# Finite Element Error Estimates for Optimal Control Problems with Pointwise Tracking

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**Abstract**
We consider a linear-quadratic elliptic optimal control problem with point evaluations of the state variable in the cost functional. The state variable is discretized by conforming linear finite elements. For control discretization, three different approaches are considered. The main goal of the paper is to significantly improve known a priori discretization error estimates for this problem. We prove optimal error estimates for cellwise constant control discretizations in two and three space dimensions. Further, in two space dimensions, optimal error estimates for variational discretization and for the post-processing approach are derived.

**Key Words**
Elliptic equations, optimal control, pointwise tracking, finite elements, error estimates

**AMS subject classification**
49N10, 49M25, 65N15, 65N30

## 1 Introduction

In this article, we develop a priori error estimates for the discretization of a linear-quadratic elliptic optimal control problem with point evaluation of the state variable in the cost functional. That is, we consider the following problem:

$$\text{Minimize } \frac{1}{2}\sum_{i\in I}(u(x_i)-\xi_i)^2 + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \tag{1.1a}$$

subject to

$$\begin{aligned} -\Delta u &= q && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1b}$$

and

$$a \le q(x) \le b \quad \text{for a.a } x \in \Omega. \tag{1.1c}$$

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Here, $\Omega \subset \mathbb{R}^d$ is a bounded, convex, polygonal/polyhedral domain with $d \in \{2,3\}$, $\alpha > 0$, $a, b \in \mathbb{R}$, and $a < b$. Problem (1.1) seeks to minimize the distance of the state $u$ to prescribed values $\xi_i \in \mathbb{R}$ at fixed points $x_i$ in the interior of $\Omega$ for $i \in I = \{1, 2, \dots, N\}$.

The consideration of such a cost functional involving pointwise evaluation is motivated by parameter identification problems with pointwise measurements, see, e.g., [23] for numerical analysis of a problem with a similar cost functional and finite dimensional control (parameter) variable. The optimal control problem (1.1) and its finite element discretization are considered and analyzed in recent publications [6, 2, 1]. The goal of this paper is to significantly improve the a priori error estimates from these papers, see Table 1 and the detailed discussion below.

Table 1: Comparison of the orders of convergence of $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$

| Discretization | | variational | cellwise constant | post processing |
|---|---|---|---|---|
| Known results | $d = 2$ | $h$ | $h\lvert\ln h\rvert$ | |
| | $d = 3$ | $C(\varepsilon)h^{1-\varepsilon}$ | $h^{\frac{1}{2}}\lvert\ln h\rvert^2$ | |
| Our approach | $d = 2$ | $h^2\lvert\ln h\rvert^2$ | $h\lvert\ln h\rvert$ | $h^2\lvert\ln h\rvert^2$ |
| | $d = 3$ | $h\lvert\ln h\rvert$ | $h\lvert\ln h\rvert$ | |

As in the above mentioned papers, we discretize the state equation (1.1b) with conforming linear finite elements, see Section 3 for details. With $h > 0$, we denote the discretization parameter describing the maximal mesh size. For the control discretization, we consider the following three approaches. For each of the approaches error estimates for the error between the optimal control $\bar{q}$ and the discrete optimal control $\bar{q}_h$ in terms of the discretization parameter $h$ are derived.

- *Variational discretization:* In this case, the control variable is not discretized explicitly. It is implicitly discretized through the optimality system leading to an optimal discrete control being not a mesh function, see [12]. For this case, estimates of order $\mathcal{O}(h)$ for $d = 2$ and $\mathcal{O}(h^{1-\varepsilon})$ for $d = 3$ are provided in [6, 1]. For the two-dimensional case we prove a quasi-optimal (up to a logarithmic term) estimate of order $\mathcal{O}(h^2|\ln h|^2)$, see Theorem 7.5.

- *Cellwise constant discretization:* In this case, the control variable is discretized by cellwise constant functions on the same mesh as used for the state variable. For this choice, error estimates of order $\mathcal{O}(h|\ln h|)$ and $\mathcal{O}(h^{\frac{1}{2}}|\ln h|^2)$ for the dimensions $d = 2$ and $d = 3$, respectively, are derived in [1]. We prove an error estimate $\mathcal{O}(h|\ln h|)$ for both $d = 2$ and $d = 3$ significantly improving the known result for the three-dimensional case, see Theorem 6.6.

- *Post processing approach:* In this case, we use the approach suggested in [18] for an optimal control problem with $L^2$ tracking. That is, we discretize the control variable by cellwise constant functions and define a post-processed control through a projection formula (7.9). To our best knowledge, there are no results for this approach in the context of pointwise tracking in the literature. For the two-dimensional case we prove the same rate of convergence as for the variational discretization, i.e., $\mathcal{O}(h^2|\ln h|^2)$, see Theorem 7.11.

These results in the case of cellwise constant discretizations for both $d = 2$ and $d = 3$ as well as in the cases of variational discretization and of the post-processing approach for $d = 2$ can not be further improved, see Table 1. To our best knowledge, the question is open, if it is possible to prove second order error estimates (up to logarithmic terms) for the three-dimensional case ($d = 3$) on general quasi-uniform meshes. A possible way out is to use graded meshes locally refined towards the points $\{ x_i \}$. By the techniques from [13], it seems to be directly possible to prove second order estimates on such meshes. On such meshes also the case of absence of one or both of the control bounds (i.e. $a = -\infty$ or/and $b = \infty$) can be covered.

The structure of the paper is as follows. In the next section, we discuss the functional-analytic setting of the problem, provide optimality conditions and derive regularity results. It turns out, that although the adjoint state $\bar{z}$ possesses in general only $W^{1,s}(\Omega)$ regularity with $s < d/(d-1)$, the optimal control $\bar{q}$ is Lipschitz continuous due the presence of control constraints. In Section 3, the discrete problem for different control discretizations is introduced and the corresponding optimality conditions are stated. After some estimates for an auxiliary equation in Section 4, we discuss in Section 5 properties of the continuous and the discrete Green's functions. Thanks to recent results from [14], we are able to show that the discrete Green's function has similar growth behavior close to the singularity as the continuous one. This property is an important ingredient to prove our main results but is also of an independent interest. Then, we prove error estimates of order $\mathcal{O}(h|\ln h|)$ (see Table 1) in Section 6 and finally estimates of order $\mathcal{O}(h^2|\ln h|^2)$ in Section 7. The last section is devoted to numerical results illustrating our error estimates. For both, the cellwise constant discretization and the post-processing approach in two dimensions, the numerical results are fully in agreement with the presented theory. We present also a three-dimensional example for the post-processing approach and observe second order convergence, which is not covered by our theory, see the discussion above and in Section 8 of this issue.

## 2 Continuous Problem

In this section, we give a rigorous definition of the continuous optimal control problem (1.1) and derive an optimality system as well as first regularity results for the optimal control. To this end, let $Q = L^2(\Omega)$ and

$$Q_{\text{ad}} = \{ q \in Q \mid a \leq q \leq b \text{ a.e. in } \Omega \} .$$

Further, let $\frac{2d}{d+2} < s < \frac{d}{d-1}$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Then, in particular there holds $s' > d$, $W^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$, and $W^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$. The weak formulation of the state equation (1.1b) reads as: For a given control $q \in Q$ find the state $u \in W_0^{1,s'}(\Omega)$ such that

$$(\nabla u, \nabla \varphi) = (q, \varphi) \quad \forall \varphi \in W_0^{1,s}(\Omega), \tag{2.1}$$

where here and in the sequel, $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product.

For the state equation one obtains (cf. [11], Theorem 3.2.1.2) the existence and uniqueness of a solution $u \in H^2(\Omega) \cap W_0^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$. This allows us to define a linear control-to-state mapping $S \colon Q \to C(\bar{\Omega})$ as $Sq = u$ where $u$ is the solution of (2.1). We have the following standard estimate for $Sq$:

$$\|Sq\|_{L^\infty(\Omega)} \leq C\|Sq\|_{H^2(\Omega)} \leq C\|q\|_{L^2(\Omega)}. \tag{2.2}$$

For a mutually disjoint set of points $\{ x_i \mid i \in I \} \subset \Omega$ with $I = \{ 1, 2, \ldots, N \} \subset \mathbb{N}$ and prescribed target values $\{ \xi_i \}_{i \in I} \subset \mathbb{R}$ at these points, we define the cost functional $J \colon Q \times C(\bar{\Omega}) \to \mathbb{R}$ as

$$J(q, u) = \frac{1}{2} \sum_{i \in I} (u(x_i) - \xi_i)^2 + \frac{\alpha}{2} \|q\|^2_{L^2(\Omega)}.$$

We then aim at solving the following optimal control problem:

$$\text{Minimize } J(q,u) \text{ subject to (2.1) and } (q,u) \in Q_{\text{ad}} \times C(\bar{\Omega}). \tag{2.3}$$

**Theorem 2.1.** *Problem* (2.3) *has the unique solution* $\bar{q} \in Q_{ad}$.

*Proof.* The proof can be done by standard arguments, cf., e.g., [27]. □

With the reduced cost functional $j\colon Q \to \mathbb{R}$ given by means of the control-to-state mapping $S$ as

$$j(q) = J(q, Sq),$$

it is straightforward to see that problem (2.3) is equivalent to the problem

$$\text{Minimize } j(q) \text{ subject to } q \in Q_{\text{ad}}. \tag{2.4}$$

**Lemma 2.2.** *For* $q, \delta q \in Q$*, the first Fréchet derivative of the reduced function* $j$ *is given by*

$$j'(q)(\delta q) = (\alpha q + z, \delta q),$$

*where* $z \in W_0^{1,s}(\Omega)$ *solves*

$$(\nabla z, \nabla \varphi) = \sum_{i\in I} (Sq(x_i) - \xi_i)\varphi(x_i) \quad \forall \varphi \in W_0^{1,s'}(\Omega). \tag{2.5}$$

*The second Fréchet derivative is given for* $q, \delta q, \tau q \in Q$ *by*

$$j''(q)(\delta q, \tau q) = \sum_{i\in I} S\delta q(x_i) S\tau q(x_i) + \alpha(\delta q, \tau q).$$

*Proof.* The proof is standard. The regularity of $z$ can be found, e.g., in [3, Theorem 4]. □

For $i \in I$, let $z_i \in W_0^{1,s}(\Omega)$ be given as the solution of

$$(\nabla z_i, \nabla \varphi) = \varphi(x_i) \quad \forall \varphi \in W_0^{1,s'}(\Omega). \tag{2.6}$$

Then, it holds by construction that the solution $z$ of (2.5) can be expressed as

$$z = \sum_{i\in I} (Sq(x_i) - \xi_i) z_i. \tag{2.7}$$

**Theorem 2.3.** *A control* $\bar{q} \in Q_{ad}$ *with associated state* $\bar{u} = S\bar{q} \in W_0^{1,s'}(\Omega)$ *is an optimal solution to the problem* (2.3) *if and only if there exists an adjoint state* $\bar{z} \in W_0^{1,s}(\Omega)$ *such that*

$$\begin{aligned} (\nabla \bar{u}, \nabla \varphi) &= (\bar{q}, \varphi) && \forall \varphi \in W_0^{1,s}(\Omega), \\ (\nabla \bar{z}, \nabla \varphi) &= \sum_{i\in I} (\bar{u}(x_i) - \xi_i)\varphi(x_i) && \forall \varphi \in W_0^{1,s'}(\Omega), \end{aligned} \tag{2.8}$$

$$(\alpha \bar{q} + \bar{z}, \delta q - \bar{q}) \geq 0 \qquad \forall \delta q \in Q_{ad}. \tag{2.9}$$

*Proof.* It holds by a standard result (cf. [16, Theorem 1.4]) that $\bar{q}$ is a solution to (1.1) if and only if $j'(\bar{q})(\delta q - \bar{q}) \geq 0$ for all $\delta q \in Q_{\text{ad}}$. Then, Lemma 2.2 implies the assertion. □

**Proposition 2.4.** *A control $\bar{q} \in Q_{ad}$ is the solution to the optimal control problem* (2.3) *if and only if $\bar{q}$ and the solution $\bar{z}$ of* (2.8) *fulfill the projection formula*

$$\bar{q} = P_{[a,b]}\left(-\frac{1}{\alpha}\bar{z}\right).$$

*Here the projection $P_{[a,b]}$ is given by*

$$P_{[a,b]}(g(x)) = \min\left(b, \max\left(a, g(x)\right)\right)$$

*for $g(x) \in \mathbb{R} \cup \{-\infty, \infty\}$ at $x \in \Omega$.*

*Proof.* A proof of the equivalence of (2.9) and this projection formula can be found in [27, Theorem 2.28]. □

**Proposition 2.5.** *For the solution $\bar{q} \in Q_{ad}$ of the optimal control problem* (2.3)*, it holds*

$$\bar{q} \in H^1(\Omega).$$

*Proof.* This result follows directly from [5, Lemma 3.3]. □

# 3 Discrete Problem

We approximate the continuous state equation (2.1) using a Galerkin finite element discretization. For this discretization, we use a family of triangulations $\{ \mathcal{T}_h \}$. A cell $K \in \mathcal{T}_h$ has the diameter $h_K$. The discretization parameter $h$ is given as $h = \max_{K \in \mathcal{T}_h} h_K$. We also require the triangulation to be regular and quasi-uniform.

