proof. This can be deduced from Theorem 7.1.2 and Theorem 7.1.3 in [23] by setting $n = 2$, $l = 2$ and $s = 1$. □

Remark 2.3. *Lemma 2.1 and 2.2 remain valid for $\beta_j = 0$, $j = 1, \dots, m$, if the function v is equal to zero in the neighborhood N_j of the corner $x^{(j)}$.*

Next we state a lemma concerning the solvability and the regularity of the elliptic boundary value problem (1.2).

Lemma 2.4. *Let $\lambda_j = \pi/\omega_j$ and $1 > \beta_j > 1 - \lambda_j$, $j = 1, \dots, m$. Moreover, let $f \in W^{0,2}_\beta(\Omega)$ and $g \in W^{1/2,2}_\beta(\Gamma)$. Then the problem*

$$\begin{aligned} -\Delta y + y &= f \qquad \text{in } \Omega, \\ \partial_n y &= g \qquad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{2.1}$$

has a unique weak solution and the a priori estimate

$$\|y\|_{W^{2,2}_\beta(\Omega)} \le c \left(\|f\|_{W^{0,2}_\beta(\Omega)} + \|g\|_{W^{1/2,2}_\beta(\Gamma)} \right)$$

is valid with a constant c independent of y.

Proof. The technique is adapted from the proof of Theorem 3.2. in [4], where the special case $g = 0$ for $\Omega \subset \mathbb{R}^3$ is considered. We get from the Lax-Milgram Theorem the unique solvability of the elliptic boundary value problem in $H^1(\Omega)$. Thus we can conclude for $\beta_j > 1 - \lambda_j$ the a priori estimate

$$\|y\|_{W^{2,2}_\beta(K_j)} \le c \left(\|f\|_{W^{0,2}_\beta(K_j)} + \|g\|_{W^{1/2,2}_\beta(\Gamma^1_j)} + \|g\|_{W^{1/2,2}_\beta(\Gamma^2_j)} \right)$$

and the existence of a unique solution $y \in W^{2,2}_\beta(K_j)$ by setting $k = 0$ in Theorem 3.2. of [34]. The Neumann problem (2.1) can be locally transformed near each corner $x^{(j)}$ into a boundary value problem in the infinite angle K_j. Thus we obtain the desired result by using the partition of unity method and the local results in K_j. □

We set $\lambda = (\lambda_1, \dots, \lambda_m)$ a real-valued vector. In the sequel all inequalities containing vectorial parameters must be understood component-by-component. For $\omega_j < \pi$ we obtain $\lambda_j > 1$ and we use the freedom to choose $\beta_j = 0$ in the following. This means weights are active in concave corners only. Moreover, we will always choose $\beta_j < 1$. This is always possible for concave corners since $\lambda_j > 0$ (even $> 1/2$). Now we apply the results given so far to the optimal control problem. The exposition follows [25]. The solution operator $S : L^2(\Gamma) \to L^2(\Omega)$ which associates a state $y = Su$ to a control u is given via (1.2). We also introduce the problem

$$\begin{aligned} -\Delta p + p &= z \qquad \text{in } \Omega, \\ \partial_n p &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{2.2}$$

and the according solution operator $P : L^2(\Omega) \to H^1(\Omega)$ which associates a function $p = Pz$ with a function z via (2.2). The adjoint operator of S is $S^* : L^2(\Omega) \to L^2(\Gamma)$, where we have $S^*z = (Pz)_{|\Gamma} = p_{|\Gamma}$ [25]. Applying the Green formula shows that

$$(Su, z)_{L^2(\Omega)} = (u, S^*z)_{L^2(\Gamma)} \quad \forall u \in L^2(\Gamma),\ z \in L^2(\Omega).$$

We can also relate an adjoint state $p_u := P(Su - y_d)$ to every control u, which is the unique solution of

$$\begin{aligned} -\Delta p_u + p_u &= y - y_d && \text{in } \Omega, \\ \partial_n p_u &= 0 && \text{on } \Gamma. \end{aligned} \tag{2.3}$$

Note that $S^*(Su - y_d) = (P(Su - y_d))_{|\Gamma} = p_{u|\Gamma}$. Before finishing this section with the theorem about the solvability and regularity of the optimal control problem, we define the projection

$$\Pi_{[a,b]} f(x) := \max(a, \min(b, f(x))). \tag{2.4}$$

Theorem 2.5. *The optimal control problem* (1.1) *has a unique solution $\bar{u}$. The corresponding state $\bar{y} = S\bar{u}$ and adjoint state $\bar{p} = p_{\bar{u}}$ are given by* (1.2) *and* (2.3). *The variational inequality*

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad} \tag{2.5}$$

is necessary and sufficient for the optimality of $\bar{u}$. This condition can be expressed equivalently by

$$\bar{u} = \Pi_{[a,b]} \left(-\frac{1}{\nu} \bar{p}_{|\Gamma} \right). \tag{2.6}$$

Moreover, there holds for $\beta > 1 - \lambda$, $u \in L^2(\Gamma)$ and $y_d \in W^{0,2}_\beta(\Omega)$ that $\bar{y} \in W^{2,2}_\beta(\Omega) \cap C(\bar{\Omega})$, $\bar{p} \in W^{2,2}_\beta(\Omega) \cap C(\bar{\Omega})$, $\bar{p}_{|\Gamma} \in W^{3/2,2}_\beta(\Gamma) \cap C(\Gamma)$ and $\bar{u} \in C(\Gamma) \bigcap_{j=1}^m H^1(\Gamma_j)$ and there exists a constant c which depends only on the data of the problem such that

$$\|\bar{u}\|_{L^\infty(\Gamma)} + \sum_{j=1}^m \|\bar{u}\|_{H^1(\Gamma_j)} + \|\bar{y}\|_{W^{2,2}_\beta(\Omega)} + \|\bar{p}\|_{W^{2,2}_\beta(\Omega)} + \|\bar{p}\|_{W^{3/2,2}_\beta(\Gamma)} \leq c. \tag{2.7}$$

Proof. First we recall some results from the literature which hold for linear quadratic and strictly convex optimal control problems. The unique solvability of the optimal control problem, the necessity and sufficiency of the variational inequality for an optimal control and the equivalence between the variational inequality and the projection formula have been discussed in several papers, see e.g. [12, 11, 25]. We have $y \in H^{3/2}(\Omega)$ for any function $u \in L^2(\Gamma)$, see Theorem 2.1. in [12]. Since the space $H^{3/2}(\Omega)$ is embedded in the space $W^{0,2}_\beta(\Omega)$ (definitely for $\beta \geq 0$) we get with Lemma 2.4 that $p \in W^{2,2}_\beta(\Omega)$ for $\beta > 1 - \lambda$. This implies that the restriction of p on every side Γ_j, $j = 1, \ldots, m$, belongs to $H^1(\Gamma_j) \cap W^{3/2,2}_\beta(\Gamma_j)$ and the restriction on the complete boundary belongs to $H^{1-\epsilon}(\Gamma)$, $\epsilon > 0$ arbitrary, due to the trace theorem [33, Theorem 8.7]. Since $\bar{u}$ is determined by the projection formula we get that $\bar{u}$ belongs to the space $\bigcap_{j=1}^m H^1(\Gamma_j)$.

