ERROR ESTIMATES FOR SPACE-TIME DISCRETIZATION OF PARABOLIC TIME-OPTIMAL CONTROL PROBLEMS WITH BANG-BANG CONTROLS*

LUCAS BONIFACIUS†, KONSTANTIN PIEPER‡, AND BORIS VEXLER†

Abstract. In this paper a priori error estimates are derived for full discretization (in space and time) of time-optimal control problems. Various convergence results for the optimal time and the control variable are proved under different assumptions. Especially the case of bang-bang controls is investigated. Numerical examples are provided to illustrate the results.

Key words. Time-optimal control, Error estimates, Galerkin method, Bang-bang controls

AMS subject classifications. 49K20, 49M25, 65M15, 65M60

1. Introduction. In this article, we consider time-optimal control problems subject to a linear parabolic partial differential equation. More precisely, we study the following model problem, where u denotes the state, q the control, and T the terminal time:

$$(P) \qquad \text{Minimize } T \quad \text{subject to} \quad \begin{cases} T > 0, & \\ \partial_t u - \Delta u = Bq, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{in } \Omega, \\ G(u(T)) \le 0, & \\ q_a \le q(t) \le q_b, & \text{in } \omega,\ t\in(0,T). \end{cases}$$

Here, we consider either distributed controls $q(t) \in L^2(\omega)$ for some appropriate subset $\omega \subset \Omega$ of the domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, or time-dependent parameter controls $q(t) \in \mathbb{R}^{N_c}$, $N_c \in \mathbb{N}$. Moreover, B is an appropriate control operator and $q_a, q_b \in \mathbb{R}$ are the control bounds; see Section 2 for the precise assumptions. The terminal constraint on the state is given by

$$G(u) \coloneqq \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} - \frac{\delta_0^2}{2}, \tag{1.1}$$

where u_d denotes the desired state and $\delta_0 > 0$ is a given tolerance. Thus, the goal is to steer the heat-equation from an initial heat distribution u_0 as fast as possible into a ball of radius δ_0 around the desired state u_d. Without doubt, time-optimal control is a classical subject in control theory and we refer to, e.g., the monographs [19, 24, 15] for a general overview.

The aim of this article is to describe, for the first time, an appropriate fully space-time discrete version of (P) and to prove *a priori* discretization error estimates. We note that the problem is posed on a variable time-horizon, which introduces a nonlinear dependency on the additional variable T. Furthermore, the optimal solutions

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†Fakultät für Mathematik, Technische Universität München (lucas.bonifacius@tum.de; vexler@ma.tum.de).

‡Department of Scientific Computing, Florida State University (kpieper@fsu.edu).

to (P) are typically bang-bang (i.e. the set where the control does not equal the control bounds is a set of zero measure), since there are no control costs in the objective. This significantly complicates the numerical analysis of (P) compared to linear-quadratic problems with a fixed final time T considered in, e.g, [27, 28, 26] with control costs in the objective or [34] without control costs.

Time-optimal control problems have been extensively studied, and there are a few publications concerning their discretization in the context of parabolic equations. In [30, 21, 22, 37, 38, 17, 31, 20] the state equation is discretized in space only; see also the introduction of [5] for a detailed comparison. To the best of our knowledge, the only paper considering a full space-time discretization is [5] by the authors. However, in contrast to the aforementioned articles, the additional cost term

$$\frac{\alpha}{2}\|q\|^2_{L^2((0,T)\times\omega)} \quad \text{with } \alpha > 0 \tag{1.2}$$

is added to the objective functional in [5]. Unfortunately, the analysis given there does not apply in the case $\alpha = 0$. Moreover, the derived error estimates depend essentially on a *second order sufficient condition*, and the constants in the error estimates explode for $\alpha \to 0$. For this reason we cannot directly rely on those results.

To deal with the variable time horizon, the state and control variables are transformed to a reference interval. The state equation is discretized by means of the discontinuous Galerkin scheme in time and linear finite elements in space. We prove various convergence results; see also Table 1.1 for an overview. First, we show existence of solutions to the discrete problem and optimal order convergence of the optimal times $T_{kh} \to T$, where we only suppose that a linearized Slater condition on the continuous level holds. We emphasize that the latter condition is automatically satisfied in the setting with $u_d = 0$ and $0 \in Q_{ad}$ as considered in [21, 37, 31]; see [5, Theorem 3.10]. For example, in the important special case of purely time-dependent controls, we obtain the optimal convergence rate $\mathcal{O}(k + h^2)$ for the optimal time (up to logarithmic factors); Theorem 4.17. Here, k and h denote the temporal and spatial mesh size, respectively. In addition, we then assume that the following nodal set condition

$$|\{\, (t,x) \in I \times \omega \colon (B^*\bar{z})(t,x) = 0 \,\}| = 0 \tag{1.3}$$

holds. It requires the nodal set of the observation associated to the optimal adjoint state $\bar{z}$ (see Lemma 3.2) to be of measure zero, where $|\cdot|$ denotes the measure associated with the product set $I \times \omega$. Based on (1.3) we prove further convergence results for the controls. Note that this condition, which is guaranteed for, e.g., the linear heat-equation with a distributed control (cf. also [31]), ensures uniqueness and the bang-bang property for the optimal control. Here, we generalize a technique based on a structural assumption of the adjoint state. Precisely, we show that the nodal set condition (1.3) implies the existence of a continuous function $\Psi \colon [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that

$$|\{\, (t,x) \in I \times \omega \colon -\varepsilon \le (B^*\bar{z})(t,x) \le \varepsilon \,\}| \le \Psi(\varepsilon) \tag{1.4}$$

holds for all $\varepsilon > 0$. Based on (1.4), we derive an (abstract) growth condition. Furthermore, we prove that the nodal set condition (1.3) is a sufficient condition for local optimality (see Theorem 3.8), which seems to be a new result. Finally, assuming that the structural assumption (1.4) is valid with $\Psi(\varepsilon) = C\varepsilon^\kappa$ for some constants $C, \kappa > 0$, we obtain the convergence rate $\left(k + h^2\right)^\kappa$ in L^1 for the control variable. In this way,

Assumptions	$\lvert T_{kh} - T\rvert$	Control variable	Results
Linearized Slater condition	$\mathcal{O}\left(k+h^2\right)$	$\bar{q}_{kh} \rightharpoonup \bar{q}$ in L^s, $s<\infty$	Lemma 4.9
+ nodal set condition (1.3)	$\mathcal{O}\left(k+h^2\right)$	$\bar{q}_{kh} \to \bar{q}$ in L^s, $s<\infty$	Theorem 4.11
+ (1.4) with $\Psi(\varepsilon)=C\varepsilon^\kappa$	$\mathcal{O}\left(k+h^2\right)$	$\lVert\bar{q}_{kh}-\bar{q}\rVert_{L^1} \in \mathcal{O}\left((k+h^2)^\kappa\right)$	Theorem 4.17, Theorem 4.19

Table 1.1

Summary of convergence results, neglecting logarithmic terms. For simplicity we assume purely time-dependent control or distributed control with variational control discretization; see Corollary 4.23 for distributed control with piecewise and cellwise constant control.

we are able to prove results which directly apply to the global solutions of the discrete problem, without requiring that they are chosen close to the (unique) continuous optimal solution.