For discretizing the state and adjoint equations, we consider the conforming space $V_h \subset W^{1,\infty}(\Omega)$ of linear finite elements on the triangulation $\mathcal{T}_h$

$$V_h = \left\{ v_h \in C(\bar{\Omega}) \;\middle|\; v_h|_K \in \mathcal{P}_1(K)\ \forall K \in \mathcal{T}_h \text{ and } v_h\big|_{\partial\Omega} = 0 \right\}.$$

We consider two types of discretizations for the control variable. The first type is the so-called variational discretization introduced by [12]. Here, the control variable is not explicitly discretized. As second possibility, we consider a piecewise constant control discretization on the family of triangulations $\{ \mathcal{T}_h \}$ introduced for the discretization of the states. Then, we define the space of discrete controls as

$$Q_h^c = \{ q_h \in Q \mid q_h|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathcal{T}_h \}.$$

where $\mathcal{P}_0(K)$ denotes the space of piecewise constant polynomials on a cell $K$. The discrete admissible set is then defined as $Q^c_{h,\text{ad}} = Q^c_h \cap Q_{\text{ad}}$.

In the following, we introduce properties for the discrete problem similar to the continuous case in Section 2 before. To this end, $Q_{h,\text{ad}}$ serves as placeholder for either $Q_{\text{ad}}$ (for variational discretization) or $Q^c_{h,\text{ad}}$ (for cellwise constant discretization).

The discrete state equation for $u_h \in V_h$ with given $q \in Q$ reads as

$$(\nabla u_h, \nabla \varphi_h) = (q, \varphi_h) \quad \forall \varphi_h \in V_h \tag{3.1}$$

and the discrete analog to (2.3) has the form

$$\text{Minimize } J(q_h, u_h) \text{ subject to (3.1) and } (q_h, u_h) \in Q_{h,\text{ad}} \times V_h. \tag{3.2}$$

Again, we can define the discrete control-to-state mapping with $S_h\colon Q \to V_h$ as $S_h q = u_h$. We define the discrete reduced cost functional $j_h\colon Q \to \mathbb{R}$ by

$$j_h(q) = J(q, S_h q).$$

We start with the following stability result for the discrete solution operator $S_h$.

**Lemma 3.1.** *For $q \in Q$, it holds*

$$\|S_h q\|_{L^\infty(\Omega)} \le C\|q\|_{L^2(\Omega)}.$$

*Proof.* The proof is standard. The sub-optimal $L^\infty$ error estimate [7, p. 168] implies with (2.2) that

$$\|Sq - S_h q\|_{L^\infty(\Omega)} \le Ch^{2-\frac{d}{2}}\|Sq\|_{H^2(\Omega)} \le Ch^{2-\frac{d}{2}}\|q\|_{L^2(\Omega)}.$$

Hence,

$$\|S_h q\|_{L^\infty(\Omega)} \le \|Sq\|_{L^\infty(\Omega)} + \|Sq - S_h q\|_{L^\infty(\Omega)} \le C(1 + h^{2-\frac{d}{2}})\|q\|_{L^2(\Omega)}$$

implies the assertion. □

**Theorem 3.2.** *Problem* (3.2) *admits the unique solution $\bar{q}_h \in Q_{h,ad}$.*

*Proof.* As Theorem 2.1, this can be shown by standard arguments. □

As in the continuous case, we have the following expressions for the first and second derivatives of $j_h$.

**Lemma 3.3.** *For $q, \delta q \in Q$, the first Fréchet derivative of the reduced function $j$ is given by*

$$j_h'(q)(\delta q) = (\alpha q + z_h, \delta q),$$

*where $z_h \in V_h$ solves*

$$(\nabla z_h, \nabla \varphi_h) = \sum_{i \in I} (S_h q(x_i) - \xi_i)\varphi_h(x_i) \quad \forall \varphi_h \in V_h. \tag{3.3}$$

*The second Fréchet derivative is given for $q, \delta q, \tau q \in Q$ by*

$$j_h''(q)(\delta q, \tau q) = \sum_{i \in I} S_h \delta q(x_i) S_h \tau q(x_i) + \alpha(\delta q, \tau q).$$

For $i \in I$, let $z_{h,i} \in V_h$ be given as the solution of

$$(\nabla z_{h,i}, \nabla \varphi_h) = \varphi_h(x_i) \quad \forall \varphi_h \in V_h. \tag{3.4}$$

Then, it holds by construction that the solution $z_h$ of (3.3) can be expressed as

$$z_h = \sum_{i \in I} (S_h q(x_i) - \xi_i) z_{h,i}. \tag{3.5}$$

In the following theorem, we formulate the optimality conditions for the discrete problem.

**Theorem 3.4.** *A control* $\bar q_h \in Q_{h,ad}$ *with the associated state* $\bar u_h = S_h \bar q_h \in V_h$ *is an optimal solution to the problem* (3.2) *if and only if there exists an adjoint state* $\bar z_h \in V_h$ *such that*

$$(\nabla \bar u_h, \nabla \varphi_h) = (\bar q_h, \varphi_h) \qquad \forall \varphi_h \in V_h,$$

$$(\nabla \bar z_h, \nabla \varphi_h) = \sum_{i\in I} (\bar u_h(x_i) - \xi_i)\varphi_h(x_i) \quad \forall \varphi_h \in V_h, \tag{3.6}$$

$$(\alpha \bar q_h + \bar z_h, \delta q_h - \bar q_h) \geq 0 \qquad \forall \delta q_h \in Q_{h,ad}. \tag{3.7}$$

*Proof.* The assertion of the theorem can be proved in the same way as its continuous analogue of Theorem 2.3. □

*Remark* 3.5. Similar to Proposition 2.4 on the continuous level, condition (3.7) can be formulated by the means of the projection $P_{[a,b]}$. For $Q_{h,\mathrm{ad}} = Q_{\mathrm{ad}}$, it holds

$$\bar q_h = P_{[a,b]}\left(-\frac{1}{\alpha}\bar z_h\right).$$

In the case $Q_{h,\mathrm{ad}} = Q^c_{h,\mathrm{ad}}$, the corresponding formula reads as

$$\bar q_h = P_{[a,b]}\left(-\frac{1}{\alpha}\pi_h \bar z_h\right),$$

where $\pi_h$ denotes the $L^2$ projection on $Q^c_h$.

## 4 Finite Element Error Analysis for an Auxiliary Equation

For the numerical analysis carried out later, we first need to bound the error $u(x_i) - u_h(x_i)$ for given $q \in Q_{\mathrm{ad}}$. The are multiple results available for the state equation in case of a bounded right-hand side, see e.g., [19, 21, 22] but mostly for $C^2$ smooth boundaries in contrast to our case.

For given a $f \in L^2(\Omega)$ let $w \in H^1_0(\Omega)$ be the solution of the auxiliary problem

$$(\nabla w, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in H^1_0(\Omega). \tag{4.1}$$

**Lemma 4.1.** *Let* $w \in H^1_0(\Omega)$ *be the solution of* (4.1)*. Provided that* $f \in L^p(\Omega)$ *with* $p > d$*, it holds* $w \in W^{1,\infty}(\Omega)$ *with*

$$\|\nabla w\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

*Proof.* For $d = 2$, the assertion follows from [11, Theorems 4.3.2.4 and 4.4.3.7] using the convexity of $\Omega$ which ensures the existence of $p > 2$ with $w \in W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$.

For $d = 3$ and given $y \in \Omega$, let $G(\cdot, y)$ be the Green's function associated to $y$. Then, there holds by the convexity of $\Omega$ and [17, Theorem 5.1.8] that

$$|\nabla_y G(x,y)| \leq |x-y|^{-2} \quad \forall x \in \Omega,\ x \neq y.$$

This allows us to estimate by the Hölder inequality for $1 = \frac{1}{p} + \frac{1}{p'}$ that

$$|\nabla w(y)| = \left|\int_\Omega \nabla_y G(x,y) f(x)\, dx\right| \leq \int_\Omega |f(x)||x-y|^{-2}\, dx \leq \|f\|_{L^p(\Omega)} \left\| |x-y|^{-2} \right\|_{L^{p'}(\Omega)}.$$

Since $p > 3$, the second factor is bounded, which completes the proof. □

We continue with a regularity result for the solution $w$ of (4.1) in a subdomain $\Omega_1 \Subset \Omega_0 \Subset \Omega$, provided that the right-hand side $f$ has higher regularity in $\Omega_0$.

**Lemma 4.2.** *Let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with $\Omega_0$ smooth and $w$ be the solution of* (4.1).