The space $H^1(\Gamma_j)$ is embedded in the space $W_\beta^{1/2,2}(\Gamma_j)$ since $\beta \geq 0$. Thus we can conclude $\bar{y} \in W_\beta^{2,2}(\Omega)$ by means of Lemma 2.4. Moreover we have $\bar{p}, \bar{u} \in C(\Gamma)$ due to the embedding $H^{1-\epsilon}(\Gamma) \hookrightarrow C(\Gamma)$, $\epsilon < \frac{1}{2}$, and $\bar{y}, \bar{p} \in C(\bar{\Omega})$ due to the embedding $H^{3/2}(\Omega) \hookrightarrow C(\bar{\Omega})$. □

3 Discretization and fully discrete approach

We discretize the optimal control problem by a finite element method. To state the discretization we introduce a family of graded triangulations $\{\mathcal{T}_h\}$ of Ω, which is admissible in the sense of Ciarlet [14] with h being the the global mesh parameter. Note that there is a segmentation $\mathcal{G}_h$ of the boundary given by the triangulation $\mathcal{T}_h$ in a natural way. Furthermore we denote by $\mu_j \in (0,1]$, $j = 1,\ldots,m$, the grading parameters which are summarized in the vector μ. With $r_{T,j} := \inf_{(x_1,x_2)\in T} |x - x^{(j)}|$ and $r_{G,j} := \inf_{(x_1,x_2)\in G} |x - x^{(j)}|$ we denote the distance of the triangle T and of the edge G to the corner $x^{(j)}$, respectively. For edges G with $G \subset \bar{T}$ there holds $r_{T,j} \leq r_{G,j}$. We assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$\begin{aligned} c_1 h^{1/\mu_j} \leq h_T \leq c_2 h^{1/\mu_j} \quad & \text{for } r_{T,j} = 0, \\ c_1 h r_{T,j}^{1-\mu_j} \leq h_T \leq c_2 h r_{T,j}^{1-\mu_j} \quad & \text{for } 0 < r_{T,j} \leq R_j, \\ c_1 h \leq h_T \leq c_2 h \quad & \text{for } r_{T,j} > R_j \end{aligned} \tag{3.1}$$

for $j = 1,\ldots,m$ and some constants R_j. We emphasize that the number of elements of such a discretization is of order h^{-2}, see e.g. [3]. Moreover, we introduce the finite dimensional spaces

$$\begin{aligned} V_h &:= \{y_h \in C(\bar{\Omega}) : y_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \\ U_h &:= \{u_h \in L^\infty(\Gamma) : u_h|_G \in \mathcal{P}_0 \ \forall G \in \mathcal{G}_h\}, \\ U_h^{ad} &:= U_h \cap U^{ad}, \end{aligned}$$

where $\mathcal{P}_k,\ k = 0,1$ is the space of polynomials of degree less than or equal to k.

Now we can state the discrete version of the state equation. Find for each $u \in L^2(\Gamma)$ the unique element $y_h = S_h u \in V_h$ satisfying

$$a(y_h, v_h) = (u, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h,$$

where $S_h : L^2(\Gamma) \to L^2(\Omega)$ is the discrete solution operator, $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(y,v) = \int_\Omega \nabla y \cdot \nabla v + yv \, \mathrm{d}x.$$

The finite dimensional approximation of the optimal control problem reads as follows

$$\begin{aligned} J_h(\bar{u}_h) &= \min_{u_h \in U_h^{ad}} J_h(u_h), \\ J_h(u_h) &:= \frac{1}{2}\|S_h u_h - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|u_h\|_{L^2(\Gamma)}. \end{aligned} \tag{3.2}$$

To be coherent with the definitions of the continuous optimal control problem, we also introduce the discrete solution operator $P_h : L^2(\Omega) \to H^1(\Omega)$ defined by $p_h = P_h z$ with $p_h \in V_h$ such that

$$a(v_h, p_h) = (z, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h,$$

The adjoint of the discrete solution operator is equal to the discretized version of the adjoint solution operator. Thus we can write $S_h^* : L^2(\Omega) \to L^2(\Gamma)$ with $S_h^* z = (P_h z)_{|\Gamma}$. The discrete adjoint state is the unique element $p_h(u) \in V_h$ such that

$$a(v_h, p_h(u)) = (S_h u - y_d, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h.$$

Note that $S_h^*(S_h u - y_d) = p_h(u)_{|\Gamma}$ and

$$(S_h u, z)_{L^2(\Omega)} = (u, S_h^* z)_{L^2(\Gamma)} \qquad \forall z \in L^2(\Omega),\ \forall u \in L^2(\Gamma).$$

Remark 3.1. *The discrete optimal control problem* (3.2) *admits a unique solution* $\bar{u}_h$. *The optimal discrete state* $S_h \bar{u}_h$ *is denoted by* $\bar{y}_h$ *and the optimal discrete adjoint state* $\bar{p}_h$ *is given by* $p_h(\bar{u}_h)$. *The first order optimality condition reads as follows*

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \qquad \forall u_h \in U_{ad}^h. \tag{3.3}$$

For the rest of the paper we make the following assumption to state the superconvergence properties of the fully discrete approach. Since the optimal control $\bar{u}$ is given by the projection formula (2.6) we can easily classify the edges $G \in \mathcal{G}_h$ in two sets K_1 an K_2.

Assumption 3.2. *Let* K_1 *and* K_2 *be defined as follows*

$$K_1 := \bigcup_{G \in \mathcal{G}_h : \bar{u} \notin W^{3/2,2}_{3(1-\mu)/2}(G)} G, \qquad K_2 := \bigcup_{G \in \mathcal{G}_h : \bar{u} \in W^{3/2,2}_{3(1-\mu)/2}(G)} G.$$

We assume

$$\operatorname{meas}(K_1) \leq ch$$

with a constant c *greater than zero.*

Note that this assumption is satisfied in many practical cases, see Section 4 in [25] for details. Now we state the main results for the fully discrete approach.

Theorem 3.3. *Assume that Assumption 3.2 is valid. If the grading parameters satisfy* $\mu < \lambda$ *then the estimates*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^{3/2}, \tag{3.4}$$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq ch^{3/2} \tag{3.5}$$

hold true.

The proof is postponed to section 5. Following [27, 25], we construct a control in a postprocessing step and compute a control $\tilde{u}_h$ by a projection of the discrete adjoint state $\bar{p}_h$ to the set of admissible controls U_{ad}, i.e. $\tilde{u}_h := \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}_h\right)$. Thus this projection is piecewise linear and continuous, but in general $\tilde{u}_h \notin U_h$. However we can prove the following superconvergence property.

Theorem 3.4. *Assume that Assumption 3.2 is fulfilled. For a family of meshes with grading parameters $\mu < \lambda$, the estimate*

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \leq ch^{3/2} \tag{3.6}$$

is valid.

Proof. The projection operator $\Pi_{[a,b]}$ is a bounded continuous operator from $L^2(\Gamma)$ to $L^2(\Gamma)$. Thus we obtain

$$\begin{aligned} \nu\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} &= \nu \left\| \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}\right) - \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}_h\right) \right\|_{L^2(\Gamma)} \\ &\leq c\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq ch^{3/2}, \end{aligned}$$

where we used (3.5) in the last step. □

4 Results from the finite element theory

In this section we collect some results from the finite element theory for elliptic equations and from numerical integration to prove the supercloseness and superconvergence results in the next section.

Lemma 4.1. *Let $u \in L^2(\Gamma)$, and $z \in W^{0,2}_\beta(\Omega)$ with $\beta > 1 - \lambda$. The discretization error for the state and the adjoint state can be estimated by*

$$\|Su - S_hu\|_{L^2(\Omega)} + h\|Su - S_hu\|_{W^{1,2}(\Omega)} \leq ch^2, \tag{4.1}$$

$$\|Pz - P_hz\|_{L^2(\Omega)} + h\|Pz - P_hz\|_{W^{1,2}(\Omega)} \leq ch^2, \tag{4.2}$$

provided that the mesh grading parameters fulfil $\mu < \lambda$.