Our results are improvements over existing contributions in different aspects. First, and most importantly, we deal with fully discrete problem formulations, which is crucial since it directly reflects how the problems are solved in practice. Neglecting this fact we compare our results to the literature in the following. In [21] an error estimate for the optimal times is proved. In the particular case considered there, the linearized Slater condition holds uniformly for the discrete problem; see also [2, Section 5.6] for a generalization of [21] to fully discrete problems. Here, we obtain an optimal rate for the optimal times without any assumptions on the discrete solutions. For the case of a distributed control with the variational control discretization we can improve the result of [17] (see also [38] for a semilinear state equation) and obtain an optimal rate $\mathcal{O}(k+h^2)$. While the corresponding result from [37] with an explicit control discretization requires certain conditions $(H1)$ and $(H2)$, which so far could only be verified in very special situations, we assume a condition on the set of switchings which can be justified from practical observations; see Corollary 4.23. In [20] an error estimate of order $h^{2-\varepsilon}$ is obtained for a globally acting control and a semilinear state equation, whereas we can only prove $\mathcal{O}(k+h^{3/2})$. The reduced rate is due to pointwise control constraints in time and space instead of control bounds in $L^\infty((0,T);L^2)$. Last, to the best of our knowledge, this article is the first one dealing with quantitative error estimates for the control variable in the context of time-optimal problems, using the structural assumption (1.4).

Finally, we comment on the validity of (1.4) with $\Psi(\varepsilon)=C\varepsilon^\kappa$ for some $\kappa\in(0,1]$. Although it is difficult to quantify the structural assumption a priori, we try to check it numerically, which serves as an indicator for the assumption for the continuous problem. In case of purely time-dependent controls, $\kappa=1$ is valid in our examples and we observe the optimal order of convergence $\mathcal{O}(k+h^2)$ for the controls in L^1. This is related to the fact that the N_c time-dependent functions constituting $B^*\bar{z}$ have only a finite number of simple roots; cf. Remark 3.6. In contrast, in case of a distributed control, the structural assumption only appears to be satisfied with $\Psi(\varepsilon)=C\varepsilon^\kappa$ for some $\kappa<1$ in our numerical tests, which restricts the rate of convergence. Here, we observe a better rate of convergence than expected for the value of κ that we estimated numerically. However, the optimal theoretical value of κ remains an open problem.

Concerning the numerical realization, we use the bilevel algorithm from [3] that is based on an equivalent reformulation of (P). In the outer loop we employ a Newton method to find the root of a certain value function. For the inner loop, we use an accelerated conditional gradient method. It is worth mentioning that this approach does not require additional regularization terms such as (1.2) in the objective.

This paper is organized as follows. In Section 2 we introduce the notation and main assumptions. Necessary and sufficient optimality conditions are discussed in Section 3. Section 4 is devoted to the discretization of the optimal control problem and the corresponding error estimates. Last, in Section 5 we conclude with some numerical examples.

2. Notation and main assumptions. We generally work with the same notation and assumptions as in [5] that will be summarized in the following for the convenience of the reader. For a Lipschitz domain $\Omega \subset \mathbb{R}^d$, let $H_0^1(\Omega)$ denote the usual Sobolev space with zero trace on the boundary. Its dual space is $H^{-1}(\Omega)$. We use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Usually we drop the spatial domain Ω from the notation of the function spaces, if ambiguity is not to be expected. For a Hilbert space Z, $(\cdot,\cdot)_Z$ stands for its inner product. Last, c is a generic constant that may have different values at different appearances.

Throughout this paper we impose the following assumptions.

ASSUMPTION 2.1. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a polygonal or polyhedral and convex domain. Moreover, the initial value satisfies $u_0 \in H_0^1(\Omega)$.*

Concerning the control operator B we consider one of the following situations:

(i) Distributed control: Let $\omega \subseteq \Omega$ be the control domain that is polygonal or polyhedral as well. The control operator $B\colon L^2(\omega) \to L^2(\Omega)$ is the extension by zero and its adjoint $B^*\colon L^2(\Omega) \to L^2(\omega)$ is the restriction to ω operator.

(ii) Purely time-dependent control: For $N_c \in \mathbb{N}$, let $\omega = \{1, 2, \dots, N_c\}$ be equipped with the counting measure. The control operator is defined by $Bq = \sum_{n=1}^{N_c} q_n e_n$, where $e_n \in L^2(\Omega)$ are given form functions. Then we have $L^2(\omega) \cong \mathbb{R}^{N_c}$ and $B^*\colon L^2(\Omega) \to \mathbb{R}^{N_c}$ with $(B^*\varphi)_n = (e_n, \varphi)_{L^2(\Omega)}$ for $n = 1, 2 \dots, N_c$.

The space of admissible controls is defined as

$$Q_{ad} \coloneqq \left\{ q \in L^2(\omega)\colon q_a \le q \le q_b \ \text{ a.e. in } \ \omega \right\} \subset L^\infty(\omega)$$

for $q_a, q_b \in \mathbb{R}$ with $q_a < q_b$. Moreover, for $T > 0$ we set $Q(0,T) \coloneqq L^2((0,T)\times\omega)$ and

$$Q_{ad}(0,T) \coloneqq \{ q \in Q(0,T)\colon q(t) \in Q_{ad} \text{ a.e. } t \in (0,T) \} \subset L^\infty((0,T)\times\omega).$$

The set $(0,T)\times\omega$ is always equipped with the completion of the product measure. Furthermore, we use $W(0,T)$ to abbreviate $H^1((0,T); H^{-1}) \cap L^2((0,T); H_0^1)$, endowed with the canonical norm and inner product. The symbol $i_T\colon W(0,T) \to L^2$ denotes the continuous trace mapping $i_T u = u(T)$. Last, the control operator B is extended to $Q(0,T)$ by $(Bq)(t) = Bq(t)$ for any $q \in Q(0,T)$.