*(i) Provided that $f\big|_{\Omega_0} \in L^p(\Omega_0)$ for some $2 \le p < \infty$, it holds $w \in W^{2,p}(\Omega_1)$ and*

$$\|w\|_{W^{2,p}(\Omega_1)} \le C_p\{\|f\|_{L^p(\Omega_0)} + \|f\|_{L^2(\Omega)}\}$$

*where $C_p \sim Cp$ for $p \to \infty$.*

*(ii) Provided that $f\big|_{\Omega_0} = 0$, it holds $w \in W^{2,\infty}(\Omega_1)$ and*

$$\|w\|_{W^{2,\infty}(\Omega_1)} \le C\|f\|_{L^2(\Omega)}.$$

*Proof.* The proof of the first assertion follows along the lines of the proof of [15, Lemma 2.5]. We choose a smooth $\tilde{\Omega}_1$ such that $\Omega_1 \Subset \tilde{\Omega}_1 \Subset \Omega_0$. We first prove that $w \in W^{2,6}(\tilde{\Omega}_1)$. To this end, let $\tilde{\omega} \in [0,1]$ be a smooth cutoff function with the properties

$$\begin{aligned} \tilde{\omega} &= 1 \qquad \text{in } \tilde{\Omega}_1, \\ \tilde{\omega} &= 0 \qquad \text{in } \Omega \setminus \Omega_0. \end{aligned}$$

Set $v$ as $v = \tilde{\omega} w$. Then, there holds by the product rule

$$\begin{aligned} (\nabla v, \nabla \varphi) &= (\nabla(\tilde{\omega} w), \nabla \varphi) = (\tilde{\omega} \nabla w, \nabla \varphi) + (w \nabla \tilde{\omega}, \nabla \varphi) \\ &= (\nabla w, \nabla(\tilde{\omega}\varphi)) - (\nabla w, \varphi \nabla \tilde{\omega}) - (\nabla \cdot (w \nabla \tilde{\omega}), \varphi) \\ &= (\tilde{\omega} f, \varphi) - 2(\nabla \tilde{\omega} \nabla w, \varphi) - (w \Delta \tilde{\omega}, \varphi) \end{aligned}$$

and therefore $v$ satisfies the following equation

$$\begin{aligned} -\Delta v &= \tilde{\omega} f - 2\nabla\tilde{\omega}\nabla w - w\Delta\tilde{\omega} \qquad && \text{in } \Omega_0, \\ v &= 0 && \text{on } \partial\Omega_0. \end{aligned}$$

By [10, Corollary 9.10], the $W^{2,6}(\Omega_0)$ norm of $v$ is bounded by the $L^6(\Omega_0)$ norm of the right-hand side above. Using the smoothness of $\tilde{\omega}$, it follows

$$\begin{aligned} \|w\|_{W^{2,6}(\tilde{\Omega}_1)} = \|v\|_{W^{2,6}(\tilde{\Omega}_1)} \le \|v\|_{W^{2,6}(\Omega_0)} &\le C\|\tilde{\omega} f - 2\nabla\tilde{\omega}\nabla w - w\Delta\tilde{\omega}\|_{L^6(\Omega_0)} \\ &\le C\{\|f\|_{L^6(\Omega_0)} + \|\nabla w\|_{L^6(\Omega_0)} + \|w\|_{L^6(\Omega_0)}\}. \end{aligned}$$

By (2.2) and the continuous embedding $w \in H^2(\Omega) \hookrightarrow W^{1,6}(\Omega)$ we get

$$\|w\|_{W^{2,6}(\tilde{\Omega}_1)} \le C\{\|f\|_{L^6(\Omega_0)} + \|f\|_{L^2(\Omega)}\}.$$

For $p \le 6$, this already states the assertion.

For $p > 6$, we iterate the previous steps with

$$\begin{aligned} \omega(x) &= 1 \qquad \text{in } \Omega_1, \\ \omega(x) &= 0 \qquad \text{in } \Omega \setminus \tilde{\Omega}_1, \end{aligned}$$

and use the smoothness of $\tilde{\Omega}_1$ to estimate

$$\begin{aligned}\|w\|_{W^{2,p}(\Omega_1)} = \|v\|_{W^{2,p}(\Omega_1)} \le \|v\|_{W^{2,p}(\tilde{\Omega}_1)} &\le C_p\|\omega f - 2\nabla\omega\nabla w - w\Delta\omega\|_{L^p(\tilde{\Omega}_1)} \\ &\le C_p\{\|f\|_{L^p(\tilde{\Omega}_1)} + \|\nabla w\|_{L^p(\tilde{\Omega}_1)} + \|w\|_{L^p(\tilde{\Omega}_1)}\},\end{aligned}$$

where $C_p$ can be traced from the proof of [10, Theorem 9.8, Theorem 9.9]. Exploiting $W^{2,6}(\tilde{\Omega}_1) \hookrightarrow W^{1,p}(\tilde{\Omega}_1)$ for all $1 < p < \infty$, we have

$$\|w\|_{L^p(\tilde{\Omega}_1)} + \|\nabla w\|_{L^p(\tilde{\Omega}_1)} \le \|w\|_{W^{2,6}(\tilde{\Omega}_1)} \le C\{\|f\|_{L^6(\Omega_0)} + \|f\|_{L^2(\Omega)}\},$$

which concludes the proof of the first assertion.

The second assertion follows similarly. Noting that $\tilde{\omega} f = 0$ on the whole domain $\Omega$ and the smoothness of $\Omega_0$, the first step implies $v \in H^3(\Omega_0)$ and hence $w \in H^3(\tilde{\Omega}_1)$. Then, since also $\omega f = 0$ on the whole $\Omega$ and $\tilde{\Omega}_1$ is smooth, the next step yields $v \in H^5(\tilde{\Omega}_1)$ and consequently $w \in H^5(\Omega_1)$. This implies the second assertion. □

Let $w_h \in V_h$ be the Ritz projection of $w$ given by

$$(\nabla w_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall v \in V_h. \tag{4.2}$$

We will use the following Schatz-Wahlbin-type estimate to bound the error between the solutions $w$ of (4.1) and $w_h$ of (4.2) on a subset of $\Omega$.

**Proposition 4.3.** *Let be $\Omega_1 \Subset \Omega_0 \Subset \Omega$, $v \in H_0^1(\Omega) \cap C(\bar{\Omega})$, and $v_h \in V_h$ satisfying*

$$(\nabla(v - v_h), \nabla\varphi_h) = 0 \quad \forall \varphi_h \in V_h.$$

*Then, there are constants $C, C' > 0$, $0 < h_0 < 1$, and $r > 0$ such that $C'h \le r$, $\operatorname{dist}(\Omega_1, \partial\Omega_0) \ge r$, and $\operatorname{dist}(\Omega_0, \partial\Omega) \ge r$. For $0 < h \le h_0$ and any $\varphi_h \in V_h$, there holds*

$$\|v - v_h\|_{L^\infty(\Omega_1)} \le C\{|\ln rh|\|v - \varphi_h\|_{L^\infty(\Omega_0)} + r^{-\frac{d}{2}}\|v - v_h\|_{L^2(\Omega_0)}\}.$$

*Proof.* See [26, Corollary 5.1]. The corollary there is stated in a slightly different form. Applying it to $v - \varphi_h - v_h + \varphi_h$ yields the stated assertion. □

**Lemma 4.4.** *Let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with smooth $\Omega_0$. Further, let $w \in H_0^1(\Omega)$ be the solution of (4.1) and $w_h \in V_h$ be the solution of (4.2).*

*(i) Provided that $f\big|_{\Omega_0} \in L^\infty(\Omega_0)$ there is $h_0 > 0$ such that*

$$\|w - w_h\|_{L^\infty(\Omega_1)} \le Ch^2|\ln h|^2\{\|f\|_{L^\infty(\Omega_0)} + \|f\|_{L^2(\Omega)}\}.$$

*holds for $h \le h_0$.*

*(ii) Provided that $f\big|_{\Omega_0} = 0$, it holds*

$$\|w - w_h\|_{L^\infty(\Omega_1)} \le Ch^2|\ln h|\|f\|_{L^2(\Omega)}.$$

*Proof.* We start with the first assertion. By Proposition 4.3, we have for a suitable chosen smooth $\tilde{\Omega}_1$ with $\Omega_1 \Subset \tilde{\Omega}_1 \Subset \Omega_0$ and any $\varphi_h \in V_h$

$$\|w - w_h\|_{L^\infty(\Omega_1)} \le C\{|\ln rh| \|w - \varphi_h\|_{L^\infty(\tilde{\Omega}_1)} + r^{-\frac{d}{2}} \|w - w_h\|_{L^2(\tilde{\Omega}_1)}\}.$$

Since $r$ is constant, we can use a standard result to estimate the second term, i.e.,

$$r^{-\frac{d}{2}} \|w - w_h\|_{L^2(\tilde{\Omega}_1)} \le r^{-\frac{d}{2}} \|w - w_h\|_{L^2(\Omega)} \le Ch^2 \|f\|_{L^2(\Omega)}.$$

For the first term, we note that Lemma 4.2 ensures $w \in W^{2,p}(\tilde{\Omega}_1)$ for all $2 \le p < \infty$. For $\varphi_h = i_h w$ being the standard nodal interpolant of $w$, it holds

$$\|w - i_h w\|_{L^\infty(\tilde{\Omega}_1)} \le Ch^{2-\frac{d}{p}} \|\nabla^2 w\|_{L^p(\tilde{\Omega}_1)} \le Cph^{2-\frac{d}{p}} \{\|f\|_{L^\infty(\Omega_0)} + \|f\|_{L^2(\Omega)}\}.$$

Choosing $p = |\ln h|$ for small $h$, we get $ph^{2-\frac{d}{p}} \le Ch^2 |\ln h|$, which implies the stated estimate.

The second assertion follows similarly by using the second assertion of Lemma 4.2 ensuring $w \in W^{2,\infty}(\tilde{\Omega}_1)$:

$$\|w - i_h w\|_{L^\infty(\tilde{\Omega}_1)} \le Ch^2 \|\nabla^2 w\|_{L^\infty(\tilde{\Omega}_1)} \le Ch^2 \|f\|_{L^2(\Omega)}.$$

□

## 5 Estimates for the continuous and discrete Green's functions

In this section, we consider a point $x_0 \in \Omega$ and the associated Green's function solving

$$\begin{aligned} -\Delta g &= \delta_{x_0} && \text{in } \Omega, \\ g &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

It is well-known that $g \in W_0^{1,s}(\Omega)$ for any $s < \frac{d}{d-1}$ with

$$\|\nabla g\|_{L^s(\Omega)} \le C, \tag{5.2}$$

see, e.g., [3, Theorem 4].

**Lemma 5.1.** *Let $g$ be the solution of (5.1). Then, for every $M > 0$ there exists an open ball $B \subset \Omega$ containing $x_0$ such that*

$$g(x) \ge M \quad \forall x \in B.$$

*Proof.* It is well-known, see, e.g., [4, (3.11)] that the asymptotic behavior of $g$ for $x \to x_0$ is of type

$$g(x) \approx \begin{cases} C_1 \ln \frac{1}{|x-x_0|} + C_2 & \text{for } d = 2, \\ C_1 \frac{1}{|x-x_0|} + C_2 & \text{for } d = 3 \end{cases} \tag{5.3}$$

with $C_1 > 0$. This directly implies the assertion. □

**Lemma 5.2.** *For the solution $g$ of (5.1) and any open ball $B \subset \Omega$ containing $x_0$, it holds*

$$g \in W^{1,\infty}(\Omega \setminus \bar{B}) \cap H^2(\Omega \setminus \bar{B}).$$

*Proof.* There exists an open ball $B' \Subset B$ with $x_0 \in B'$. As before, we consider a smooth cutoff function $\omega\colon \Omega \to [0,1]$ with the following properties

$$\begin{aligned} \omega &= 1 && \text{in } \Omega \setminus \bar{B}, \\ \omega &= 0 && \text{in } B'. \end{aligned}$$

Let $\varphi \in W_0^{1,s'}(\Omega)$. As in the proof of Lemma 4.2, it holds for $v = \omega g$ that

$$(\nabla v, \nabla \varphi) = \omega(x_0)\varphi(x_0) - 2(\nabla\tilde{\omega}\nabla g, \varphi) - (g\Delta\tilde{\omega}, \varphi) = -2(\nabla\omega\nabla g, \varphi) - (g\Delta\omega, \varphi),$$

since $\omega(x_0) = 0$. Therefore, $v$ satisfies the following equation

$$\begin{aligned} -\Delta v &= -g\Delta\omega - 2\nabla\omega\nabla g && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

For any $s < \frac{d}{d-1}$, we have $g \in W^{1,s}(\Omega)$ from (5.2) and it follows $v \in W^{2,s}(\Omega)$. Hence, by the smoothness of $\omega$, we obtain $g \in W^{2,s}(\Omega \setminus \bar{B}) \hookrightarrow H^1(\Omega \setminus \bar{B})$ .

Iterating the previous steps using $g \in H^1(\Omega \setminus \bar{B})$, we obtain $v \in H^2(\Omega)$ and hence $g \in H^2(\Omega \setminus \bar{B}) \hookrightarrow W^{1,6}(\Omega \setminus \bar{B})$.