Proof. To prove (4.1) we use standard techniques for estimates on finite element errors. The result for the adjoint state holds analogously. Due to the continuity and the coercivity of the bilinear form a we can conclude by means of Cea's Lemma

$$\|y - y_h\|_{W^{1,2}(\Omega)} \leq c \inf_{v_h \in V_h} \|y - v_h\|_{W^{1,2}(\Omega)}. \tag{4.3}$$

Since $y \in C(\bar{\Omega})$ the linear Lagrange interpolant I_hy is well defined and belongs to V_h. We have

$$\|y - y_h\|_{W^{1,2}(\Omega)} \leq c\|y - I_hy\|_{W^{1,2}(\Omega)}.$$

To derive an estimate for the interpolation error we write according to Lemma 2.1

$$y = \sum_{j=0}^{m} \xi_j y_j = \xi_0 y + \sum_{j\in C^e} \xi_j y + \sum_{j\in C^a} \xi_j y_s + \sum_{j\in C^a} \xi_j y_j$$

with $\xi_j y_s \in V^{2,2}_\beta(\Omega)$, $y_j \in \Pi_0$, $j \in C^a$, and ξ_j being the cut-off functions defined in Section 2. Hence we can estimate

$$\|y - y_h\|_{W^{1,2}(\Omega)} \le c \Bigg(\|\xi_0 y - I_h(\xi_0 y)\|_{W^{1,2}(\Omega)} + \sum_{j\in C^e} \|\xi_j y - I_h(\xi_j y)\|_{W^{1,2}(\Omega)}$$
$$+ \sum_{j\in C^a} \|\xi_j y_s - I_h(\xi_j y_s)\|_{W^{1,2}(\Omega)} + \sum_{j\in C^a} \|\xi_j y_j - I_h(\xi_j y_j)\|_{W^{1,2}(\Omega)} \Bigg).$$

Since $\xi_0 y$, $\xi_j y$ for $j \in C^e$ and $\xi_j y_j$ for $j \in C^a$ belong to $W^{2,2}(\Omega)$ we can use standard interpolation error estimates, see e.g. [9], for the first term and for the terms in the first and third sum. The function $\xi_j y_s$ belongs to $V^{2,2}_{\beta_j}(K_j)$ for $j \in C^a$. Thus we can use interpolation error estimates in weighted Sobolev spaces with homogeneous norm in angles to estimate the terms in the second sum, see e.g. [28, 29, 8]. Collecting everything we arrive at

$$\begin{aligned}
&\|y - y_h\|_{W^{1,2}(\Omega)} \\
&\quad\le ch \Bigg(|\xi_0 y|_{W^{2,2}(\Omega)} + \sum_{j\in C^e} |\xi_j y|_{W^{2,2}(\Omega)} + \sum_{j\in C^a} \Big(\|\xi_j y_s\|_{V^{2,2}_{\beta_j}(K_j)} + |\xi_j y_j|_{W^{2,2}(\Omega)} \Big) \Bigg) \\
&\quad\le ch \Bigg(|\xi_0 y|_{W^{2,2}(\Omega)} + \sum_{j\in C^e} |\xi_j y|_{W^{2,2}(\Omega)} + \sum_{j\in C^a} \Big(\|\xi_j y_s\|_{V^{2,2}_{\beta_j}(K_j)} + c_j |y_j| \Big) \Bigg),
\end{aligned}$$

where we used $y_j \in \Pi_0$, $\mu < \lambda$ and that $c_j := |\xi_j|_{W^{2,2}(\Omega)}$ is constant independent of the mesh parameter h. With Lemma 2.1 and Theorem 2.5 we get

$$\|y - y_h\|_{W^{1,2}(\Omega)} \le ch\|y\|_{W^{2,2}_\beta(\Omega)} \le ch.$$

Finally we can use the Nitsche method to double the order of convergence in the L^2-norm, see e.g. [8]. □

Lemma 4.2. *The norms of the discrete solution operators S_h and S_h^* are bounded in several norms by a constant c always independent of h. We have*

$$\begin{aligned}
\|S_h u\|_{L^2(\Omega)} &\le c\|u\|_{L^2(\Gamma)}, \\
\|S_h^* z\|_{H^{1/2}(\Gamma)} &\le c\|z\|_{L^2(\Omega)}, \\
\|S_h^* z\|_{L^\infty(\Gamma)} &\le c\|z\|_{L^2(\Omega)}.
\end{aligned}$$

Proof. Due to the variational formulation of the elliptic boundary value problem we have

$$\|S_h u\|^2_{H^1(\Omega)} = (u, S_h u)_{L^2(\Gamma)} \le \|u\|_{L^2(\Gamma)} \|S_h u\|_{L^2(\Gamma)} \le c\|u\|_{L^2(\Gamma)} \|S_h u\|_{H^1(\Omega)},$$

where we used the trace theorem [9, Theorem 1.6.6] in the last step. With the inclusion $H^1(\Omega) \subset L^2(\Omega)$ we get the first inequality. Analogously we get

$$\|P_h z\|_{H^1(\Omega)} \le c\|z\|_{L^2(\Omega)}. \tag{4.4}$$

To prove the second inequality we apply the trace theorem [33, Theorem 8.7]. We have that

$$\|S_h^* z\|_{H^{1/2}(\Gamma)} \le c\|P_h z\|_{H^1(\Omega)} \le c\|z\|_{L^2(\Omega)}, \tag{4.5}$$

where we inserted (4.4). For the proof of the third inequality we begin with introducing intermediate functions

$$\|S_h^* z\|_{L^\infty(\Gamma)} \le \|S_h^* z - I_h S^* z\|_{L^\infty(\Gamma)} + \|I_h S^* z\|_{L^\infty(\Gamma)}.$$

For the first term we get with Lemma 3 of [2]

$$\|S_h^* z - I_h S^* z\|_{L^\infty(\Gamma)} \le \|S_h^* z - I_h S^* z\|_{L^\infty(\Omega)} \le c\|z\|_{L^2(\Omega)}.$$

The second term can be estimated by

$$\|I_h S^* z\|_{L^\infty(\Gamma)} \le \|S^* z\|_{L^\infty(\Omega)} \le c\|S^* z\|_{H^{1+\epsilon}(\Omega)} \le c\|S^* z\|_{W^{2,2}_{1-\epsilon}(\Omega)} \le c\|z\|_{L^2(\Omega)},$$

where we used the boundedness of I_h, usual embedding theorems, the trace theorem [33, Theorem 8.7], Theorem 1.3 in [32] and Lemma 2.4 with $0 < \epsilon < \lambda$. □

Lemma 4.3. *Let $G \subset \bar{T}$ be an arbitrary element in $\mathcal{G}_h$ and $\hat{G}$ the reference element. Denote $F : \hat{x} \in \hat{G} \to F(\hat{x}) = B\hat{x} + b \in G$ the affine linear change of variables such that $F\hat{G} = G$ and denote $\hat{v} = v \circ F$ with $v \in H^s(G)$, $s \in [0, 2]$. Then there exists a constant c such that*

$$|\hat{v}|_{H^s(\hat{G})} \le c|G|^{-1/2} h_T^s |v|_{H^s(G)}. \tag{4.6}$$

Proof. One can deduce the inequality for integer s from Theorem 15.1 in [14]. For real-valued s we refer to Example 3 in [16]. Notice, that $\|B\| \le ch_T$ and $|\det(B)| \sim h_T$. □

Next we define a projection operator from the space of the continuous functions into the space of piecewise constant functions on the boundary. Furthermore we state some properties of this operator which will be used in the error analysis in the sequel.

Definition 4.4. *Let $\mathcal{G}_h$ be the triangulation of the boundary of Ω which is induced by the triangulation $\mathcal{T}_h$. The projection operator R_h is the 0-interpolator onto U_h. We define*

$$R_h : C(\Gamma) \to U_h \quad f \to R_h f$$

with

$$(R_h f)(x) := f(S_G) \quad \text{if } x \in G,$$

where S_G is the midpoint of the edge G.