ASSUMPTION 2.2. *The terminal constraint G is defined by* (1.1) *for a fixed desired state $u_d \in H_0^1(\Omega)$ and a fixed $\delta_0 > 0$.*

REMARK 2.1. *The error analysis is also valid for more general terminal constraints. Concretely, we require that G is strictly convex, two times continuously Fréchet-differentiable, G'' is bounded on bounded sets in L^2, and $G'(u) \in H_0^1$ for any $u \in H_0^1$. We focus on* (1.1) *to make the main ideas clearly visible to the reader.*

In order to ensure existence of optimal solutions, we require the following basic conditions, which we assume to hold throughout this article.

Assumption 2.3. *There exist a finite time $T > 0$ and a feasible control $q \in Q_{ad}(0,T)$ such that the solution to the state equation of (P) satisfies $G(u(T)) \leq 0$. To exclude the trivial case, we additionally assume $G(u_0) > 0$.*

We also refer to [5, Remark 2.2] for a discussion of several situations where Assumption 2.3 is guaranteed to hold. Using these assumptions, existence of globally optimal solutions is guaranteed. In the following, "optimal solution" generally refers to a globally optimal solution, unless otherwise specified.

Proposition 2.2. *There exists an optimal solution $(T, \bar{q}) \in \mathbb{R}_+ \times Q_{ad}(0,T)$ to (P). Additionally, it holds $G(\bar{u}(T)) = 0$, where $\bar{u}$ is the state solution associated to the optimal control. Moreover, the final time T and the observation $\bar{u}(T)$ are unique.*

Proof. Existence follows by the direct method; cf. [5, Proposition 3.1]. Continuity of the trajectory $\bar{u}\colon [0,T] \to L^2$ implies both that $T > 0$ (using $G(u_0) > 0$) and that $G(\bar{u}(T)) = 0$ (otherwise a feasible control with a shorter time exists, which contradicts the optimality of the solution). Moreover, T is unique, since T is the objective functional. Now, assume that there exist $q_1, q_2 \in Q_{ad}(0,T)$ such that the corresponding states fulfill $G(u_1(T)) \leq 0$ and $G(u_2(T)) \leq 0$ with $u_1(T) \neq u_2(T)$. Then, by strict convexity of G, it holds $G(u_{1/2}(T)) = G((u_1(T) + u_2(T))/2) < 0$ for the state $u_{1/2}$ corresponding to $q_{1/2} = (q_1 + q_2)/2 \in Q_{ad}(0,T)$, which contradicts $G(\bar{u}(T)) = 0$. Thus, $\bar{u}(T)$ is unique. ∎

For the following arguments and error estimates, we require additional regularity of the state, which follows from maximal parabolic regularity; see, e.g., [14, Theorem 2.9b)].

Proposition 2.3. *Assume that $f \in L^s((0,T); L^p)$ for $s, p \in (1, \infty)$ and u_0 lies in the real interpolation space $(L^p, \mathcal{D}_{L^p}(-\Delta))_{1-1/s,s}$, where L^p is the domain of maximal definition of $-\Delta$ equipped with homogeneous Dirichlet boundary conditions on L^p. Then, the solution to the equation*

$$\partial_t u - \Delta u = f, \quad u(0) = u_0,$$

lies in the space $W^{1,s}((0,T); L^p) \cap L^s((0,T); \mathcal{D}_{L^p}(-\Delta))$, which is continuously embedded into $C([0,T]; (L^p, \mathcal{D}_{L^p}(-\Delta))_{1-1/s,s})$.

Concerning the state equation, we have $Bq \in L^s((0,T), L^p) \hookrightarrow L^2((0,T), L^2)$ (for any $2 \leq s < \infty$ and $2 \leq p < \infty$ for distributed controls and $p = 2$ for purely time-dependent controls). Under the general assumptions, we obtain that any state solution has the additional regularity $u \in H^1((0,T); L^2) \cap L^2((0,T); H^2 \cap H^1_0)$, using Proposition 2.3 with $p = s = 2$ and using the identifications $(L^2, \mathcal{D}_{L^2}(-\Delta))_{1/2,2} = H^1_0$ and $\mathcal{D}_{L^2}(-\Delta) = H^2 \cap H^1_0$. Moreover, the state solution is continuous with values in H^1_0. Higher regularity of u requires additional assumptions on the initial value, which will be required only for the optimal error estimates for distributed controls; see Subsection 4.3.

3. The time-optimal control problem.

In this section we introduce the transformation approach, which forms the basis of the discretization concept, and collect results for the continuous problem (P) which are fundamental for the error analysis.

3.1. Change of variables.

We transform the state equation to a fixed reference time interval in order to deal with the variable time horizon of (P). For $\nu \in \mathbb{R}_+$ we set $T_\nu(t) = \nu t$ and obtain the transformed state equation

$$\partial_t u - \nu \Delta u = \nu B q, \quad u(0) = u_0. \tag{3.1}$$

For each pair $(\nu, q) \in \mathbb{R}_+ \times Q(0,1)$ there exists a unique solution to the transformed state equation; see, e.g., [12, Theorem 2, Chapter XVIII, §3]. Abbreviating $I = (0,1)$, let $S\colon \mathbb{R}_+ \times Q_{ad}(0,1) \to W(0,1)$, $(\nu, q) \mapsto u$, denote the corresponding control-to-state mapping. We define the reduced terminal constraint by

$$g(\nu, q) \coloneqq G(i_1 S(\nu, q)),$$

where i_1 denotes the trace mapping. The transformed optimal control problem is

$$(\hat{P}) \qquad \text{Minimize } \nu \text{ subject to } g(\nu, q) \le 0, (\nu, q) \in \mathbb{R}_+ \times Q_{ad}(0,1).$$

Note that both problems (P) and ($\hat{P}$) are equivalent; see, e.g., [4, Proposition 4.6]. Throughout the paper, we will need the following differentiability property, which is obtained by standard arguments; cf. also [5, Section 3.1]. Concerning the stability estimates for the control-to-state mapping we also refer to [5, Proposition 3.3], where in particular the explicit dependence with respect to ν is given.

Lemma 3.1. *Let $\nu \in \mathbb{R}_+$ and $q \in Q(0,1)$. The control-to-state mapping S is twice continuously Fréchet-differentiable. Moreover, $\delta u = S'(\nu, q)(\delta\nu, \delta q) \in W(0,1)$ is the unique solution to*

$$\partial_t \delta u - \nu \Delta \delta u = \delta\nu (Bq + \Delta u) + \nu B \delta q, \quad \delta u(0) = 0,$$

for $(\delta\nu, \delta q) \in \mathbb{R} \times L^2(I \times \omega)$ and $\delta\tilde{u} = S''(\nu, q)(\delta\nu_1, \delta q_1; \delta\nu_2, \delta q_2) \in W(0,1)$ is the unique solution to

$$\partial_t \delta\tilde{u} - \nu \Delta \delta\tilde{u} = \delta\nu_1 \left(B\delta q_2 + \Delta \delta u_2\right) + \delta\nu_2 \left(B \delta q_1 + \Delta \delta u_1\right), \quad \delta\tilde{u}(0) = 0,$$

for $(\delta\nu_i, \delta q_i) \in \mathbb{R} \times L^2(I \times \omega)$ and $\delta u_i = S'(\nu, q)(\delta\nu_i, \delta q_i)$, $i = 1, 2$.