Iterating again using $g \in W^{1,6}(\Omega \setminus \bar{B})$ implies $v \in W^{1,\infty}(\Omega)$ by Lemma 4.1. Consequently, we obtain the assertion $g \in W^{1,\infty}(\Omega \setminus \bar{B}) \cap H^2(\Omega \setminus \bar{B})$. □

Let $g_h \in V_h$ be the Ritz projection of $g$ given as solution of

$$(\nabla g_h, \nabla \varphi_h) = \varphi_h(x_0) \quad \forall \varphi_h \in V_h. \tag{5.4}$$

**Lemma 5.3.** *For the solutions $g \in W_0^{1,s}(\Omega)$ of (5.1) and $g_h \in V_h$ of (5.4), it holds*

$$\|g - g_h\|_{L^1(\Omega)} \le Ch^2|\ln h|^2,$$

*where $C$ is independent of $h$.*

*Proof.* The assertion is a direct consequence of [20, Lemma 3.3 (ii)], cf. also [8] for $d = 2$. However, the exponent of the log-term in [20, Lemma 3.3 (ii)] is different from 2 in the three-dimensional case. Therefore, we give a proof here, which is a direct consequence of Lemma 4.4, (i).

Let $f = \operatorname{sgn}(g - g_h)$ and let $w \in H_0^1(\Omega)$ and $w_h \in V_h$ be the corresponding solutions of (4.1) and (4.2). There holds

$$\|g - g_h\|_{L^1(\Omega)} = (f, g - g_h) = w(x_0) - w_h(x_0) \le Ch^2|\ln h|^2\|f\|_{L^\infty(\Omega)} \le Ch^2|\ln h|^2,$$

where we have used $\|f\|_{L^\infty(\Omega)} \le 1$. □

**Lemma 5.4.** *For the solution $g_h \in V_h$ of (5.4), it holds*

$$\|g_h\|_{L^2(\Omega)} \le C$$

*with $C$ independent of $h$.*

*Proof.* We write the discretization error as $e = g - g_h$ and define $w \in H_0^1(\Omega)$ as the solution of

$$(\nabla w, \nabla \varphi) = (e, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

and $w_h \in V_h$ as the solution of

$$(\nabla w_h, \nabla \varphi_h) = (e, \varphi_h) \quad \forall \varphi_h \in V_h.$$

Then, the $L^2$ error on $\Omega$ can be expressed as

$$\|e\|_{L^2(\Omega)}^2 = (e, g) - (e, g_h) = (\nabla w, \nabla g) - (\nabla w_h, \nabla g_h) = w(x_0) - w_h(x_0)$$

and the sub-optimal $L^\infty$ error estimate [7, p. 168] implies

$$\|e\|_{L^2(\Omega)}^2 \le Ch^{2-\frac{d}{2}} \|e\|_{L^2(\Omega)}.$$

Hence, we get

$$\|g_h\|_{L^2(\Omega)} \le Ch^{2-\frac{d}{2}} + \|g\|_{L^2(\Omega)} \le C$$

due to (5.2). □

**Lemma 5.5.** *For the solutions $g \in W_0^{1,s}(\Omega)$ of* (5.1) *and $g_h \in V_h$ of* (5.4) *and any open ball $B \subset \Omega$ containing $x_0$, there is $h_0 > 0$ such that*

$$\|g - g_h\|_{L^2(\Omega \setminus \bar{B})} \le Ch^2 |\ln h|$$

*for all $h \le h_0$.*

*Proof.* We proceeded as in the proof of Lemma 5.4 by writing $e = g - g_h$ and introducing the indicator function $\chi_B$ of $B$. Further, we define $w \in H_0^1(\Omega)$ as the solution of

$$(\nabla w, \nabla \varphi) = ((1 - \chi_B)e, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \tag{5.5}$$

and $w_h \in V_h$ the solution of

$$(\nabla w_h, \nabla \varphi_h) = ((1 - \chi_B)e, \varphi_h) \quad \forall \varphi_h \in V_h.$$

Then, the $L^2$ error on $\Omega \setminus \bar{B}$ can be expressed as

$$\|e\|_{L^2(\Omega \setminus \bar{B})}^2 = ((1 - \chi_B)e, g) - ((1 - \chi_B)e, z_{h,i}) = (\nabla w, \nabla g) - (\nabla w_h, \nabla g_h) = w(x_0) - w_h(x_0).$$

Since the right-hand side of (5.5) is zero on $B$, Lemma 4.4 implies by choosing a suitable subdomain $B' \Subset B$ containing $x_0$ that

$$\|e\|_{L^2(\Omega \setminus \bar{B})}^2 = w(x_0) - w_h(x_0) \le \|w - w_h\|_{L^\infty(B')} \le Ch^2 |\ln h| \|e\|_{L^2(\Omega \setminus \bar{B})},$$

which concludes the proof. □

**Lemma 5.6.** *For the solution $g_h \in V_h$ of* (5.4) *and any open ball $B \subset \Omega$ containing $x_0$ there is $h_0 > 0$ such that*

$$\|g_h\|_{L^\infty(\Omega \setminus \bar{B})} \le C \quad \textit{and} \quad \|\nabla g_h\|_{L^2(\Omega \setminus \bar{B})} \le C,$$

*for all $h \le h_0$ with a constant $C$ independent of $h$.*

*Proof.* On $\Omega \setminus \bar{B}$, we get by an inverse inequality (cf. [7, Theorem 3.2.6]), Lemma 5.5, and by inserting the nodal interpolant $i_h g$ of $g$ that

$$\begin{aligned}\|g-g_h\|_{L^\infty(\Omega\setminus\bar{B})} &\le \|g_h - i_h g\|_{L^\infty(\Omega\setminus\bar{B})} + \|i_h g - g\|_{L^\infty(\Omega\setminus\bar{B})}\\ &\le Ch\|\nabla g\|_{L^\infty(\Omega\setminus\bar{B})} + Ch^{-\frac{d}{2}}\|i_h g - g_h\|_{L^2(\Omega\setminus\bar{B})}\\ &\le Ch\|\nabla g\|_{L^\infty(\Omega\setminus\bar{B})} + Ch^{-\frac{d}{2}}\{\|i_h g - g\|_{L^2(\Omega\setminus\bar{B})} + \|g-g_h\|_{L^2(\Omega\setminus\bar{B})}\}\\ &\le Ch^{2-\frac{d}{2}}|\ln h|.\end{aligned}$$

Here, we used that $g \in W^{1,\infty}(\Omega\setminus\bar{B}) \cap H^2(\Omega\setminus\bar{B})$ according to Lemma 5.2. Hence, we get

$$\|g_h\|_{L^\infty(\Omega\setminus\bar{B})} \le \|g\|_{L^\infty(\Omega\setminus\bar{B})} + \|g-g_h\|_{L^\infty(\Omega\setminus\bar{B})},$$

which implies the first assertion again by means of Lemma 5.2. For the second assertion, we similarly obtain

$$\begin{aligned}\|\nabla(g-g_h)\|_{L^2(\Omega\setminus\bar{B})} &\le Ch\|\nabla^2 g\|_{L^2(\Omega\setminus\bar{B})} + Ch^{-1}\{\|i_h g - g\|_{L^2(\Omega\setminus\bar{B})} + \|g-g_h\|_{L^2(\Omega\setminus\bar{B})}\}\\ &\le Ch|\ln h|.\end{aligned}$$

Here, we again used for the interpolation estimates that $g \in H^2(\Omega\setminus\bar{B})$. Then,

$$\|\nabla g_h\|_{L^2(\Omega\setminus\bar{B})} \le \|\nabla g\|_{L^2(\Omega\setminus\bar{B})} + \|\nabla(g-g_h)\|_{L^2(\Omega\setminus\bar{B})}$$

together with Lemma 5.2 implies the second assertion. □

We close this section by a discrete analogue of Lemma 5.1 for $d = 2$ only.

**Lemma 5.7.** *Let $d = 2$ and $g_h \in V_h$ be the solution of* (5.4)*. Then, for every $M > 0$ there exists an open ball $B \Subset \Omega$ containing $x_0$ and $h_0 > 0$ such that for all $h \le h_0$, it holds*

$$g_h(x) \ge M \quad \forall x \in B.$$

*Proof.* The main idea for the proof stems from the proof of [14, Theorem 4.6]. While in [14], smoothness of the domain is required we do not need this for our particular result since $x_0$ lies in the interior of $\Omega$ and we are only interested in the behavior of $g_h$ in a neighborhood of $x_0$ and not close to the boundary. Hence, case 3 of the proof of [14, Theorem 4.6] does not need to be considered here. We adapt the technique to our case.

We distinguish the following two cases:

$\boldsymbol{|x-x_0| \ge \kappa h|\ln h|^{\frac{1}{2}}}$**:** We show, that for this case we have pointwise convergence of $g_h$. This setting fulfills the assumptions of [26, Theorem 6.1] which states the existence of constants $\kappa$ and $C_\kappa$ such that for $h$ sufficiently small and $x, x_0 \in \Omega_0 \Subset \Omega$ with $|x-x_0| \ge \kappa h$, it holds

$$|g(x) - g_h(x)| \le C_\kappa \frac{h^2}{|x-x_0|^2} \ln\frac{|x-x_0|}{h}. \tag{5.6}$$

Abbreviating $\eta = |x-x_0|h^{-1}$, the right-hand side of (5.6) becomes $C_\kappa \eta^{-2}\ln\eta$. Since $\kappa h|\ln h|^{\frac{1}{2}} \le |x-x_0|$, we have that $\eta \ge \kappa|\ln h|^{\frac{1}{2}}$ which means that for $h$ small enough $C_\kappa\eta^{-2}\ln\eta$ is maximal at $\eta = \kappa|\ln h|^{\frac{1}{2}}$. Combined, we have

$$|g(x) - g_h(x)| \le C_\kappa \eta^{-2}\ln\eta \le C_\kappa |\ln h|^{-1}\ln|\ln h|^{\frac{1}{2}}.$$

Hence, for $h \le h_0$ sufficiently small, we obtain

$$g_h(x) = g(x) + (g_h(x) - g(x)) \ge \frac{1}{2} g(x),$$

which implies the assertion in this case by Lemma 5.1.

$\boldsymbol{|x - x_0| \le \kappa h |\ln h|^{\frac{1}{2}}}$**:** From [14, Lemma 4.8], we know that $g_h(x_0) \ge C(1 + |\ln h|)$. Due to [14, Lemma 4.9], the first derivative of $g_h$ is bounded by $\|\nabla g_h\|_{L^\infty(\Omega)} \le Ch^{-1}$. Then, by the mean value theorem, it holds

$$g_h(x) \ge g_h(x_0) - \|\nabla g_h\|_{L^\infty(\Omega)} |x - x_0| \ge C|\ln h| - C|\ln h|^{\frac{1}{2}}.$$

That implies $|g_h(x)| \ge M$ for $h \le h_0$ sufficiently small.

Combination of these cases yields the assertion with a set $B$ which can be chosen independently of $h \le h_0$. □

*Remark* 5.8. To our best knowledge, the question is open, if it is possible to prove a similar estimate for the discrete Green's function $g_h$ in the three-dimensional case ($d = 3$) on general quasi-uniform meshes.

# 6 Error Analysis for the Optimal Control Problem

In this section, we derive estimates for the discretization error between the continuous optimal control $\bar{q} \in Q_{\text{ad}}$ and the discrete optimal control $\bar{q}_h \in Q_{h,\text{ad}}$ for the case of variational control (i.e. $Q_{h,\text{ad}} = Q_{\text{ad}}$) and piecewise constant control discretization (i.e. $Q_{h,\text{ad}} = Q^c_{h,\text{ad}}$). In both cases, we derive error estimates of order $h|\ln h|$, cf. the Theorems 6.5 and 6.6.

Before doing so, we introduce an additional piece of notation. So far we have only discussed the parts of the adjoint equation related to the singular behavior. Now we also consider the coefficient $S\bar{q}(x_i) - \xi_i$ accompanying each $z_i$. If this coefficient becomes zero, the singularity at that point vanishes. For the rest of the paper we define

$$L = \{\, i \in I \mid S\bar{q}(x_i) - \xi_i = 0 \,\}$$

to be the set of such indices.