Lemma 4.5. *Let $\mathcal{E}_h \subset \mathcal{G}_h$ be a triangulation of Γ_j with grading parameters μ_j and μ_{j+1}. Then for any element $G \in \mathcal{E}_h$ the following estimates hold true*

$$\left|\int_G (f - R_h f) ds_x\right| \leq \begin{cases} ch^{3/2}|G|^{1/2}\|f\|_{V^{3/2,2}_{3(1-\mu_j)/2}(G)} & \text{for } f \equiv \xi_j g,\ g \in V^{3/2,2}_{3(1-\mu_j)/2}(\Gamma_j^2) \\ ch^{3/2}|G|^{1/2}\|f\|_{V^{3/2,2}_{3(1-\mu_{j+1})/2}(G)} & \text{for } f \equiv \xi_{j+1} g,\ g \in V^{3/2,2}_{3(1-\mu_{j+1})/2}(\Gamma_{j+1}^1) \\ ch^{3/2}|G|^{1/2}|f|_{W^{3/2,2}(G)} & \text{for } f \in W^{3/2,2}(\Gamma_j) \\ ch|G|^{1/2}|f|_{H^1(G)} & \text{for } f \in H^1(\Gamma_j). \end{cases}$$

Proof. We prove the first inequality and deduce the second one analogously. The basic idea is that the formula is a exact numerical scheme for polynomials of order one. We distinguish between edges G with $r_{G,j} > 0$ and $r_{G,j} = 0$. For $r_{G,j} > 0$ we get

$$\begin{aligned} \left|\int_G (f - R_h f) ds_x\right| &\leq c|G| \left|\int_{\hat{G}} (\hat{f} - \hat{R}_h \hat{f}) ds_{\hat{x}}\right| \\ &= c|G| \left|\int_{\hat{G}} (\hat{f} - \hat{p} - \hat{R}_h(\hat{f} - \hat{p})) ds_{\hat{x}}\right| \\ &\leq c|G| \left(\left|\int_{\hat{G}} (\hat{f} - \hat{p}) ds_{\hat{x}}\right| + \left|\int_{\hat{G}} (\hat{R}_h(\hat{f} - \hat{p})) ds_{\hat{x}}\right|\right) \\ &\leq c|G| \left(\|\hat{f} - \hat{p}\|_{L^1(\hat{G})} + \|\hat{f} - \hat{p}\|_{L^\infty(\hat{G})}\right) \\ &\leq c|G| \|\hat{f} - \hat{p}\|_{H^1(\hat{G})}, \end{aligned}$$

where $\hat{p}$ is a arbitrary polynomial of order one. Next we use Theorem 6.1 in [16] to arrive at

$$\|\hat{f} - \hat{p}\|_{H^1(\hat{G})} \leq c\|\hat{f} - \hat{p}\|_{H^{3/2}(\hat{G})} \leq c|\hat{f}|_{H^{3/2}(\hat{G})}. \tag{4.7}$$

Using (4.6) we can continue with

$$\left|\int_G (f - R_h f) ds_x\right| \leq c|G|^{1/2} h_T^{3/2} |f|_{H^{3/2}(G)}. \tag{4.8}$$

Inserting the definition of h_T for $r_{T,j} > 0$ we get with $\alpha := 3(1 - \mu_j)/2$

$$\begin{aligned} \left|\int_G (f - R_h f) ds_x\right| &\leq c|G|^{1/2} h^{3/2} r_{T,j}^\alpha |f|_{H^{3/2}(G)} \\ &\leq c|G|^{1/2} h^{3/2} \left(\int_G \int_G \frac{|r_j(x)^\alpha D_x f(x) - r_j(y)^\alpha D_y f(y)|^2}{|x-y|^2} ds_x ds_y \right. \\ &\quad \left. + \int_G \int_G \frac{|r_j(x)^\alpha - r_j(y)^\alpha|^2}{|x-y|^2} ds_x |D_y f(y)|^2 ds_y\right)^{1/2}. \end{aligned}$$

A Taylor series expansion of $r_j(y)^\alpha$ yields with $\theta \in (0,1)$

$$\left|\int_G (f - R_h f) ds_x\right| \le c|G|^{1/2}h^{3/2}\Bigg(\int_G\int_G \frac{|r_j(x)^\alpha D_x f(x) - r_j(y)^\alpha D_y f(y)|^2}{|x-y|^2} ds_x ds_y$$
$$+ \int_G\int_G r_j((1-\theta)x+\theta y)^{2(\alpha-1)} ds_x |D_y f(y)|^2 ds_y\Bigg)^{1/2}.$$

For every G with $r_{G,j} > 0$ there holds $\min_{x\in G} r_j(x) \sim \max_{x\in G} r_j(x)$. Thus we get

$$\left|\int_G (f - R_h f) ds_x\right| \le c|G|^{1/2}h^{3/2}\Bigg(\int_G\int_G \frac{|r_j(x)^\alpha D_x f(x) - r_j(y)^\alpha D_y f(y)|^2}{|x-y|^2} ds_x ds_y$$
$$+ \int_G r_j(y)^{2(\alpha-1)}|G||D_y f(y)|^2 ds_y\Bigg)^{1/2}$$
$$\le c|G|^{1/2}h^{3/2}\Bigg(\int_G\int_G \frac{|r_j(x)^\alpha D_x f(x) - r_j(y)^\alpha D_y f(y)|^2}{|x-y|^2} ds_x ds_y$$
$$+ \int_G r_j(y)^{2(\alpha-3/2+1)}|D_y f(y)|^2 ds_y\Bigg)^{1/2}$$
$$= c|G|^{1/2}h^{3/2}\|f\|_{V^{3/2,2}_{3(1-\mu_j)/2}(G)},$$

where we used $|G| \sim h_T$ and $h_T \le c r_{T,j}$ which is satisfied for $r_{T,j} > 0$. For the case that $r_{G,j} = 0$ we can proceed by using that R_h is a bounded operator from $L^\infty(G)$ to $L^\infty(G)$ with norm 1 and $V^{3/2,2}_{3(1-\mu_j)/2}(G) \hookrightarrow W^{3\mu_j/2,2}(G) \hookrightarrow L^\infty(G)$ for all μ_j in the possible range $(1/3, 1)$. This leads to the estimate

$$\left|\int_G (f - R_h f) ds_x\right| \le c|G|\|f\|_{L^\infty(G)} \le c|G|\|\hat f\|_{L^\infty(\hat G)} \le c|G|\|\hat f\|_{V^{3/2,2}_{3(1-\mu_j)/2}(\hat G)}$$
$$\le c|G|^{1/2}h_T^{3/2-3(1-\mu_j)/2}\|f\|_{V^{3/2,2}_{3(1-\mu_j)/2}(G)}.$$

This can analogously be proven to (4.6) together with $\hat r_j^\gamma \sim h_T^{-\gamma} r_j^\gamma$ for $r_{T,j} = 0$. The first inequality is proven by observing that $h_T \le c h^{1/\mu_j}$ for $r_{T,j} = 0$. One can directly deduce the third inequality from (4.8), since f belongs to $W^{3/2,2}(\Gamma_j)$ and $h_T \le ch$. The inequality for functions in $H^1(\Gamma_j)$ is similar, only $\|\hat f - \hat p\|_{H^1(\hat G)} \le c|\hat f|_{H^1(\hat G)}$ is used instead of (4.7). □

In the following we introduce the L^2-projection in the space of piecewise constant functions on the boundary and state some properties of this projection operator.

Definition 4.6. *Let $\mathcal{G}_h$ be the triangulation of the boundary. The L^2-projection of a function $f \in L^2(\Gamma)$ is the piecewise constant function that fulfills*

$$Q_h f \equiv \frac{1}{|G|}\int_G f(x) ds_x$$

on any element $G \in \mathcal{G}_h$.