By means of Lemma 3.1, the reduced constraint mapping $g\colon \mathbb{R}_+ \times Q(0,1) \to \mathbb{R}$ is twice continuously Fréchet-differentiable. Moreover, the expressions

$$g'(\nu, q)(\delta\nu, \delta q) = (u(1) - u_d, \delta u(1))_{L^2}, \tag{3.2}$$

$$g''(\nu, q)(\delta\nu_1, \delta q_1; \delta\nu_2, \delta q_2) = (\delta u_1(1), \delta u_2(1))_{L^2} + (u(1) - u_d, \delta\tilde{u}(1))_{L^2}, \tag{3.3}$$

hold, where δu_1, δu_2, and $\delta\tilde{u}$ are defined as in Lemma 3.1. Last, for $\nu \in \mathbb{R}_+$, $q \in Q(0,1)$, $u = S(\nu, q)$, and $\mu \in \mathbb{R}$ we have the representation

$$\mu\, g'(\nu, q)^* = \begin{pmatrix} \int_0^1 \langle Bq + \Delta u, z\rangle \\ \nu B^* z \end{pmatrix}, \tag{3.4}$$

where $z \in W(0,1)$ is the unique solution to the *adjoint* equation

$$-\partial_t z - \nu \Delta z = 0, \quad z(1) = \mu(u(1) - u_d). \tag{3.5}$$

Here, we have used that $G'(u(1)) = u(1) - u_d$. Due to $u(1) \in H_0^1$ it also holds $z \in H^1(I; L^2) \cap L^2(I; H^2 \cap H_0^1) \hookrightarrow C(I; H_0^1)$.

3.2. First order necessary optimality conditions. General first order conditions for ($\hat{P}$) are given as follows (see, e.g., [29, 4]): For any optimal solution $(\bar{\nu}, \bar{q})$ of ($\hat{P}$) with associated $\bar{u} = S(\bar{\nu}, \bar{q})$, there exists a multiplier $\bar{\mu} > 0$ such that the following variational inequality holds:

$$\int_0^1 \langle B^* \bar{z}(t), q(t) - \bar{q}(t)\rangle \,\mathrm{d}t \ge 0 \quad \text{for all } q \in Q_{ad}(0,1), \tag{3.6}$$

where the optimal adjoint state $\bar{z} \in W(0,1)$ solves (3.5) with $\nu = \bar{\nu}$, $\mu = \bar{\mu}$, and $u(1) = \bar{u}(1)$. Note that (3.6) characterizes the optimal control on the set where $B^*\bar{z} \neq 0$, since it follows

$$\bar{q}(t,x) = \begin{cases} q_a & \text{if } (B^*\bar{z})(t,x) > 0, \\ q_b & \text{if } (B^*\bar{z})(t,x) < 0. \end{cases} \tag{3.7}$$

The optimality of the time variable is characterized by the Hamiltonian condition

$$\mu_0 + \langle B\bar{q}(t) + \Delta\bar{u}(t), \bar{z}(t)\rangle = 0 \quad \text{for all } t \in [0,1], \tag{3.8}$$

which holds true for either $\mu_0 = 0$ or $\mu_0 = 1$. In the latter case, we call the optimality conditions *qualified*. The Hamiltonian condition (3.8) can be derived to hold almost everywhere by introducing the time transformation ν as a function with $\nu, 1/\nu \in L^\infty(0,1)$ (instead of $\nu \in \mathbb{R}_+$); see, e.g., [4, Section 4.2] for details. Using continuity of $\bar{u}(t)$ and $\bar{z}(t)$ with values in H_0^1, and the fact that with (3.7) we have $\langle B\bar{q}(t), \bar{z}(t)\rangle = (q_a, (B^*\bar{z}(t))_+)_{L^2(\omega)} - (q_b, (B^*\bar{z}(t))_-)_{L^2(\omega)}$, we conclude that the Hamiltonian condition (3.8) holds for all $t \in [0,1]$.

For the purposes of the error analysis, we require the following *linearized Slater* condition, which will turn out not only to be sufficient, but also to be necessary for the qualified conditions:

Assumption 3.1. *Let $(\bar{\nu}, \bar{q})$ be a solution of $(\hat{P})$. We assume that*

$$\bar{\eta} := -\partial_\nu g(\bar{\nu}, \bar{q}) > 0. \tag{3.9}$$

Note that, for now, the constant $\bar{\eta}$ depends on the potentially non-unique optimal control $\bar{q}$. However, as a consequence of the optimality conditions, it will turn out that it is uniquely defined. Let us comment on the relation of (3.9) to a linearized Slater condition: Due to $g(\bar{\nu}, \bar{q}) = 0$, the point $\breve{\chi}^\gamma = (\bar{\nu} + \gamma, \bar{q}) \in \mathbb{R}_+ \times Q_{ad}(0,1)$ defined for any $\gamma > 0$ satisfies

$$g(\bar{\chi}) + g'(\bar{\chi})(\breve{\chi}^\gamma - \bar{\chi}) = -\bar{\eta}\,\gamma < 0.$$

Thus, in (3.9) we assume that a special form of the linearized Slater condition holds. However, since we will see that this condition is equivalent to qualified optimality conditions, this is not restrictive.

To state the optimality conditions in an abstract way, we also introduce the Lagrange function as

$$\mathcal{L}\colon \mathbb{R}_+ \times Q(0,1) \times \mathbb{R} \to \mathbb{R}, \quad \mathcal{L}(\nu, q, \mu) := \nu + \mu\, g(\nu, q).$$

Now, optimality conditions for $(\hat{P})$ in qualified form can be given as follows: there exists a $\bar{\mu} \geq 0$, such that

$$\partial_{(\nu,q)}\mathcal{L}(\bar{\nu}, \bar{q}, \bar{\mu})(\delta\nu, q - \bar{q}) \geq 0 \quad \text{for all } (\delta\nu, q) \in \mathbb{R} \times Q_{ad}(0,1). \tag{3.10}$$

Finally, we connect the different conditions and prove the previously claimed results.

Lemma 3.2. *For any optimal solution $(\bar{\nu}, \bar{q})$ of $(\hat{P})$ the following conditions are equivalent:*

(i) Assumption 3.1 is satisfied.