**Lemma 6.1.** *Let $\bar{q} \in Q_{ad}$ be the solution of* (2.3)*. Then, for each $i \in I \setminus L$ there is an open ball $B_i \subset \Omega$ containing $x_i$ such that, depending on the sign of $S\bar{q}(x_i) - \xi_i$, either $\bar{q}(x) = a$ or $\bar{q}(x) = b$ holds for all $x \in B_i$ . Moreover, it holds $\bar{q} \in W^{1,\infty}(\Omega)$ with*

$$\|\nabla \bar{q}\|_{L^\infty(\Omega)} \le C.$$

*Proof.* For the solution $\bar{z}$ of the adjoint equation (2.8), it holds

$$\bar{z} = \sum_{i \in I} (S\bar{q}(x_i) - \xi_i) z_i$$

with $z_i$ being the solution of (2.6). The Lemmas 5.1 and 5.2 applied to $z_i$ for any $i \in I \setminus L$ ensure that for such $i$ and any $M > 0$ there are open balls $B_i$ containing $x_i$ such that

$$|z_i(x)| \ge M \quad \forall x \in B_i \qquad \text{and} \qquad \|z_j\|_{L^\infty(B_i)} \le C \quad \text{for } j \in I \setminus \{\, i \,\}\,.$$

Hence, by using $S\bar{q}(x_i) - \xi_i \neq 0$, we can choose $B_i$ for $i \in I \setminus L$ such that either

$$-\frac{1}{\alpha}\bar{z} = -\frac{1}{\alpha}\sum_{j\in I}(S\bar{q}(x_j) - \xi_j)z_j \leq a \quad \text{or} \quad -\frac{1}{\alpha}\bar{z} = -\frac{1}{\alpha}\sum_{j\in I}(S\bar{q}(x_j) - \xi_j)z_j \geq b$$

holds. Hence, by Proposition 2.4, we obtain the first assertion.

For the second assertion, we note that $\bar{q} \in H^1(\Omega)$ by Proposition 2.5. Hence, to prove $\bar{q} \in W^{1,\infty}(\Omega)$ it is sufficient to ensure $\bar{q} \in W^{1,\infty}(B_i)$ for every $i \in I \setminus L$ and $\bar{q} \in W^{1,\infty}(\Omega \setminus \bigcup_{i\in I\setminus L} \bar{B}_i)$. The first result follows directly from the previous discussion. Lemma 5.2 yields $\bar{q} \in W^{1,\infty}(\Omega \setminus \bigcup_{i\in I\setminus L} \bar{B}_i)$, which completes the proof. □

As preparation for deriving the error estimates for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$, we prove the following three lemmas.

**Lemma 6.2.** *For $p, q, q_h \in Q$, it holds*

$$\alpha\|q - q_h\|^2_{L^2(\Omega)} \leq j_h''(p)(q - q_h, q - q_h).$$

*Proof.* This follows directly from Lemma 3.3. □

**Lemma 6.3.** *For $q, \delta q \in Q_{ad}$, it holds*

$$|j'(q)(\delta q) - j_h'(q)(\delta q)| \leq Ch^2|\ln h|^2\|\delta q\|_{L^\infty(\Omega)}$$

*with a constant $C$ independent of $h$.*

*Proof.* By the Lemmas 2.2 and 3.3, we arrive at

$$|j'(q)(\delta q) - j_h'(q)(\delta q)| = |(z - z_h, \delta q)| \leq |(z - \tilde{z}_h, \delta q)| + |(\tilde{z}_h - z_h, \delta q)|, \tag{6.1}$$

where $z \in W_0^{1,s}(\Omega)$ and $z_h \in V_h$ are the solutions of (2.5) and (3.3), respectively and $\tilde{z}_h \in V_h$ denotes the solution of

$$(\nabla\tilde{z}_h, \nabla\varphi_h) = \sum_{i\in I}(Sq(x_i) - \xi_i)\varphi_h(x_i) \quad \forall\varphi_h \in V_h.$$

By construction, it holds

$$\tilde{z}_h = \sum_{i\in I}(Sq(x_i) - \xi_i)z_{h,i}$$

with $z_{h,i}$ given by (3.4).

For the first term on the right-hand side of (6.1), we get by (2.7) and Lemma 5.3

$$\begin{aligned}|(z - \tilde{z}_h, \delta q)| &\leq \|z - \tilde{z}_h\|_{L^1(\Omega)}\|\delta q\|_{L^\infty(\Omega)} \leq \sum_{i\in I}|Sq(x_i) - \xi_i|\|z_i - z_{h,i}\|_{L^1(\Omega)}\|\delta q\|_{L^\infty(\Omega)}\\ &\leq Ch^2|\ln h|^2\{\|Sq\|_{L^\infty(\Omega)} + |\xi|\}\|\delta q\|_{L^\infty(\Omega)} \leq Ch^2|\ln h|^2\{\|q\|_{L^2(\Omega)} + |\xi|\}\|\delta q\|_{L^\infty(\Omega)}\end{aligned}$$

with $|\xi|^2 = \sum_{i\in I}\xi_i^2$.

For estimating the second term on the right-hand side of (6.1), let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with smooth $\Omega_0$ such that $\{\, x_i \mid i \in I \,\} \subset \Omega_1$. Then, (3.5) and the Lemmas 5.4 and 4.4 yield

$$\begin{aligned}|(\tilde{z}_h - z_h, \delta q)| &\leq \|\tilde{z}_h - z_h\|_{L^2(\Omega)}\|\delta q\|_{L^2(\Omega)} \leq \sum_{i\in I}|Sq(x_i) - S_hq(x_i)|\|z_{h,i}\|_{L^2(\Omega)}\|\delta q\|_{L^2(\Omega)}\\ &\leq C\|Sq - S_hq\|_{L^\infty(\Omega_1)}\|\delta q\|_{L^2(\Omega)} \leq Ch^2|\ln h|^2\|q\|_{L^\infty(\Omega)}\|\delta q\|_{L^2(\Omega)}.\end{aligned}$$

Inserting this back in (6.1) proves the lemma. □

**Lemma 6.4.** *Let $q, p, \delta q \in Q$. Then, it holds*

$$|j_h'(q)(\delta q) - j_h'(p)(\delta q)| \le C\|q-p\|_{L^2(\Omega)}\|\delta q\|_{L^2(\Omega)}.$$

*Proof.* Due to Lemma 3.3, it holds by the mean value theorem for any $\rho \in Q$ that

$$j_h'(q)(\delta q) - j_h'(p)(\delta q) = j_h''(\rho)(q-p,\delta q) = \sum_{i\in I} S_h(q-p)(x_i)S_h\delta q(x_i) + \alpha(q-p,\delta q).$$

Therefore, using Lemma 3.1, we can estimate

$$\begin{aligned}|j_h'(q)(\delta q) - j_h'(p)(\delta q)| &\le C\|S_h(q-p)\|_{L^\infty(\Omega)}\|S_h\delta q\|_{L^\infty(\Omega)} + \alpha|(q-p,\delta q)| \\ &\le C\|q-p\|_{L^2(\Omega)}\|\delta q\|_{L^2(\Omega)},\end{aligned}$$

which states the estimate. □

## 6.1 Variational Discretization

As first discretization, we consider the variational discretization approach as introduced by [12], i.e., we choose $Q_{h,\mathrm{ad}} = Q_{\mathrm{ad}}$. Note, that the following error estimate extends [2, Theorem 5.2] to domains with polygonal or polyhedral boundary and for $d = 3$ improves convergence rate by $h^{\frac{1}{2}}$ compared to [2, Theorem 5.2] and [6, Theorem 3.2].

**Theorem 6.5.** *Let $\bar q \in Q_{ad}$ be the solution of the continuous problem* (2.3) *and $\bar q_h \in Q_{ad}$ the solution of the corresponding discrete problem* (3.2) *with $Q_{h,ad} = Q_{ad}$. Then, it holds*

$$\|\bar q - \bar q_h\|_{L^2(\Omega)} \le Ch|\ln h|$$

*with a constant $C$ independent of $h$.*

*Proof.* Using Lemma 6.2, we have by the optimality conditions (2.9) and (3.7) that

$$\begin{aligned}\alpha\|\bar q - \bar q_h\|^2_{L^2(\Omega)} &\le j_h''(p)(\bar q - \bar q_h, \bar q - \bar q_h) = j_h'(\bar q)(\bar q - \bar q_h) - j_h'(\bar q_h)(\bar q - \bar q_h) \\ &\le j_h'(\bar q)(\bar q - \bar q_h) - j'(\bar q)(\bar q - \bar q_h)\end{aligned}$$

for arbitrary $p \in Q$. Since $\bar q, \bar q_h \in Q_{\mathrm{ad}}$ are bounded, Lemma 6.3 implies

$$\alpha\|\bar q - \bar q_h\|^2_{L^2(\Omega)} \le j_h'(\bar q)(\bar q - \bar q_h) - j'(\bar q)(\bar q - \bar q_h) \le Ch^2|\ln h|^2\|\bar q - \bar q_h\|_{L^\infty(\Omega)} \le Ch^2|\ln h|^2,$$

which yields the result. □

## 6.2 Cellwise Constant Control Discretization

We now consider the fully discretized case where the discrete state and adjoint are approximated by functions in $V_h$ and the discrete control is searched for in $Q_{h,\mathrm{ad}} = Q^c_{h,\mathrm{ad}}$. Note, that the following error estimate improves for $d = 3$ the—to our knowledge—best known error estimate from [1, Theorem 4.3] by $h^{\frac{1}{2}}$.

**Theorem 6.6.** *Let $\bar{q} \in Q_{ad}$ be the solution of the continuous problem (2.3) and $\bar{q}_h \in Q_{h,ad}$ the corresponding solution of the discrete problem (3.2) with $Q_{h,ad} = Q^c_{h,ad}$. Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq Ch|\ln h|$$

*with a constant $C$ independent of $h$.*

*Proof.* We split the error as

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq \|\bar{q} - \pi_h\bar{q}\|_{L^2(\Omega)} + \|\pi_h\bar{q} - \bar{q}_h\|_{L^2(\Omega)},$$

where $\pi_h$ denotes the $L^2$ projection on $Q^c_h$ given for $v \in L^1(\Omega)$ by

$$(\pi_h v)\big|_K = \frac{1}{|K|}\int_K v\,dx \qquad \forall K \in \mathcal{T}_h. \tag{6.2}$$

Standard estimates yield by Lemma 6.1 that

$$\|\bar{q} - \pi_h\bar{q}\|_{L^2(\Omega)} \leq Ch\|\nabla\bar{q}\|_{L^2(\Omega)} \leq Ch. \tag{6.3}$$

To bound $\|\pi_h\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$, note that $\pi_h\bar{q} \in Q^c_{h,\text{ad}}$. We derive by means of Lemma 6.2, as well as the optimality conditions (2.9) and (3.7) that

$$\begin{aligned}\alpha\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} &\leq j''_h(p)(\pi_h\bar{q} - \bar{q}_h, \pi_h\bar{q} - \bar{q}_h) = j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'_h(\bar{q}_h)(\pi_h\bar{q} - \bar{q}_h)\\ &\leq j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h).\end{aligned}$$

By further splitting, we obtain

$$\begin{aligned}\alpha\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} &\leq j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \pi_h\bar{q})\\ &= \{j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'_h(\bar{q})(\pi_h\bar{q} - \bar{q}_h\}\\ &\quad + \{j'_h(\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\pi_h\bar{q} - \bar{q}_h)\} - j'(\bar{q})(\bar{q} - \pi_h\bar{q}).\end{aligned} \tag{6.4}$$