Lemma 4.7. *For any element $G \in \mathcal{G}_h$ and any function $f \in H^s(G)$, $s \in [0,1]$, the estimate*

$$\|f - Q_h f\|_{L^2(G)} \le c h_T^s |f|_{H^s(G)}$$

is valid.

Proof. First we observe that $Q_h p = p$ for any $p \in \mathcal{P}_0(G)$. Thus we can write for $f \in H^s(G)$

$$\begin{aligned}\|f - Q_h f\|_{L^2(G)} &= \|f - p - Q_h(f-p)\|_{L^2(G)} \le c\|f-p\|_{L^2(G)} \\ &\le c|G|^{1/2}\|\hat f - \hat p\|_{L^2(\hat G)} \le c|G|^{1/2}\|\hat f - \hat p\|_{H^s(\hat G)} \le c|G|^{1/2}|\hat f|_{H^s(\hat G)} \\ &\le c h_T^s |f|_{H^s(G)}\end{aligned}$$

where we used Theorems 3.2 and 6.1 of [16] and (4.6). □

Corollary 4.8. *For any element $G \in \mathcal{G}_h$ and any functions $f \in H^s(G)$, $s = [0,1]$, and $v \in H^1(G)$ the estimate*

$$(f - Q_h f, v)_{L^2(G)} \le c h_T^{s+1} |f|_{H^s(G)} |v|_{H^1(G)}$$

is valid.

Proof. Due to the definition of Q_h we have the orthogonality $(f - Q_h f, p)_{L^2(G)} = 0$ for all $p \in \mathcal{P}_0(G)$. Thus we get

$$\begin{aligned}(f - Q_h f, v)_{L^2(G)} &= (f - Q_h f, v - Q_h v)_{L^2(G)} \le \|f - Q_h f\|_{L^2(G)} \|v - Q_h v\|_{L^2(G)} \\ &\le c h_T^{s+1} |f|_{H^s(G)} |v|_{H^1(G)},\end{aligned}$$

where we used the Cauchy-Schwarz inequality and Lemma 4.7. □

Lemma 4.9. *With mesh parameters $\mu < \frac{1}{3} + \frac{2}{3}\lambda$ and under Assumption 3.2 the estimate*

$$\|S_h \bar u - S_h R_h \bar u\|_{L^2(\Omega)} \le c h^{3/2} \tag{4.9}$$

holds.

Proof. Let $z_h = S_h \bar u - S_h R_h \bar u$ and $v_h = S_h^* z_h$. Due to the relation between S_h and S_h^* and by inserting $Q_h \bar u$ we can write

$$\begin{aligned}\|S_h \bar u - S_h R_h \bar u\|_{L^2(\Omega)}^2 &= (z_h, z_h)_{L^2(\Omega)}^2 = (\bar u - R_h \bar u, S_h^* z_h)_{L^2(\Gamma)} = (\bar u - R_h \bar u, v_h)_{L^2(\Gamma)} \\ &= (\bar u - Q_h \bar u, v_h)_{L^2(\Gamma)} + (Q_h \bar u - R_h \bar u, v_h)_{L^2(\Gamma)}.\end{aligned} \tag{4.10}$$

We estimate both terms in (4.10) separately. For the first term we get

$$\begin{aligned}(\bar u - Q_h \bar u, v_h)_{L^2(\Gamma)} &= \sum_{G \in \mathcal{G}_h} (\bar u - Q_h \bar u, v_h)_{L^2(G)} \le c \sum_{G \in \mathcal{G}_h} h_T^{3/2} |\bar u|_{H^1(G)} |v_h|_{H^{1/2}(G)} \\ &\le c h^{3/2} \sum_{j=1}^m |\bar u|_{H^1(\Gamma_j)} |v_h|_{H^{1/2}(\Gamma_j)} \le c h^{3/2} \|z_h\|_{L^2(\Omega)},\end{aligned}$$

where we used Corollary 4.8, the discrete Cauchy-Schwarz inequality, $h_T \le ch$, Theorem 2.5 and Lemma 4.2. For the second term in (4.10) we split the integral into two integrals over the sets K_1 and K_2

$$(Q_h\bar{u} - R_h\bar{u}, v_h)_{L^2(\Gamma)} = (Q_h\bar{u} - R_h\bar{u}, v_h)_{L^2(K_1)} + (Q_h\bar{u} - R_h\bar{u}, v_h)_{L^2(K_2)}. \tag{4.11}$$

For the second term in (4.11) we begin with applying the Cauchy-Schwarz inequality and the estimate $\|v_h\|_{L^2(K_2)} \le \|v_h\|_{L^2(\Gamma)} = \|S_h^* z_h\|_{L^2(\Gamma)} \le c\|z_h\|_{L^2(\Omega)}$ to get

$$(Q_h\bar{u} - R_h\bar{u}, v_h)_{L^2(K_2)} \le \|Q_h\bar{u} - R_h\bar{u}\|_{L^2(K_2)}\|v_h\|_{L^2(K_2)} \le c\|Q_h\bar{u} - R_h\bar{u}\|_{L^2(K_2)}\|z_h\|_{L^2(\Omega)}.$$

For $\|Q_h\bar{u} - R_h\bar{u}\|_{L^2(K_2)}$ we write

$$\|Q_h\bar{u} - R_h\bar{u}\|^2_{L^2(K_2)} = \int_{K_2} (Q_h\bar{u} - R_h\bar{u})^2 ds_x \le \sum_{j=1}^m \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j \\ G\subset K_2}} \int_G (Q_h\bar{u} - R_h\bar{u})^2 ds_x.$$

Using the definition of Q_h and that $R_h\bar{u}$ is constant on each edge G we can continue

$$\begin{aligned} I_j &:= \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j \\ G\subset K_2}} \int_G (Q_h\bar{u} - R_h\bar{u})^2 ds_x = \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j \\ G\subset K_2}} \int_G \left(\frac{1}{|G|}\int_G \bar{u} ds_\xi - R_h\bar{u}\right)^2 ds_x \\ &= \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j \\ G\subset K_2}} \int_G \left(\frac{1}{|G|}\int_G (\bar{u} - R_h\bar{u})\, ds_\xi\right)^2 ds_x = \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j \\ G\subset K_2}} \frac{1}{|G|}\left(\int_G (\bar{u} - R_h\bar{u})\, ds_\xi\right)^2. \end{aligned} \tag{4.12}$$

Now we consider the case that both corners $x^{(j)}$ and $x^{(j+1)}$ are concave. The case of only one concave corner or no concave corner can analogously be deduced by simple modifications. In particular, for a convex corner $x^{(j)}$ we don't need the following splitting, since $\xi_j\bar{u}$ belongs to $H^{3/2}(\Gamma_j)$. For $\mu_j, \mu_{j+1} \in (1/3, 1)$ we have $3/2(1-\mu_j), 3/2(1-\mu_{j+1}) \in (0, 1)$. Thus we get

$$\bar{u} = \xi_0\bar{u} + \xi_j\bar{u} + \xi_{j+1}\bar{u} = \xi_0\bar{u} + \xi_j\bar{u}_s + \xi_{j+1}\bar{u}_s + \xi_j\bar{u}_j + \xi_{j+1}\bar{u}_{j+1} \tag{4.13}$$

on Γ_j according to Lemma 2.2 with $\xi_0\bar{u} \in H^{3/2}(\Gamma_j)$, $\xi_j\bar{u}_s \in V^{3/2,2}_{3/2(1-\mu_j)}(\Gamma_j^2)$, $\xi_{j+1}\bar{u}_s \in V^{3/2,2}_{3/2(1-\mu_{j+1})}(\Gamma^1_{j+1})$ and $\bar{u}_j, \bar{u}_{j+1} \in \Pi_0$. Inserting (4.13) in (4.12) yields

$$\begin{aligned} I_j = \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j \\ G\subset K_2}} \frac{1}{|G|} \Bigg(&\int_G (\xi_0\bar{u} - R_h\xi_0\bar{u})\, ds_\xi + \sum_{i=j}^{j+1}\int_G (\xi_i\bar{u}_s - R_h\xi_i\bar{u}_s)\, ds_\xi \\ &+ \sum_{i=j}^{j+1}\int_G (\xi_i\bar{u}_i - R_h\xi_i\bar{u}_i)\, ds_\xi \Bigg)^2. \end{aligned}$$