(ii) There exists a $\bar{\mu} > 0$ such that the conditions (3.6) and (3.8) are fulfilled with $\mu_0 = 1$, where $\bar{u} = S(\bar{\nu}, \bar{q})$ and $\bar{z}$ solves (3.5).

(iii) There exists a $\bar\mu > 0$ *such that the Lagrangian condition* (3.10) *holds. Moreover, the constant* $\bar\eta$ *from* (3.9) *is independent of* $(\bar\nu, \bar q)$*, the multiplier is given as* $\bar\mu = 1/\bar\eta$ *and the adjoint state* $\bar z$ *is uniquely determined.*

Proof. Let $(\bar\nu, \bar q)$ be any optimal solution of $(\hat P)$.

(ii) ⇒ *(iii).* Clearly, (3.10) is equivalent to the conditions $\partial_q \mathcal{L}(\bar\nu, \bar q, \bar\mu)(q - \bar q) \geq 0$ for all $q \in Q_{ad}(0,1)$ and $\partial_\nu \mathcal{L}(\bar\nu, \bar q, \bar\mu) = 1 + \bar\mu \partial_\nu g(\bar\nu, \bar q) = 0$. The first is (3.6) and the second follows from (3.8) by taking the integral over the time interval. Thus, *(ii)* implies *(iii)*.

(iii) ⇒ *(i).* From (3.10) we deduce that $0 = \partial_\nu \mathcal{L}(\bar\nu, \bar q, \bar\mu) = 1 + \bar\mu\, \partial_\nu g(\bar\nu, \bar q)$. Hence, condition (3.9) from Assumption 3.1 holds.

(i) ⇒ *(ii).* First note that the linearized Slater condition (3.9) allows for exact penalization of $(\hat P)$; see [6, Theorem 2.87, Proposition 3.111]. The qualified optimality conditions for some $\bar\mu > 0$ now follow as in the proof of [4, Theorem 4.12].

To obtain uniqueness of $\bar\eta$, we divide (3.8) at $t = 1$ by $\bar\mu$ to obtain that

$$\begin{aligned} -\bar\eta = -1/\bar\mu &= \langle B\bar q(1) + \Delta \bar u(1), \bar z(1)/\bar\mu \rangle \\ &= (q_a, (B^*\zeta)_+)_{L^2(\omega)} - (q_b, (B^*\zeta)_-)_{L^2(\omega)} - (\nabla \bar u(1), \nabla\zeta)_{L^2(\Omega)}, \end{aligned}$$

where $\zeta = \bar z(1)/\bar\mu = \bar u(1) - u_d$ and $(\cdot)_+$ and $(\cdot)_-$ denote the positive and negative part, respectively. Here, we have used $\bar u, \bar z \in C([0,1], H_0^1)$ and the expression (3.7) for the first two terms. Since $\zeta = \bar u(1) - u_d$ is uniquely defined, independently of $\bar q$ (see Proposition 2.2), the same holds for $\bar\eta$ and $\bar\mu$. ◻

Assumption 3.1 will be sufficient to provide optimal order error estimates for the optimal time of (P). In this article, we also derive error estimates for the controls. This requires the following condition, which also implies that the controls are bang-bang, i.e. they take only the values q_a or q_b, except on a set of measure zero.

Assumption 3.2. *We assume that the* nodal set condition

$$|\{\, (t,x) \in I \times \omega \colon (B^*\bar z)(t,x) = 0 \,\}| = 0$$

holds, where $|\cdot|$ *denotes the measure associated with* $I \times \omega$*.*

Proposition 3.3. *If Assumption 3.2 holds, then* $\bar q$ *is bang-bang and unique.*

Proof. From the optimality condition (3.7), the uniqueness of $\bar z$ and Assumption 3.2 we immediately infer that $\bar q$ is uniquely determined as either q_a or q_b, except on the zero level set of $B^*\bar z$ of measure zero. ◻

Remark 3.4. *We comment on situations in which Assumption 3.2 is guaranteed to hold.*

- *(i) In the case of a distributed control on an open subset* $\omega \subset \Omega$*, Assumption 3.2 is satisfied; see [15, Theorem 4.7.12]. Note that [18, Theorem 1.1] is only applied for interior subsets of the cylinder* $I \times \Omega$*. Employing interior regularity of the solution to the heat-equation with zero right-hand side, the general boundary regularity of this article is sufficient for the argument.*
- *(ii) Suppose purely time-dependent controls, i.e.* $B \colon \mathbb{R}^{N_c} \to H^{-1}$*,* $Bq = \sum_{i=1}^{N_c} q_i e_i$ *and set* $B_i \colon \mathbb{R} \to H^{-1}$*,* $B_i q = q e_i$*. If* $(-\Delta, B_i)$ *is approximately controllable for all* $i = 1, 2, \ldots, M$*, i.e.* $q \mapsto \int_0^1 \mathrm{e}^{(1-s)\Delta} B_i q(s)\, \mathrm{d}s$ *has dense range in* $L^2(\Omega)$*, then Assumption 3.2 holds. This follows from analyticity of the semigroup generated by* Δ *and [32, Theorem 11.2.1, Definition 6.1.1]. In the context of time-optimal*

control of ODEs, approximate controllability of $(-\Delta, B_i)$ *for all* i *is referred to as normality; see, e.g., [19, Section II.16] or [24, Section III.3].*
We note that the assumption of normality implies that the Dirichlet Laplacian on the domain Ω *has simple spectrum (all eigenvalues have geometric multiplicity one); see, e.g., [1, Theorem 1.3]. Unfortunately, this is not fulfilled for all domains – we refer to [1, Section 3.4] for a thorough discussion. While this limits the applicability of the above criterion to certain domains, we emphasize that it is only a sufficient condition.*

3.3. Sufficient optimality conditions for bang-bang controls. Let $(\bar\nu, \bar q) \in \mathbb{R}_+ \times Q_{ad}(0,1)$ such that the necessary optimality conditions from Lemma 3.2 hold with $\bar z \in W(0,1)$ the adjoint state $\bar\mu > 0$ the Lagrange multiplier.