For the first term in (6.4) we apply Lemma 6.4 and (6.3) to end up with

$$\begin{aligned}|j'_h(\pi_h\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'_h(\bar{q})(\pi_h\bar{q} - \bar{q}_h)| &\leq C\|\pi_h\bar{q} - \bar{q}\|_{L^2(\Omega)}\|\pi_h\bar{q} - \bar{q}_h\|_{L^2(\Omega)}\\ &\leq Ch^2 + \frac{\alpha}{2}\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)}.\end{aligned}$$

The second difference in (6.4) can be dealt with as in Theorem 6.5. We apply Lemma 6.3 leading by $\|\pi_h\bar{q}\|_{L^\infty(\Omega)} \leq \|\bar{q}\|_{L^\infty(\Omega)}$ to

$$\begin{aligned}|j'_h(\bar{q})(\pi_h\bar{q} - \bar{q}_h) - j'(\bar{q})(\pi_h\bar{q} - \bar{q}_h)| &\leq Ch^2|\ln h|^2\|\pi_h\bar{q} - \bar{q}_h\|_{L^\infty(\Omega)}\\ &\leq Ch^2|\ln h|^2\{\|\bar{q}\|_{L^\infty(\Omega)} + \|\bar{q}_h\|_{L^\infty(\Omega)}\} \leq Ch^2|\ln h|^2.\end{aligned}$$

By the $L^2$ orthogonality of the projection $\pi_h$, the last term in (6.4) amounts to

$$\begin{aligned}|j'(\bar{q})(\bar{q} - \pi_h\bar{q})| &= |(\alpha\bar{q} + \bar{z}, \bar{q} - \pi_h\bar{q})| = |((\alpha\bar{q} + \bar{z}) - \pi_h(\alpha\bar{q} + \bar{z}), \bar{q} - \pi_h\bar{q})|\\ &\leq \|(\alpha\bar{q} + \bar{z}) - \pi_h(\alpha\bar{q} + \bar{z})\|_{L^1(\Omega)}\|\bar{q} - \pi_h\bar{q}\|_{L^\infty(\Omega)}.\end{aligned}$$

Since $\bar{z}$ is in $W^{1,1}(\Omega)$ (cf. [3, Theorem 4]), we can use standard estimates for the $L^2$ projection in $L^1(\Omega)$ and $L^\infty(\Omega)$ to get

$$|j'(\bar{q})(\bar{q} - \pi_h \bar{q})| \leq Ch^2 \|\nabla(\alpha\bar{q} + \bar{z})\|_{L^1(\Omega)} \|\nabla\bar{q}\|_{L^\infty(\Omega)} \leq Ch^2 \{\|\nabla\bar{z}\|^2_{L^1(\Omega)} + \|\nabla\bar{q}\|^2_{L^\infty(\Omega)}\} \leq Ch^2$$

by (5.2) applied to $\bar{z}$ and Lemma 6.1. Collecting the estimated for the right-hand side of (6.4) and absorbing the term $\frac{\alpha}{2}\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)}$ to the left-hand side, yields

$$\alpha\|\pi_h\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} \leq Ch^2 |\ln h|^2,$$

which implies the assertion together with (6.3). □

# 7 Improved Error Analysis for the Optimal Control Problem in two Dimensions

In this section, we improve the error estimates derived in Section 6 using a more detailed analysis of the behavior of the discrete optimal control in two space dimensions. Moreover, when discretizing the controls by cellwise constants (i.e. for $Q_{h,\text{ad}} = Q^c_{h,\text{ad}}$), we employ a post processing strategy to overcome the limitations to first order convergence in this case. For variational discretization and for cellwise constant control discretization with post processing, we derive error estimates of order $h^2|\ln h|^2$, cf. the Theorems 7.5 and 7.11.

Throughout this section, the analysis is restricted to $d = 2$. We start by proving a result for the discrete optimal control $\bar{q}_h$ similar to Lemma 6.1. This is possible due to the Lemmas 5.6 and 5.7.

**Lemma 7.1.** *Let $d = 2$. Further, let $\bar{q}_h \in Q_{h,ad}$ be the solution of (3.2). Then, for each $i \in I \setminus L$ there is an open ball $B_i \subset \Omega$ containing $x_i$ such that, depending on the sign of $S_h\bar{q}(x_i) - \xi_i$, either $\bar{q}_h(x) = a$ or $\bar{q}_h(x) = b$ holds for all $x \in B_i$ and $h \leq h_0$.*

*Proof.* We first show that for $i \in I \setminus L$ the difference $|S_h\bar{q}_h(x_i) - \xi_i|$ is bounded away from zero provided that $h$ is sufficiently small. For $i \in I \setminus L$, it holds

$$\begin{aligned} 0 &< |S\bar{q}(x_i) - \xi_i| \\ &\leq |S_h\bar{q}_h(x_i) - S\bar{q}_h(x_i)| + |S(\bar{q}_h - \bar{q})(x_i)| + |S_h\bar{q}_h(x_i) - \xi_i| \\ &\leq Ch^2|\ln h|\|\bar{q}_h\|_{L^\infty(\Omega)} + C\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} + |S_h\bar{q}_h(x_i) - \xi_i| \\ &\leq Ch|\ln h| + |S_h\bar{q}_h(x_i) - \xi_i|. \end{aligned}$$

Here we used Lemma 4.4, (2.2), and Theorem 6.6. Thereby, we conclude that there is $h_0 > 0$ such that $S_h\bar{q}_h(x_i) - \xi_i \neq 0$ for $i \in I \setminus L$ and $h < h_0$.

For the solution $\bar{z}_h \in V_h$ of the discrete adjoint equation (3.6), it holds

$$\bar{z}_h = \sum_{i \in I} (S_h\bar{q}_h(x_j) - \xi_j) z_{h,i}$$

with the solutions $z_{h,i} \in V_h$ of (3.4). Lemma 5.7 in combination with Lemma 5.6 applied to $z_{h,i}$ for any $i \in I \setminus L$ ensures (by possibly reducing $h_0$) that for such $i$ and any $M > 0$ there are open balls $B_i$ containing $x_i$ such that

$$|z_{h,i}(x)| \geq M \quad \forall x \in B_i \qquad \text{and} \qquad \|z_{h,j}\|_{L^\infty(B_i)} \leq C \quad \text{for } j \in I \setminus \{\, i \,\}$$

for all $h < h_0$. Hence, by using $S_h\bar q_h(x_i) - \xi_i \neq 0$, we obtain on $B_i$ for $i \in I \setminus L$ that either

$$-\frac{1}{\alpha}\bar z_h = -\frac{1}{\alpha}\sum_{j\in I}(S_h\bar q_h(x_j) - \xi_j) z_{h,j} \le a \quad \text{or} \quad -\frac{1}{\alpha}\bar z_h = -\frac{1}{\alpha}\sum_{j\in I}(S_h\bar q_h(x_j) - \xi_j) z_{h,j} \ge b$$

holds. In the case of variational discretization, i.e., for $Q_{h,\text{ad}} = Q_{\text{ad}}$, the assertion follows immediately from Remark 3.5 as in Lemma 6.1. For cellwise discrete control discretization where $Q_{h,\text{ad}} = Q^c_{h,\text{ad}}$, let $\pi_h$ be the $L^2$ projection on $Q^c_h$ as given by (6.2). Then, again by Remark 3.5, it holds for $K \in \mathcal{T}_h$ that

$$\bar q_h\big|_K = P_{[a,b]}\left(-\frac{1}{\alpha}(\pi_h \bar z_h)\big|_K\right) = P_{[a,b]}\left(-\frac{1}{\alpha}\bar z_h(S_K)\right),$$

where $S_K$ denotes the centroid of the cell $K$, cf. Section 7.2. This implies the assertion also in this case. ☐

**Lemma 7.2.** *Let $d = 2$, $\bar q \in Q_{ad}$ be the solution of* (2.3)*, and $\bar q_h \in Q_{h,ad}$ the solution of* (3.2)*. Then, there exist $h_0 > 0$ and open balls $B_i \subset \Omega$ containing $x_i$ for $i \in I \setminus L$, such that for all $h \le h_0$, it holds either*

$$\bar q(x) = \bar q_h(x) = a \qquad \text{or} \qquad \bar q(x) = \bar q_h(x) = b \qquad \forall x \in B_i.$$

*Proof.* The assertion is a direct consequence of the Lemmas 6.1 and 7.1. ☐

**Lemma 7.3.** *Let $d = 2$ and $B_i \subset \Omega$ open balls with $x_i \in B_i$ for $i \in I \setminus L$. For a $q \in Q_{ad}$, let $z$ be the solution of* (2.5) *and $\tilde z_h \in V_h$ be given as the solution of*

$$(\nabla \tilde z_h, \nabla \varphi_h) = \sum_{i\in I}(Sq(x_i) - \xi_i)\varphi_h(x_i) \quad \forall \varphi_h \in V_h. \tag{7.1}$$

*Then, it holds for $B = \bigcup_{i\in I\setminus L} B_i$ that*

$$\|z - \tilde z_h\|_{L^2(\Omega\setminus\bar B)} \le Ch^2|\ln h|^2,$$

*with a constant $C$ independent of $h$.*

*Proof.* Let $z_i$ be defined by (2.6) and $z_{h,i}$ be defined by (3.4). We begin with the splitting

$$\begin{aligned}\|z - \tilde z_h\|_{L^2(\Omega\setminus\bar B)} \le &\sum_{i\in L}\|(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega\setminus\bar B)} \\ &+ \sum_{i\in I\setminus L}\|(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega\setminus\bar B)}.\end{aligned} \tag{7.2}$$

To discuss the first term of (7.2), we note that for $i \in L$ there holds $Sq(x_i) - \xi_i = 0$. Therefore, we have for such $i$ that

$$(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i} = (S_h q(x_i) - \xi_i)z_{h,i} = (S_h q(x_i) - Sq(x_i))z_{h,i}.$$

Then, we get by the Lemmas 4.4 and 5.4 that

$$\begin{aligned}\|(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega\setminus\bar B)} &= \|(S_h q(x_i) - Sq(x_i))z_{h,i}\|_{L^2(\Omega\setminus\bar B)} \\ &\le |S_h q(x_i) - Sq(x_i)|\|z_{h,i}\|_{L^2(\Omega)} \\ &\le Ch^2|\ln h|^2\|q\|_{L^\infty(\Omega)}.\end{aligned} \tag{7.3}$$

For the second term of (7.2), we split

$$(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i} = (Sq(x_i) - \xi_i)(z_i - z_{h,i}) + (Sq(x_i) - S_h q(x_i))z_{h,i} \qquad (7.4)$$

and obtain for the first term by (2.2) and Lemma 5.5 applied to $z_i$ and $z_{h,i}$ separately on $B_i$ that

$$\begin{aligned}\|(Sq(x_i) - \xi_i)(z_i - z_{h,i})\|_{L^2(\Omega\setminus\bar{B})} &\le \{\|q\|_{L^2(\Omega)} + |\xi|\}\|z_i - z_{h,i}\|_{L^2(\Omega\setminus\bar{B})} \\ &\le Ch^2|\ln h|\{\|q\|_{L^2(\Omega)} + |\xi|\}.\end{aligned}$$

For the second term of the right-hand side of (7.4), we can proceed as for (7.3) to obtain

$$\|(Sq(x_i) - S_h q(x_i))z_{h,i}\|_{L^2(\Omega\setminus\bar{B})} \le Ch^2|\ln h|^2\|q\|_{L^\infty(\Omega)}.$$

Collecting the terms, we get for the second term on the right-hand side of (7.2)

$$\|(S\bar{q}(x_i) - \xi_i)z_i - (S_h\bar{q}(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega\setminus\bar{B})} \le Ch^2|\ln h|^2\{\|q\|_{L^\infty(\Omega)} + |\xi|\}.$$

By adding up the two estimates for (7.2), we arrive at the result. □

Based on the previous lemma, we can improve Lemma 6.3 for the case $d = 2$. The improvement consists in the fact that the $L^\infty$ norm of $\delta q$ can be replaced by its $L^2$ norm provided that $\delta q$ vanishes in the neighborhood of the points $x_i$ for $i \in I \setminus L$.