With Lemma 4.5 we arrive at

$$I_j \le c \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j\\ G\subset K_2}} h^3 \left(\|\xi_0 \bar u\|_{W^{3/2,2}(G)} + \sum_{i=j}^{j+1} \|\xi_i \bar u_s\|_{V^{3/2,2}_{3/2(1-\mu_i)}(G)} + \sum_{i=j}^{j+1} \bar u_i \|\xi_i\|_{W^{3/2,2}(G)} \right)^2$$
$$\le ch^3 \left(\|\xi_0 \bar u\|_{W^{3/2,2}(K_2\cap\Gamma_j)} + \sum_{i=j}^{j+1} \|\xi_i \bar u_s\|_{V^{3/2,2}_{3/2(1-\mu_i)}(K_2\cap\Gamma_j)} + \sum_{i=j}^{j+1} \bar u_i \right)^2,$$

since $\|\xi_i\|_{W^{3/2,2}(K_2\cap\Gamma_j)} \le c$ independent of the mesh parameter h. Using the norm equivalence of Lemma 2.2 we get

$$\sum_{j=1}^{m} \sum_{\substack{G\in\mathcal{G}_h\cap\Gamma_j\\ G\subset K_2}} \frac{1}{|G|} \left(\int_G (\bar u - R_h \bar u)\, ds_\xi \right)^2 \le ch^3 \|\bar u\|^2_{W^{3/2,2}_\beta(K_2)} \le ch^3, \tag{4.14}$$

where we used $\mu_j < \frac{1}{3} + \frac{2}{3}\lambda_j$, $j = 1, \ldots, m$, together with Theorem 2.5 in the last steps. For the estimation of the first term in (4.11) we begin with

$$\begin{aligned}
\left|(Q_h\bar u - R_h\bar u, v_h)_{L^2(K_1)}\right| &\le \|v_h\|_{L^\infty(\Gamma)} \sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} \int_G |Q_h\bar u - R_h\bar u| ds_x \\
&= \|v_h\|_{L^\infty(\Gamma)} \sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} \int_G \left| \frac{1}{|G|}\int_G \bar u ds_\xi - \frac{1}{|G|}\int_G R_h \bar u ds_\xi \right| ds_x \\
&= \|v_h\|_{L^\infty(\Gamma)} \sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} \left| \int_G \bar u - R_h\bar u ds_\xi \right|.
\end{aligned}$$

using the definition of Q_h. We can continue by means of Lemma 4.5 and (3.1)

$$\begin{aligned}
\sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} \left| \int_G \bar u - R_h\bar u ds_\xi \right| &\le c \sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} h|G|^{1/2} |\bar u|_{H^1(G)} \\
&\le ch \left(\sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} |G| \right)^{1/2} \left(\sum_{\substack{G\in\mathcal{G}_h\\ G\subset K_1}} |\bar u|^2_{H^1(G)} \right)^{1/2} \\
&\le c \sum_{j=1}^{m} h^{3/2} |\bar u|_{H^1(\Gamma_j)} \le ch^{3/2},
\end{aligned}$$

where we used the discrete Cauchy-Schwarz inequality, Assumption 3.2 and Theorem 2.5. We finally get with Lemma 4.2

$$(Q_h\bar u - R_h\bar u, v_h)_{L^2(K_1)} \le ch^{3/2} \|z_h\|_{L^2(\Omega)}.$$

This finishes the proof. □

Lemma 4.10. *With mesh parameters $\mu < \lambda$ and under Assumption 3.2 the estimate*

$$\|\bar{p} - S_h^*(S_h R_h \bar{u} - y_d)\|_{L^2(\Gamma)} \leq ch^{3/2} \tag{4.15}$$

is valid.

Proof. We begin by introducing intermediate functions and use the triangle inequality to obtain

$$\begin{aligned}\|\bar{p} - S_h^*(S_h R_h \bar{u} - y_d)\|_{L^2(\Gamma)} \leq \|\bar{p} - S_h^*(\bar{y} - y_d)\|_{L^2(\Gamma)} \\ + \|S_h^*(\bar{y} - y_d) - S_h^*(S_h \bar{u} - y_d)\|_{L^2(\Gamma)} \\ + \|S_h^*(S_h \bar{u} - y_d) - S_h^*(S_h R_h \bar{u} - y_d)\|_{L^2(\Gamma)}.\end{aligned}$$

For the first term we use the trace theorem (Theorem 1.6.6 in [9]) and Lemma 4.1 to get

$$\begin{aligned}\|\bar{p} - S_h^*(\bar{y} - y_d)\|_{L^2(\Gamma)} &\leq c\|\bar{p} - P_h(\bar{y} - y_d)\|_{L^2(\Omega)}^{1/2} \|\bar{p} - P_h(\bar{y} - y_d)\|_{H^1(\Omega)}^{1/2} \\ &\leq (h^2 \|\bar{p}\|_{W_\beta^{2,2}(\Omega)})^{1/2} (h\|\bar{p}\|_{W_\beta^{2,2}(\Omega)})^{1/2} \leq ch^{3/2}.\end{aligned}$$

For the second term we use Lemma 4.2 and again the results of Lemma 4.1 to arrive at

$$\begin{aligned}\|S_h^*(\bar{y} - y_d) - S_h^*(S_h \bar{u} - y_d)\|_{L^2(\Gamma)} &\leq c\|\bar{y} - y_d - (S_h \bar{u} - y_d)\|_{L^2(\Omega)} = c\|\bar{y} - S_h \bar{u}\|_{L^2(\Omega)} \\ &\leq ch^2 \|\bar{y}\|_{W_\beta^{2,2}(\Omega)} \leq ch^2.\end{aligned}$$

For the third term we use Lemma 4.2 and Lemma 4.9 to obtain

$$\begin{aligned}\|S_h^*(S_h \bar{u} - y_d) - S_h^*(S_h R_h \bar{u} - y_d)\|_{L^2(\Gamma)} &= \|S_h^*(S_h \bar{u} - S_h R_h \bar{u})\|_{L^2(\Gamma)} \\ &\leq c\|S_h \bar{u} - S_h R_h \bar{u}\|_{L^2(\Omega)} \leq ch^{3/2}.\end{aligned}$$

□

5 Supercloseness result and proof of Theorem 3.3

In this section the supercloseness result $\|\bar{u}_h - R_h \bar{u}\|_{L^2(\Gamma)} \leq ch^{3/2}$ and Theorem 3.3 will be proven. Before doing this we recall the following lemma which is nested in Proposition 4.5 of [25]. The proof does not require quasi uniform meshes.

Lemma 5.1. *The inequality*

$$\nu\|R_h \bar{u} - \bar{u}_h\|_{L^2(\Gamma)}^2 \leq (R_h \bar{p} - \bar{p}_h, \bar{u}_h - R_h \bar{u})_{L^2(\Gamma)} \tag{5.1}$$

is valid.

Lemma 5.2 (Supercloseness)**.** *Assume that Assumption 3.2 is fulfilled. Let $\bar{u}_h$ be the solution of the discrete optimal control problem* (3.2) *on a family of meshes with grading parameters $\mu < \lambda$. Then the estimate*

$$\|\bar{u}_h - R_h \bar{u}\|_{L^2(\Gamma)} \leq ch^{3/2} \tag{5.2}$$

holds true.