PROPOSITION 3.5. *Let Assumption 3.2 hold. Then there exists a concave, continuous, strictly monotonically increasing function* $\Psi\colon [0,\infty) \to [0,\infty)$ *with* $\Psi(0) = 0$ *and* $\lim_{\varepsilon\to\infty} \Psi(\varepsilon) = \infty$ *such that for all* $\varepsilon > 0$ *it holds*

$$|\{\,(t,x) \in I \times \omega\colon\ -\varepsilon \le (B^*\bar z)(t,x) \le \varepsilon\,\}| \le \Psi(\varepsilon). \tag{3.11}$$

Proof. Define $\Phi(\varepsilon) := |\{\,(t,x) \in I \times \omega\colon\ -\varepsilon \le (B^*\bar z)(t,x) \le \varepsilon\,\}|$, which is the left-hand side of (3.11). Then, $0 \le \Phi(\varepsilon) \le |I \times \omega| < \infty$. Moreover, one easily shows that Φ is continuous from the right, thus, in particular we have $\lim_{\varepsilon\searrow 0} \Phi(0) = 0$. Now, we define the concave hull of Φ by $\widetilde\Phi = -\left((-\Phi)^*\right)^*$, where $(\cdot)^*$ denotes the convex conjugate (also known as Fenchel-Legendre transform). Concretely,

$$\widetilde\Phi(\varepsilon) = -\sup_{\gamma\in\mathbb{R}}\left[\varepsilon\gamma - \sup_{\varepsilon'\ge 0}(\varepsilon'\gamma + \Phi(\varepsilon'))\right] = \inf_{\gamma\ge 0}\left[\varepsilon\gamma + \sup_{\varepsilon'\ge 0}(\Phi(\varepsilon') - \varepsilon'\gamma)\right],$$

where, in the last line we have substituted γ by $-\gamma$. By the standard properties of the concave hull (see, e.g., [11, Corollary 4.22]), we have

$$\widetilde\Phi(\varepsilon) \ge \Phi(\varepsilon) \quad \text{for all } \varepsilon \ge 0, \tag{3.12}$$

and $\widetilde\Phi$ is upper semi-continuous. Furthermore, we can verify that

$$\widetilde\Phi(0) = \inf_{\gamma\ge 0}\sup_{\varepsilon'\ge 0}(\Phi(\varepsilon') - \varepsilon'\gamma) = 0.$$

First, $\widetilde\Phi(0) \ge 0$ follows from (3.12) and Assumption 3.2. Assume that $\widetilde\Phi(0) > 0$. Then for each $\gamma > 0$ there is $\varepsilon = \varepsilon(\gamma) > 0$ such that $\widetilde\Phi(0) \le \Phi(\varepsilon) - \varepsilon\gamma + \widetilde\Phi(0)/2$. Hence, $\widetilde\Phi(0)/2 \le \Phi(\varepsilon)$ and $\varepsilon < \Phi(\varepsilon)/\gamma$. Using the boundedness of Φ, we find $\varepsilon(\gamma) \to 0$ for $\gamma \to \infty$. However, $\lim_{\varepsilon\searrow 0} \Phi(\varepsilon) = 0$ which is a contradiction to $\widetilde\Phi(0)/2 \le \Phi(\varepsilon)$. Finally, $\widetilde\Phi$ is continuous, since it is Lipschitz-continuous on $(0,\infty)$ (see, e.g., [11, Theorem 2.34]) and $\lim_{\varepsilon\to 0} \widetilde\Phi(\varepsilon) = 0$ with upper semi-continuity. We conclude the proof by setting $\Psi(\varepsilon) = \widetilde\Phi(\varepsilon) + \varepsilon$ to guarantee strict monotonicity and $\lim_{\varepsilon\to\infty} \Psi(\varepsilon) = \infty$. ∎

REMARK 3.6. *(i) In related contexts, the condition (3.11) is an assumption; see, e.g., [36, 13, 35, 10, 33, 34, 9] for the special case* $\Psi(\varepsilon) = C\varepsilon^\kappa$ *with constants* $C > 0$ *and* $\kappa > 0$*. However, we derived the existence of such a function* Ψ*, requiring only the nodal set condition from Assumption 3.2, which is guaranteed in many examples.*

(ii) *In the context of a distributed control, where $B^*\bar{z} = \bar{z}|_{I\times\omega}$, a sufficient condition for a strong form of (3.11) is often given as follows (see, e.g., [13, Lemma 3.2]): Assume $\bar{z} \in C^1(\overline{I\times\omega})$ and that there exists a constant $c > 0$ such that*

$$\|\nabla_{(t,x)}\bar{z}(t,x)\|_{\mathbb{R}^{d+1}} \geq c$$

for all $(t,x) \in I \times \omega$ such that $\bar{z}(t,x) = 0$, then (3.11) holds with $\Psi(\varepsilon) = C\varepsilon$.

(iii) *Condition (3.11) is also compatible with purely time-dependent controls. In this case the structural condition concretely reads as*

$$\sum_{n=1}^{N_c} |\{\, t \in I \colon |(B^*\bar{z}(t))_n| \leq \varepsilon \,\}| \leq \Psi(\varepsilon).$$

In the context of optimal control problems with ODEs, the functions $t \mapsto \sigma_n(t) = (B^*\bar{z}(t))_n = (e_n, \bar{z}(t))_{L^2(\Omega)}$ are referred to as switching functions. Here, one typically assumes that each σ_n has only finitely many roots with non-vanishing first derivatives (see, e.g., [16, 25]), which again implies (3.11) with $\Psi(\varepsilon) = C\varepsilon$.

Clearly, any function Ψ with the properties as given in Proposition 3.5 possesses a convex, strictly monotonously increasing and continuous inverse $\Psi^{-1}\colon [0,\infty) \to [0,\infty)$ with $\Psi^{-1}(0) = 0$ and $\lim_{x\to\infty} \Psi^{-1}(x) = \infty$. The proof of sufficiency of the structural assumption for a pair $(\bar{\nu}, \bar{q})$ to be locally optimal relies now on the following result.

PROPOSITION 3.7. *Let $(\bar{\nu}, \bar{q}) \in \mathbb{R}_+ \times Q_{ad}(0,1)$ and $\bar{\mu} > 0$ satisfy the first order necessary optimality conditions $g(\bar{\nu}, \bar{q}) = 0$ and (3.10). Moreover, suppose that Assumption 3.2 holds. Then there is $c_0 > 0$ such that*

$$\partial_q \mathcal{L}(\bar{\nu}, \bar{q}, \bar{\mu})(q - \bar{q}) \geq \frac{\bar{\nu}}{2}\Psi^{-1}\left(c_0\|q - \bar{q}\|_{L^1(I\times\omega)}\right)\|q - \bar{q}\|_{L^1(I\times\omega)} \tag{3.13}$$

for all $q \in Q_{ad}(0,1)$, where Ψ is from Proposition 3.5.

Proof. The proof is along the lines of [10, Proposition 2.7] with slight modifications. For $q \in Q_{ad}(0,1)$, we take $\varepsilon := \Psi^{-1}((2|q_b - q_a|)^{-1}\,\|q - \bar{q}\|_{L^1(I\times\omega)})$. Now the estimate follows as in [10] with $c_0 = (2|q_b - q_a|)^{-1}$. ∎

Assumption 3.2 allows to prove the following growth condition without two norm discrepancy. In particular, we infer that the nodal set condition is sufficient for local optimality for the time-optimal control problem (P).