**Lemma 7.4.** *Let $d = 2$ and $B_i \subset \Omega$ open balls with $x_i \in B_i$ for $i \in I \setminus L$. For $B = \bigcup_{i\in I\setminus L} B_i$ and $q, \delta q \in Q_{ad}$ with $\delta q\big|_B = 0$, it holds*

$$|j'(q)(\delta q) - j_h'(q)(\delta q)| \le Ch^2|\ln h|^2\|\delta q\|_{L^2(\Omega)}.$$

*with a constant $C$ independent of $h$.*

*Proof.* The proof follows the lines of Lemma 6.3 obtaining

$$|j'(q)(\delta q) - j_h'(q)(\delta q)| \le |(z - \tilde{z}_h, \delta q)| + |(\tilde{z}_h - z_h, \delta q)|.$$

with $\tilde{z}_h$ solving (7.1). The second term is treated as in Lemma 6.3. Using Lemma 7.3 for the first term then implies the result by

$$|(z - \tilde{z}_h, \delta q)| \le \|z - \tilde{z}_h\|_{L^2(\Omega\setminus\bar{B})}\|\delta q\|_{L^2(\Omega)}$$

since $\delta q\big|_B = 0$. □

## 7.1 Variational Discretization

The following result improves Theorem 6.5 in the case $d = 2$. Since the result relies on Lemma 5.7 which is available for $d = 2$ only, an extension to $d = 3$ is not directly possible.

**Theorem 7.5.** *Let $d = 2$, $\bar{q} \in Q_{ad}$ be the solution of the continuous problem* (2.3)*, and $\bar{q}_h \in Q_{ad}$ be the solution of the corresponding discrete problem* (3.2) *with $Q_{h,ad} = Q_{ad}$. Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \le Ch^2|\ln h|^2$$

*with a constant $C$ independent of $h$.*

*Proof.* The assertion is proven as Theorem 6.5: Let $B = \bigcup_{i\in I\setminus L} B_i$ for the sets $B_i$ given by Lemma 7.2. On $B$, it holds $\delta q = \bar{q} - \bar{q}_h = 0$. Hence, by using Lemma 7.4, we conclude

$$\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} \le |j'(q)(\delta q) - j_h'(q)(\delta q)| \le Ch^2|\ln h|^2\{\|\bar{q}\|_{L^\infty(\Omega)} + |\xi|\}\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}.$$

Dividing by $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ proves the theorem. □

## 7.2 Cellwise Constant Control Discretization with Post Processing

We adapt the proof technique from [18]. We split up the mesh into subsets with respect to the regularity of $\bar{q}$. For a given mesh $\mathcal{T}_h$, we define three subsets of cells

$$\mathcal{T}_h^1 = \left\{ K \in \mathcal{T}_h \;\middle|\; \bar{q}\big|_K = a \text{ or } \bar{q}\big|_K = b \right\}, \quad \mathcal{T}_h^2 = \left\{ K \in \mathcal{T}_h \;\middle|\; a < \bar{q}\big|_K < b \right\}, \\ \text{and} \quad \mathcal{T}_h^3 = \mathcal{T}_h \setminus (\mathcal{T}_h^1 \cup \mathcal{T}_h^2).$$

Then, $\mathcal{T}_h^3$ denotes the set of cells where $\bar{q}$ is only Lipschitz continuous in contrast to $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$ which consists of cells $K$ where $\bar{q}$ is in $H^2(K)$. As in [18], we make the following assumption:

**Assumption 1.** We assume the existence of a constant $C$ independent of $h$, such that for all $h$ sufficiently small, it holds

$$\sum_{K \in \mathcal{T}_h^3} |K| \leq Ch.$$

Similar assumptions have been made in the case of cellwise linear discretization or a postprocessing approach in, e.g., [25, 18].

*Remark* 7.6. By (5.3) applied to $\bar{z}$, it seems likely that the active set $\{ x \in \Omega \mid \bar{q}(x) = a \vee \bar{q}(x) = b \}$ has rectifiable boundary. This would imply that the boundary is a curve of finite length, i.e., it can be covered by a subset $\mathcal{T}_h^3 \subset \mathcal{T}_h$ fulfilling Assumption 1.

We denote by $S_K$ the centroid of a cell $K \in \mathcal{T}_h$. For $w \in C(\Omega)$, we define the projection $R_h w \in Q_h^c$ by

$$R_h w\big|_K = w(S_K) \quad \forall K \in \mathcal{T}_h. \tag{7.5}$$

Further, we set

$$r_h = R_h \bar{q}, \tag{7.6}$$

which is well-defined by Lemma 6.1, and note that by construction, it holds $r_h \in Q_{h,\mathrm{ad}}^c$.

**Proposition 7.7.** *Let $f \in H^2(K)$ for a given cell $K \in \mathcal{T}_h$. Then, it holds*

$$\left| \int_K (f(x) - f(S_K))\, dx \right| \leq Ch^2 |K|^{\frac{1}{2}} \|\nabla^2 f\|_{L^2(K)}.$$

*Proof.* See [18, Lemma 3.2]. ☐

Similar to [18, Lemma 3.3], we prove the following result adapted to the problem considered here:

**Lemma 7.8.** *Let $d = 2$, $\bar{q} \in Q_{ad}$ be the solution of (2.3), and $r_h \in Q_{h,ad}^c$ defined by (7.6). Then, there is $h_0 > 0$ such that for all $i \in I \setminus L$ and all $h \leq h_0$, it holds*

$$|S_h(\bar{q} - r_h)(x_i)| \leq Ch^2$$

*with a constant $C$ independent of $h$.*

*Proof.* Let $z_{h,i} \in V_h$ be the solution of (3.4). Then, we can write

$$S_h(\bar{q} - r_h)(x_i) = (\nabla z_{h,i}, \nabla(S_h\bar{q} - S_h r_h)) = (z_{h,i}, \bar{q} - r_h).$$

For $K \in \mathcal{T}_h^1$, it holds $\bar{q}\big|_K \in \{ a, b \}$. Hence, we have $r_h = \bar{q}$ there and $(z_{h,i}, \bar{q} - r_h)$ vanishes on $\bigcup \mathcal{T}_h^1$. Consequently, we get

$$(z_{h,i}, \bar{q} - r_h) = \sum_{K \in \mathcal{T}_h^2} \int_K z_{h,i}(\bar{q} - r_h)\, dx + \sum_{K \in \mathcal{T}_h^3} \int_K z_{h,i}(\bar{q} - r_h)\, dx.$$

Lemma 6.1 ensures the existence of open sets $B_i \subset \Omega$ for $i \in I \setminus L$ with $x_i \in B_i$ and $\bar{q}(x) \in \{ a, b \}$ for $x \in B_i$. For $h \le h_0$, the sets $B_i$ can be chosen such that $B_i \subset \bigcup \mathcal{T}_h^1$.

We first consider the integral over $K \in \mathcal{T}_h^3$. Again by Lemma 6.1, we know that $\bar{q} \in W^{1,\infty}(K)$ which implies for $x \in K$ that

$$|\bar{q}(x) - r_h(x)| = |\bar{q}(x) - \bar{q}(S_K)| \le C|x - S_K| \|\nabla \bar{q}\|_{L^\infty(\Omega)} \le Ch\|\nabla \bar{q}\|_{L^\infty(\Omega)}.$$

For $B = \bigcup_{i \in I \setminus L} B_i$ it holds $B \subset \bigcup \mathcal{T}_h^1$. Hence, we have $\bigcup \mathcal{T}_h^3 \subset \Omega \setminus \bar{B}$ and one obtains

$$\begin{aligned}\left| \sum_{K \in \mathcal{T}_h^3} \int_K z_{h,i}(\bar{q} - r_h)\, dx \right| &\le \sum_{K \in \mathcal{T}_h^3} \int_K |z_{h,i}(\bar{q} - r_h)|\, dx \\ &\le Ch\|z_{h,i}\|_{L^\infty(\Omega\setminus\bar{B})} \|\nabla \bar{q}\|_{L^\infty(\Omega)} \sum_{K \in \mathcal{T}_h^3} |K| \le Ch^2 \|\nabla \bar{q}\|_{L^\infty(\Omega)}\end{aligned}$$

by means of Assumption 1 and Lemma 5.6.

For a cell $K \in \mathcal{T}_h^2$, we have that

$$\int_K z_{h,i} r_h\, dx = \int_K z_{h,i}\bar{q}(S_K)\, dx = \int_K z_{h,i}(S_K)\bar{q}(S_K)\, dx.$$

Using this, we obtain by Proposition 7.7 and the Cauchy–Schwarz inequality that

$$\left| \sum_{K \in \mathcal{T}_h^2} \int_K z_{h,i}(\bar{q} - r_h)\, dx \right| \le Ch^2 \left( \sum_{K \in \mathcal{T}_h^2} \|\nabla^2 (z_{h,i}\bar{q})\|_{L^2(K)}^2 \right)^{\frac{1}{2}}.$$

On $\bigcup \mathcal{T}_h^2$, we have $\bar{q} = -\alpha^{-1}\bar{z}$. By the product rule, we get since $\mathcal{T}_h^2 \subset \Omega \setminus \bar{B}$ that

$$\left( \sum_{K \in \mathcal{T}_h^2} \|\nabla^2 (z_{h,i}\bar{q})\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \le C\|z_{h,i}\|_{L^\infty(\Omega\setminus\bar{B})} \|\nabla^2 \bar{z}\|_{L^2(\Omega\setminus\bar{B})} + C\|\nabla z_{h,i}\|_{L^2(\Omega\setminus\bar{B})} \|\nabla \bar{z}\|_{L^\infty(\Omega\setminus\bar{B})}.$$

Using Lemma 5.6 for the terms involving $z_{h,i}$ and Lemma 5.2 for

$$\bar{z} = \sum_{i \in I \setminus L} (S\bar{q}(x_i) - \xi_i) z_i,$$

we obtain the assertion by collecting the previous estimates. □

**Lemma 7.9.** *Let be $d = 2$ and $B_i \subset \Omega$ open balls with $x_i \in B_i$ for $i \in I \setminus L$. For $\hat{z}_h \in V_h$ given as the solution of*

$$(\nabla \hat{z}_h, \nabla \varphi_h) = \sum_{i \in I} (S_h r_h(x_i) - \xi_i)\varphi_h(x_i) \quad \forall \varphi_h \in V_h. \tag{7.7}$$

*for $r_h$ defined by (7.6) and $B = \bigcup_{i \in I \setminus L} B_i$, it holds*

$$\|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2$$

*with a constant $C$ independent of $h$.*

*Proof.* We split

$$\|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq \|\bar{z} - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} + \|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})},$$

where $\tilde{z}_h \in V_h$ is the solution of (7.1). For the first term, Lemma 7.3 states

$$\|\bar{z} - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2.$$

For the second term, we get from Lemma 5.4 that

$$\|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq \|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega)} \leq C \sum_{i \in I} |S_h(\bar{q} - r_h)(x_i)|.$$

Then, by Lemma 7.8, we get

$$\|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2.$$

Collecting the estimates yields the assertion. □

**Lemma 7.10.** *Let $d = 2$, $\bar{q}_h \in Q^c_{h,ad}$ be the solution of (3.2) with $Q_{h,ad} = Q^c_{h,ad}$, and $r_h$ defined by (7.6). Then, there exists $h_0 > 0$ such that for $h \leq h_0$, it holds*

$$\|\bar{q}_h - r_h\|_{L^2(\Omega)} \leq Ch^2 |\ln h|^2$$

*with a constant $C$ independent of $h$.*

*Proof.* As in [18], we first derive a variational inequality for $r_h$. To this end let $h_0 > 0$ be sufficiently small such that the sets $\{ B_{0,i} \subset \Omega \mid i \in I \setminus L \}$ given by Lemma 7.2, subsets $\{ B_{1,i} \subset B_{0,i} \mid i \in I \setminus L \}$, and a subset of cells $\widetilde{\mathcal{T}}_{h_0} \subset \mathcal{T}_{h_0}$ fulfill the relation

$$B_1 \subset \Omega_{h_0} \subset B_0$$

for $B_0 = \bigcup_{i \in I \setminus L} B_{0,i}$, $\Omega_{h_0} = \bigcup \widetilde{\mathcal{T}}_{h_0}$, and $B_1 = \bigcup_{i \in I \setminus L} B_{1,i}$.