Proof. We rewrite inequality (5.1):

$$\begin{aligned}\nu\|R_h\bar u-\bar u_h\|^2_{L^2(\Gamma)} &\le (R_h\bar p-\bar p_h,\bar u_h-R_h\bar u)_{L^2(\Gamma)}\\ &= (R_h\bar p-\bar p,\bar u_h-R_h\bar u)_{L^2(\Gamma)}\\ &\quad+(\bar p-S_h^*(S_hR_h\bar u-y_d),\bar u_h-R_h\bar u)_{L^2(\Gamma)}\\ &\quad+(S_h^*(S_hR_h\bar u-y_d)-\bar p_h,\bar u_h-R_h\bar u)_{L^2(\Gamma)}.\end{aligned}$$

For the first term we can use that it represents a formula for the numerical integration, since $\bar u_h-R_h\bar u$ is piecewise constant. We obtain

$$\begin{aligned}(R_h\bar p-\bar p,\bar u_h-R_h\bar u)_{L^2(\Gamma)} &= \sum_{G\in\mathcal{G}_h}\int_G(R_h\bar p(x)-\bar p(x))(\bar u_h(x)-R_h\bar u(x))ds_x\\ &=\sum_{G\in\mathcal{G}_h}(\bar u_h(S_G)-R_h\bar u(S_G))|G|^{1/2}|G|^{-1/2}\int_G(R_h\bar p-\bar p)ds_x\\ &=\sum_{G\in\mathcal{G}_h}\|\bar u_h-R_h\bar u\|_{L^2(G)}|G|^{-1/2}\int_G(R_h\bar p-\bar p)ds_x\\ &\le c\|\bar u_h-R_h\bar u\|_{L^2(\Gamma)}\left(\sum_{G\in\mathcal{G}_h}\frac{1}{|G|}\left(\int_G(R_h\bar p-\bar p)ds_x\right)^2\right)^{1/2},\end{aligned}$$

where we used in the last step the discrete Cauchy-Schwarz inequality. Using Lemma 4.5 as in the proof of Lemma 4.9 between (4.12) and (4.14) with $\frac{1}{3}+\frac{2}{3}\lambda_j>\lambda_j>\mu_j$, $j=1,\dots,m$, we arrive at

$$(R_h\bar p-\bar p,\bar u_h-R_h\bar u)_{L^2(\Gamma)}\le ch^{3/2}\|\bar u_h-R_h\bar u\|_{L^2(\Gamma)}\|\bar p\|_{W^{3/2,2}_\beta(\Gamma)}.$$

For the second term we use the Cauchy-Schwarz inequality and Lemma 4.10 to get

$$\begin{aligned}(\bar p-S_h^*(S_hR_h\bar u-y_d),\bar u_h-R_h\bar u)_{L^2(\Gamma)} &\le \|\bar p-S_h^*(S_hR_h\bar u-y_d)\|_{L^2(\Gamma)}\|\bar u_h-R_h\bar u\|_{L^2(\Gamma)}\\ &\le ch^{3/2}\|\bar u_h-R_h\bar u\|_{L^2(\Gamma)}.\end{aligned}$$

The third term can be omitted, since it is smaller than zero,

$$\begin{aligned}&(S_h^*(S_hR_h\bar u-y_d)-\bar p_h,\bar u_h-R_h\bar u)_{L^2(\Gamma)}\\ &\qquad=(S_h^*(S_hR_h\bar u-y_d)-S_h^*(S_h\bar u_h-y_d),\bar u_h-R_h\bar u)_{L^2(\Gamma)}\\ &\qquad=(S_h^*(S_h(R_h\bar u-\bar u_h)),\bar u_h-R_h\bar u)_{L^2(\Gamma)}\\ &\qquad=(S_h(R_h\bar u-\bar u_h),S_h(\bar u_h-R_h\bar u))_{L^2(\Omega)}\\ &\qquad=-\|S_h(R_h\bar u-\bar u_h)\|_{L^2(\Omega)}\le 0,\end{aligned}$$

where we used the definition of $\bar p_h$ and S_h^*. □

Now we can state the missing proof of Theorem 3.3.

Proof of Theorem 3.3. We introduce intermediate functions and apply the triangle inequality, use $\bar{y} = S_h\bar{u}_h$ and obtain

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)} + \|S_hR_h\bar{u} - S_h\bar{u}_h\|_{L^2(\Omega)} \\ &\leq \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)} + c\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}.\end{aligned}$$

The last inequality holds due to Lemma 4.2. The three terms were estimated separately in Lemma 4.1, Lemma 4.9 and in Lemma 5.2. Thus we can conclude

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^{3/2}.$$

For the error in the adjoint state we get

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq \|\bar{p} - S_h^*(S_hR_h\bar{u} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S_hR_h\bar{u} - y_d) - \bar{p}_h\|_{L^2(\Gamma)}.$$

The first term was estimated in Lemma 4.10. For the second holds

$$\begin{aligned}\|S_h^*(S_hR_h\bar{u} - y_d) - \bar{p}_h\|_{L^2(\Gamma)} &= \|S_h^*(S_hR_h\bar{u} - y_d) - S_h^*(S_h\bar{u}_h - y_d)\|_{L^2(\Gamma)} \\ &= \|S_h^*S_h(R_h\bar{u} - \bar{u}_h)\|_{L^2(\Gamma)} \leq c\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \\ &\leq ch^{3/2},\end{aligned}$$

where we used Lemma 4.2 and Lemma 5.2 in the last two steps. □

6 Variational approach

In this section we consider the variational discretization concept first presented in [20] for distributed control problems. We consider as discretized optimal control problem:

$$\min_{u \in U_{ad}} J_h(u) = F(S_hu, u). \tag{6.1}$$

Problem (6.1) admits a unique solution u_h^*. In that what follows we denote by $y_h^* = S_hu_h^*$ the discrete state and by $p_h^* = p_h(u_h^*)$ the discrete adjoint state. The first order necessary and sufficient condition for the optimality of u_h^* to problem (6.1) can be written as

$$(p_h^* + \nu u_h^*, u - u_h^*)_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad}. \tag{6.2}$$

Lemma 6.1. *Let u_h^* be the solution of problem* (6.1) *on a family of triangulations $\mathcal{T}_h$ of Ω. Then the estimate*

$$\nu\|\bar{u} - u_h^*\|_{L^2(\Gamma)} \leq \|S^*(S\bar{u} - y_d) - S_h^*(S\bar{u} - y_d)\|_{L^2(\Gamma)} + c\|S\bar{u} - S_h\bar{u}\|_{L^2(\Omega)} \tag{6.3}$$

is valid.

Proof. The proof is given in Section 7 of [25]. It also holds for graded meshes. □

Theorem 6.2. *Let u_h^* be the solution of problem* (6.1) *on a family of meshes with mesh grading parameters $\mu < \lambda$ according to* (3.1)*. Then the estimate*

$$\nu\|\bar{u} - u_h^*\|_{L^2(\Gamma)} \leq ch^{3/2}$$

holds.