It is worth mentioning that due to the particular objective functional we do not require additional assumptions such as conditions on the second derivative of the Lagrange function; cf. [8, Theorem 2.2] and [10, Theorem 2.8].

THEOREM 3.8. *Let $(\bar{\nu}, \bar{q}) \in \mathbb{R}_+ \times Q_{ad}(0,1)$ and $\bar{\mu} > 0$ satisfy the first order necessary optimality conditions $g(\bar{\nu}, \bar{q}) = 0$ and (3.10). Moreover, suppose that Assumption 3.2 holds. Then there exists a constant $\delta > 0$ such that*

$$\frac{\bar{\nu}}{6}\Psi^{-1}\left(c_0\|q - \bar{q}\|_{L^1(I\times\omega)}\right)\|q - \bar{q}\|_{L^1(I\times\omega)} \leq \nu - \bar{\nu} \tag{3.14}$$

for all $|\nu - \bar{\nu}| \leq \delta$ and $q \in Q_{ad}(0,1)$ such that (ν, q) is admissible for ($\hat{P}$), i.e. $g(\nu, q) \leq 0$. Thus $(\bar{\nu}, \bar{q})$ is locally optimal for ($\hat{P}$).

In order to prove the result, we first observe that the second derivative of the Lagrange function can be bounded below as follows.

PROPOSITION 3.9. *Let $(\bar\nu,\bar q) \in \mathbb{R}_+ \times Q_{ad}(0,1)$, $\bar\mu > 0$, and $0 < \nu_{\min} < \nu_{\max}$. There is $c > 0$ such that*

$$\partial^2_{(\nu,q)}\mathcal{L}(\nu_\xi, q_\xi, \bar\mu)[\nu-\bar\nu, q-\bar q]^2 \geq -c|\nu-\bar\nu|^2 - c|\nu-\bar\nu|\|q-\bar q\|_{L^2(I\times\omega)}$$

for all $\nu, \nu_\xi \in \mathbb{R}_+$, $q, q_\xi \in Q_{ad}(0,1)$ with $\nu_{\min} \leq \nu, \nu_\xi \leq \nu_{\max}$.

Proof. Set $\delta\nu = \nu - \bar\nu$ and $\delta q = q - \bar q$. Define $u_\xi = S(\nu_\xi, q_\xi)$, $\delta u = S'(\nu_\xi, q_\xi)(\delta\nu, \delta q)$, and $\delta\tilde u = S''(\nu_\xi, q_\xi)[\delta\nu, \delta q]^2$. Moreover, let z_ξ be the corresponding adjoint state with terminal value $\bar\mu(u_\xi(1) - u_d)$. Then we observe

$$\begin{aligned}\bar\mu(u_\xi(1) - u_d, \delta\tilde u(1))_{L^2} &= (z_\xi(1), \delta\tilde u(1))_{L^2} - (z_\xi(0), \delta\tilde u(0))_{L^2}\\ &= \int_0^1 \langle \partial_t \delta\tilde u, z_\xi\rangle + \int_0^1 \langle \partial_t z_\xi, \delta\tilde u\rangle = \int_0^1 \langle \partial_t \delta\tilde u, z_\xi\rangle - \int_0^1 \langle \bar\nu\Delta\delta\tilde u, z_\xi\rangle\\ &= 2\delta\nu \int_0^1 \langle B\delta q + \Delta\delta u, z_\xi\rangle \,\mathrm{d}t.\end{aligned}$$

Thus, using (3.3), we find

$$\begin{aligned}\partial^2_{(\nu,q)}\mathcal{L}(\nu_\xi, q_\xi, \bar\mu)[\delta\nu, \delta q]^2 &= \bar\mu\|\delta u(1)\|^2_{L^2} + 2\delta\nu \int_0^1 \langle B\delta q + \Delta\delta u, z_\xi\rangle \,\mathrm{d}t\\ &\geq -2|\delta\nu| \int_0^1 |\langle B\delta q + \Delta\delta u, z_\xi\rangle| \,\mathrm{d}t.\end{aligned}$$

The Cauchy-Schwarz inequality and the stability estimates for $u_\xi, \delta u$, and z_ξ with Lemma 3.1 further imply

$$\begin{aligned}\partial^2_{(\nu,q)}\mathcal{L}(\nu_\xi, q_\xi, \bar\mu)[\delta\nu, \delta q]^2 &\geq -2|\delta\nu| \left(\|B\delta q\|_{L^2(I;H^{-1})} + \|\delta u\|_{L^2(I;H^1_0)}\right) \|z_\xi\|_{L^2(I;H^1_0)}\\ &\geq -c|\delta\nu| \left(\|B\delta q\|_{L^2(I;H^{-1})} + \frac{|\delta\nu|}{\nu_\xi}\left(\|Bq_\xi\|_{L^2(I;H^{-1})} + \|u_\xi\|_{L^2(I;H^1_0)}\right)\right) \|z_\xi(1)\|_{L^2}.\end{aligned}$$

Since q_ξ is uniformly bounded due to boundedness of $Q_{ad}(0,1)$ and ν_ξ is uniformly bounded from below and from above, there exists a constant $c > 0$ such that

$$\partial^2_{(\nu,q)}\mathcal{L}(\nu_\xi, q_\xi, \bar\mu)[\delta\nu, \delta q]^2 \geq -c|\delta\nu|^2 - c|\delta\nu|\|\delta q\|_{L^2(I\times\omega)}$$

proving the assertion. ∎

Last, we require a technical result, which follows from the Fenchel-Young inequality.

PROPOSITION 3.10. *Let $\varepsilon > 0$, $c_0 > 0$, and let Ψ satisfy the assumptions of Proposition 3.5. Then there exists a (convex) function $F\colon [0,\infty) \to [0,\infty)$ such that*

$$xy \leq \varepsilon\Psi^{-1}(c_0x^2)x^2 + F(y) \quad \textit{for all } x, y \in [0,\infty),$$

and $F(0) = 0$ and $\lim_{y\to 0} F(y)/y = 0$.

Proof. We abbreviate $H(x) = \varepsilon\Psi^{-1}(c_0x^2)x^2$. Note first that Ψ^{-1} is convex as the inverse of a concave function. Thus, it is Lipschitz continuous on the interior of its domain $\mathbb{R}_+$; see, e.g., [11, Theorem 2.34]. Therefore, we can apply Rademacher's theorem and the chain rule to compute the derivative

$$H'(x) = 2\varepsilon\left(c_0(\Psi^{-1})'(c_0x^2)x^3 + \Psi^{-1}(c_0x^2)x\right),$$

which is defined almost everywhere. Using again that Ψ^{-1} is convex, we verify that H' is monotonically increasing. Hence, H is convex, locally Lipschitz continuous, and strictly monotonically increasing. Now, we define $F = H^*$, where $H^*(y) = \sup_{x\geq 0}[yx - H(x)]$ is the convex conjugate of H. Clearly, $F(y) \geq F(0) = H(0) = 0$ for all $y \geq 0$. Thus, the desired inequality is given by the Fenchel-Young inequality $xy \leq H(x) + H^*(y)$.