By Lemma 5.2, we have that $\bar{z} \in C(\bar{\Omega} \setminus B_1)$. We now apply the optimality condition (2.9), which holds true for all $\delta q \in Q_{\text{ad}}$ and also pointwise on $\bar{\Omega} \setminus B_1$:

$$(\alpha \bar{q}(x) + \bar{z}(x))(\delta q - \bar{q}(x)) \geq 0 \qquad \forall \delta q \in [a, b], \ x \in \bar{\Omega} \setminus B_1.$$

We apply this formula for $x = S_K$ with $K \in \mathcal{T}_h$ and $K \subset \Omega \setminus \Omega_{h_0}$ and $\delta q = \bar{q}_h(S_K)$ which gives

$$(\alpha r_h(S_K) + \bar{z}(S_K))(\bar{q}_h(S_K) - r_h(S_K)) \geq 0.$$

Integrating this inequality over $K$ and summing this up over $K \in \mathcal{T}_h$ with $K \subset \Omega \setminus \Omega_{h_0}$ yields

$$(\alpha r_h + R_h \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega \setminus \Omega_{h_0})} \geq 0.$$

Noting that $\bar{q}_h - r_h = 0$ on $B_0$ implies

$$(\alpha r_h + R_h \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} \geq 0.$$

By testing the discrete optimality condition (3.7) with $\delta q_h = r_h$, we get

$$(\alpha \bar{q}_h + \bar{z}_h, r_h - \bar{q}_h)_{L^2(\Omega\setminus\bar{B}_0)} \geq 0$$

again by using $r_h - \bar{q}_h = 0$ on $B_0$. Adding the last two inequalities results in the estimate

$$\alpha \|r_h - \bar{q}_h\|^2_{L^2(\Omega)} = \alpha \|r_h - \bar{q}_h\|^2_{L^2(\Omega\setminus\bar{B}_0)} \leq (R_h \bar{z} - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)}. \tag{7.8}$$

We split the right-hand side of the above inequality to get

$$\begin{aligned}(R_h \bar{z} - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} = (R_h \bar{z} - \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} \\ + (\bar{z} - \hat{z}_h, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} + (\hat{z}_h - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)},\end{aligned}$$

where $\hat{z}_h \in V_h$ solves (7.7). We separately estimate the three terms on the right-hand side.

Using Proposition 7.7, the fact that $\bar{q}_h = r_h$ on $B_0$, and that $\bar{q}_h$ and $r_h$ are piecewise constant, one arrives for the first term at

$$\begin{aligned}(R_h \bar{z} - \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} &\leq \sum_{\substack{K\in\mathcal{T}_h \\ K\subset\Omega\setminus\Omega_{h_0}}} |\bar{q}_h(S_K) - r_h(S_K)| \left| \int_K (R_h \bar{z} - \bar{z})\, dx \right| \\ &\leq Ch^2 \sum_{\substack{K\in\mathcal{T}_h \\ K\subset\Omega\setminus\Omega_{h_0}}} |\bar{q}_h(S_K) - r_h(S_K)| |K|^{\frac{1}{2}} \|\nabla^2 \bar{z}\|_{L^2(K)} \\ &\leq Ch^2 \|\bar{q}_h - r_h\|_{L^2(\Omega)} \|\bar{z}\|_{H^2(\Omega\setminus\bar{B}_1)},\end{aligned}$$

where $\|\bar{z}\|_{H^2(\Omega\setminus\bar{B}_1)}$ is bounded due to Lemma 5.2.

For the second term, it follows by Lemma 7.9 that

$$(\bar{z} - \hat{z}_h, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} \leq Ch^2 |\ln h|^2 \|\bar{q}_h - r_h\|_{L^2(\Omega)}.$$

For the last term, we have since $\bar{q}_h - r_h = 0$ on $B_0$ that

$$(\hat{z}_h - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega\setminus\bar{B}_0)} = (\nabla(\hat{z}_h - \bar{z}_h), \nabla S_h(\bar{q}_h - r_h)) = -\sum_{i\in I} (S_h(\bar{q}_h - r_h)(x_i))^2 \leq 0.$$

By using the last three estimates and (7.8), we complete the proof. □

Using the previous lemmas, we can conclude this section by formulating the error estimate for the post-processed control $\hat{q}_h$ given by

$$\hat{q}_h = P_{[a,b]}(-\alpha^{-1} \bar{z}_h), \tag{7.9}$$

where $\bar{z}_h \in V_h$ is the adjoint state associated to the solution $\bar{q}_h \in Q^c_{h,\text{ad}}$ of the discrete problem (3.2) with $Q_{h,\text{ad}} = Q^c_{h,\text{ad}}$.

**Theorem 7.11.** *Let $d = 2$, $\bar{q} \in Q_{ad}$ be the solution of the continuous problem* (2.3)*, and $\hat{q}_h$ given by* (7.9)*. Furthermore, let Assumption 1 hold. Then, it holds*

$$\|\bar{q} - \hat{q}_h\|_{L^2(\Omega)} \le Ch^2 |\ln h|^2$$

*with a constant $C$ independent of $h$.*

*Proof.* By Lemma 7.2 there is $B = \bigcup_{i \in I \setminus L} B_i$ such that $\bar{q} = \hat{q}_h$ on $B$. Then, the Lipschitz continuity of the projection operator $P_{[a,b]}$ in $L^2(\Omega)$ implies

$$\|\bar{q} - \hat{q}_h\|_{L^2(\Omega)} = \|\bar{q} - \hat{q}_h\|_{L^2(\Omega \setminus B)} = \|P_{[a,b]}(-\alpha^{-1}\bar{z}) - P_{[a,b]}(-\alpha^{-1}\bar{z}_h)\|_{L^2(\Omega \setminus \bar{B})} \le C\|\bar{z} - \bar{z}_h\|_{L^2(\Omega \setminus \bar{B})}.$$

Then, we split

$$\|\bar{z} - \bar{z}_h\|_{L^2(\Omega \setminus \bar{B})} \le \|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} + \|\hat{z}_h - \bar{z}_h\|_{L^2(\Omega \setminus \bar{B})},$$

where $\hat{z}_h$ solves (7.7). For the first term, we have by Lemma 7.9

$$\|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \le Ch^2 |\ln h|^2.$$

For the second term, we get by the Lemmas 3.1, 7.10, and 5.4.

$$\begin{aligned}\|\bar{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} &\le \|\bar{z}_h - \hat{z}_h\|_{L^2(\Omega)} \le C \sum_{i \in I} |S_h(\bar{q}_h - r_h)(x_i)| \le C\|\bar{q}_h - r_h\|_{L^2(\Omega)} \\ &\le Ch^2 |\ln h|^2.\end{aligned}$$

Combining the estimates implies the assertion. □

# 8 Numerical Results

We give numerical results to confirm the results of the previous sections. To this end we consider different sample problems. The optimal control problems are solved by the optimization library RoDoBo [24] and the finite element toolkit Gascoigne [9].

We consider the optimal control problem (1.1) with the slightly modified state equation

$$-\Delta u = f + q$$

with given right-hand side $f$ in a ball $\Omega = B_{0.5}(x_1) \subset \mathbb{R}^d$ with $d \in \{2, 3\}$. The cost functional consist of one point evaluation at $x_1 = (0.5, 0.5)^T$ for $d = 2$ and respectively $x_1 = (0.5, 0.5, 0.5)^T$ for $d = 3$. The choice of $\Omega$ allows to give an exact solution of $-\Delta z_1 = \delta_{x_1}$ by

$$z_1 = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x - x_1|} - \frac{\ln 2}{2\pi} & \text{for } d = 2, \\ \frac{1}{4\pi} \frac{1}{|x - x_1|} - \frac{1}{2\pi} & \text{for } d = 3, \end{cases}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. We choose $\bar{u}(x) = \cos(\pi|x - x_1|)$ and $\xi_1 = \bar{u}(x_1) - 1$. Hence, the adjoint solution $\bar{z}$ is then given as $\bar{z} = z_1$. By choosing $\alpha = 1$, the optimal control fulfills $\bar{q} = P_{[a,b]}(-\bar{z})$. The right-hand side $f$ is chosen such that $\bar{u}$ solves the state equation. Thus, we have $f = -\Delta \cos(\pi|x|) - q$.

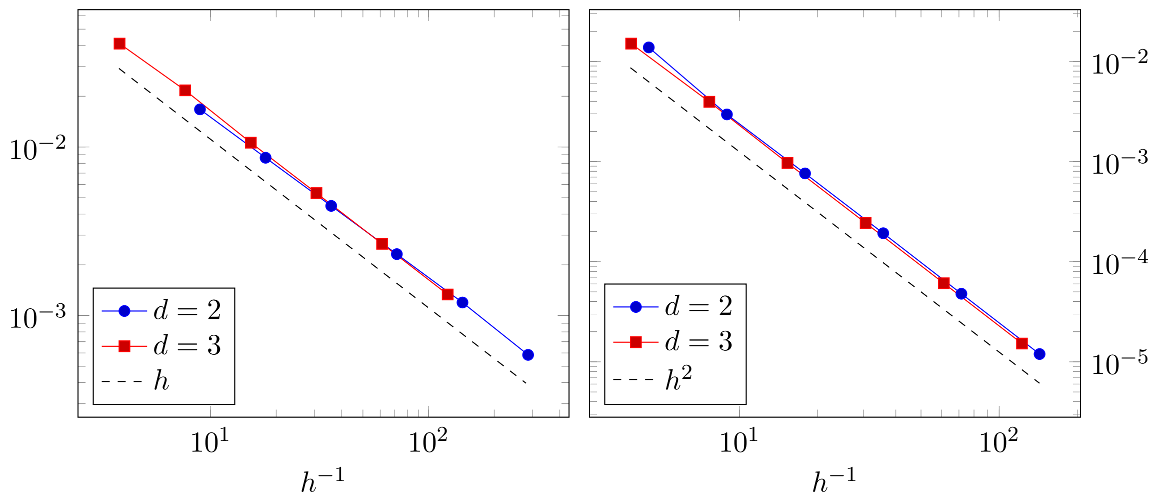


Figure 1: Errors $\|q - q_h\|_{L^2(\Omega)}$ for cellwise constant control discretization (left) and $\|q - \hat{q}_h\|_{L^2(\Omega)}$ for cellwise constant control discretization with post processing (right)

First, we present results for cellwise constant discretization of the control. Here, the bounds are chosen as $-a = b = 1$. The numerical results depicted in Figure 1 (left) confirm the estimates of Theorem 6.6.

Further, for choosing a different value for the bounds, $-a = b = 0.2$, we present in Figure 1 (right) results for cellwise constant discretization of the control with post processing. They confirm the estimate of Theorem 7.11, which was proved in $d = 2$ dimensions only. However, the numerical results for $d = 3$ indicate that a similar convergence result may also hold in three dimensions.

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