Proof. We get from (6.3)

$$\begin{aligned}
\nu\|\bar{u} - u_h^*\|_{L^2(\Gamma)} &\leq \|S^*(S\bar{u} - y_d) - S_h^*(S\bar{u} - y_d)\|_{L^2(\Gamma)} + c\|S\bar{u} - S_h\bar{u}\|_{L^2(\Omega)} \\
&\leq c\left(\|\bar{p} - P_h(S\bar{u} - y_d)\|_{L^2(\Omega)}^{1/2}\|\bar{p} - P_h(S\bar{u} - y_d)\|_{H^1(\Omega)}^{1/2} + h^2\|\bar{y}\|_{W_\beta^{2,2}(\Omega)}\right) \\
&\leq c\left((h^2\|\bar{p}\|_{W_\beta^{2,2}(\Omega)})^{1/2}(h\|\bar{p}\|_{W_\beta^{2,2}(\Omega)})^{1/2} + h^2\|\bar{y}\|_{W_\beta^{2,2}(\Omega)}\right) \\
&\leq ch^{3/2},
\end{aligned}$$

where we used the trace theorem [9, Theorem 1.6.6], the definition of S_h and S_h^*, Lemma 4.1 and Theorem 2.5 in the last steps. □

7 Numerical example and concluding remarks

In this section we present a numerical example for the fully discrete approach to compare the proven order of convergence with the experimental one. The example is taken from [25]. Let r, ϕ be the polar coordinates. For $\omega \in [0, 2\pi)$ define the circular sector $S_\omega := \{x \in \mathbb{R}^2 : (r(x), \phi(x)) \in (0, \sqrt{2}] \times [0, \omega]\}$ and the domain $\Omega_\omega := (-1, 1)^2 \cap S_\omega$ with its boundary Γ_ω. Consider the optimal control problem in the form

$$\begin{aligned}
-\Delta y + y &= 0 && \text{in } \Omega_\omega, \\
\partial_n y &= u + g_2 && \text{a.e. on } \Gamma_\omega, \\
-\Delta p + p &= y - y_d && \text{in } \Omega_\omega, \\
\partial_n p &= g_1 && \text{a.e. on } \Gamma_\omega, \\
u &= \Pi_{[a,b]}\left(-p_{|\Gamma_\omega}\right) && \text{on } \Gamma_\omega.
\end{aligned}$$

The data y_d, g_1 and g_2 are chosen in the following way

$$\begin{aligned}
y_d &= -r^\lambda \cos(\lambda\phi) && \text{in } \Omega_\omega, \\
g_1 &= -\partial_n y_d && \text{on } \Gamma_\omega, \\
g_2 &= -\Pi_{[a,b]}(y_d) && \text{on } \Gamma_\omega,
\end{aligned}$$

with $\lambda = \pi/\omega$, $a = -0.5$ and $b = 0.5$. The unique solution of this problem is given by

$$\begin{aligned}
\bar{y} &= 0 && \text{in } \Omega_\omega, \\
\bar{p} &= -y_d && \text{in } \Omega_\omega, \\
\bar{u} &= -g_2 && \text{on } \Gamma_\omega.
\end{aligned}$$

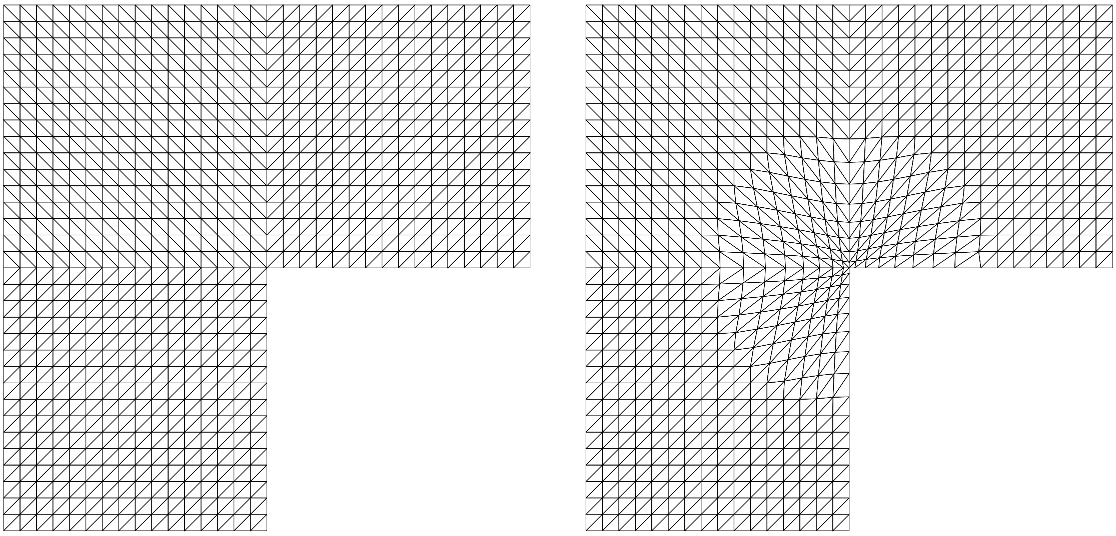

Figure 1: $\Omega_{3/2\pi}$ with ungraded and graded mesh ($\mu = 0.6666$)

The function $\bar{y}$ vanishes, but $\bar{p}$ and $\bar{u}$ have exactly the singular behaviour discussed above. To solve the optimal control problem we have used an active set strategy. The discrete solutions of the PDEs have been computed by using a finite element method with graded meshes as discussed in Section 3, see Figure 1.

In Table 1 one can find the computed errors $\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$ and the experimental order of convergence (eoc) for $\omega = 3/2\pi$ and different mesh grading parameters when the mesh parameter h becomes small. On uniform meshes ($\mu = 1$) the order of convergence for this angle is $1/2 + \lambda = 1.16$, compare Theorem 6.2 in [25]. We can see that the convergence rate can be improved by using the described mesh grading technique. For a mesh grading parameter $\mu = 0.6666 < 2/3 = \lambda$ we have a convergence order of about 1.71.

One can also see that the mesh grading parameter $\mu = 0.7777 < 7/9 = 1/3 + 2\lambda/3$ would suffice to get the proven order of convergence 3/2. This weaker condition was already stated in Lemma 4.9. Tracing through the proofs of the fully discrete approach as well as of the variational approach reveals that we only used the condition $\mu < \lambda$ to get $\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq ch^{3/2}$. The corresponding $L^2(\Omega)$-error already has quadratic convergence on such meshes, compare Lemma 4.1. We do not need mesh grading techniques at all to prove the $L^2(\Omega)$-convergence rate 3/2. Thus, the finite element error estimate on the boundary seems to be not sharp. For the corresponding interpolation error one can easily show the desired estimate $\|\bar{p} - I_h\bar{p}\|_{L^2(\Gamma)} \leq ch^{3/2}$ for $\mu < 1/3 + 2\lambda/3$, but the transfer to the finite element error is still an open question.

Numerical experiments indicate also that an approximate rate close to 2 can be obtained for mesh grading with $\mu < 1/4 + \lambda/2$. This condition can be expected from the consideration of the interpolation error on the boundary. It is stronger than the condition $\mu < \lambda$ but weaker than the condition $\mu < \lambda/2$ which is expected to be necessary and sufficient for providing an $L^\infty(\Omega)$-error estimate of order $2 - \epsilon$ which also would yield an

number of dof		$\mu = 1$		$\mu = 0.7777$		$\mu = 0.6666$	
Ω	Γ	value	eoc	value	eoc	value	eoc
225	64	4.59e-02	1.02	1.74e-02	1.23	1.53e-02	1.36
833	128	2.27e-02	1.07	6.89e-03	1.33	5.46e-03	1.48
3201	256	5.00e-03	1.11	2.63e-03	1.39	1.83e-03	1.58
12545	512	2.29e-03	1.12	9.82e-04	1.42	5.99e-04	1.61
49665	1024	1.04e-03	1.14	3.61e-04	1.45	1.92e-04	1.64
197633	2048	4.72e-04	1.14	1.31e-04	1.46	6.00e-05	1.68
788481	4096	2.13e-04	1.15	4.72e-05	1.48	1.84e-05	1.71

Table 1: L^2-error of the postprocessed control $\tilde{u}_h$ for $\omega = 3\pi/2$

$L^\infty(\Gamma)$-error estimate. But as in the discussion before there are missing results in the finite element approximation on the boundary for elliptic equations.

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