It remains to verify that the directional derivative of F at zero, i.e. $F'(0,+1) = \lim_{y\to 0}(F(y) - F(0))/(y - 0) = \lim_{y\to 0} F(y)/y$, is equal to zero. We consider the subdifferential $\partial F(0)$, which reads in this case as

$$\partial F(0) = \{\, v \in \mathbb{R} \colon F(x)/x \geq v \text{ for all } x > 0 \,\}.$$

Assume that $F'(0,+1) > 0$. Then $(-\infty, F'(0,+1)] \subset \partial F(0)$. Therefore, we deduce that $0 \in \partial H(x)$ for all $x \leq F'(0,+1)$ by the subdifferential inversion formula; see, e.g., [11, Exercise 4.27]. This implies that these points are global minima of H and thus $H(x) = 0$ for $x \leq F'(0,+1)$, which contradicts the strict monotonicity of H. ∎

Finally, we give the proof of the main result of this section.

Proof of Theorem 3.8. Let $(\nu, q) \in \mathbb{R}_+ \times Q_{ad}(0,1)$ be admissible. Set $\delta\nu = \nu - \bar\nu$ and $\delta q = q - \bar q$. Using feasibility of (ν, q), the facts that $\bar\mu > 0$ and $g(\bar\nu, \bar q) = 0$ from the necessary optimality conditions for $(\bar\nu, \bar q)$, as well as Taylor expansion we find

$$\begin{aligned}\nu - \bar\nu &\geq \nu + \bar\mu g(\nu, q) - (\bar\nu + \bar\mu g(\bar\nu, \bar q)) = \mathcal{L}(\nu, q, \bar\mu) - \mathcal{L}(\bar\nu, \bar q, \bar\mu)\\ &= \partial_{(\nu,q)}\mathcal{L}(\bar\nu, \bar q, \bar\mu)(\delta\nu, \delta q) + \frac{1}{2}\partial^2_{(\nu,q)}\mathcal{L}(\nu_\xi, q_\xi, \bar\mu)[\delta\nu, \delta q]^2,\end{aligned}$$

with appropriate $\nu_\xi = \bar\nu + \xi_\nu(\nu - \bar\nu)$ and $q_\xi = \bar q + \xi_q(q - \bar q)$ for $0 \leq \xi_\nu, \xi_q \leq 1$. Thus, according to Proposition 3.9 there is $c_1 > 0$ such that

$$\nu - \bar\nu \geq \partial_{(\nu,q)}\mathcal{L}(\bar\nu, \bar q, \bar\mu)(\delta\nu, \delta q) - c_1|\delta\nu|^2 - c_1|\delta\nu|\|\delta q\|_{L^2(I\times\omega)}.$$

Since $\partial_\nu\mathcal{L}(\bar\nu, \bar q, \bar\mu) = 0$ and using Proposition 3.7, this further implies

$$\nu - \bar\nu \geq \frac{\bar\nu}{2}\Psi^{-1}\left(c_0\|\delta q\|_{L^1(I\times\omega)}\right)\|\delta q\|_{L^1(I\times\omega)} - c_1|\delta\nu|^2 - c_1|\delta\nu|\|\delta q\|_{L^2(I\times\omega)}.$$

Clearly, we have

$$\|\delta q\|_{L^2(I\times\omega)} \leq \|\delta q\|^{1/2}_{L^\infty(I\times\omega)}\|\delta q\|^{1/2}_{L^1(I\times\omega)} \leq |q_b - q_a|^{1/2}\|\delta q\|^{1/2}_{L^1(I\times\omega)}.$$

Employing Proposition 3.10 for some $\varepsilon > 0$ to determined later, we obtain

$$|\delta\nu|\|\delta q\|^{1/2}_{L^1(I\times\omega)} \leq \varepsilon\Psi^{-1}\left(c_0\|\delta q\|_{L^1(I\times\omega)}\right)\|\delta q\|_{L^1(I\times\omega)} + F(|\delta\nu|).$$

Taking $\varepsilon = \left(4c_1|q_b - q_a|^{1/2}\right)^{-1}\bar\nu$, we obtain

$$\delta\nu + F(|\delta\nu|) + c_1|\delta\nu|^2 \geq \frac{\bar\nu}{4}\Psi^{-1}\left(c_0\|q - \bar q\|_{L^1(I\times\omega)}\right)\|q - \bar q\|_{L^1(I\times\omega)}.$$

Finally, using that $\lim_{y\to 0} F(y)/y = 0$, we deduce

$$F(|\delta\nu|) + c_1|\delta\nu|^2 \leq \frac{1}{2}|\delta\nu|, \quad |\delta\nu| \leq \delta,$$

for $\delta > 0$ sufficiently small, concluding the proof. ∎

REMARK 3.11. *For the special case $\Psi(\varepsilon) = C\varepsilon^\kappa$, in Proposition 3.7 we obtain*

$$\partial_q \mathcal{L}(\bar\nu, \bar q, \bar\mu)(q - \bar q) \geq c\|q - \bar q\|^{1+1/\kappa}_{L^1(I\times\omega)}.$$

Moreover, the growth condition from Theorem 3.8 reads as follows: There are $\delta > 0$ and $c > 0$ such that

$$c\|q - \bar q\|^{1+1/\kappa}_{L^1(I\times\omega)} \leq \nu - \bar\nu$$

for all admissible $(\nu, q) \in \mathbb{R}_+ \times Q_{ad}(0,1)$ with $|\nu - \bar\nu| \leq \delta$.

4. A priori discretization error estimates. The aim of this section is the derivation of discretization error estimates for bang-bang controls based on the different conditions of the preceding section. We consider the same assumptions concerning the temporal and spatial discretization of the partial differential equation as in [5], which will be summarized in the following for the convenience of the reader. Let

$$[0,1] = \{\,0\,\} \cup I_1 \cup I_2 \cup \ldots \cup I_M$$

be a partitioning of the reference time interval $[0,1]$ with disjoint subintervals $I_m = (t_{m-1}, t_m]$ of size k_m defined by the time points

$$0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = 1.$$

Moreover, let k denote the time discretization parameter defined as the piecewise constant function $k|_{I_m} = k_m$ for all $m = 1, 2, \ldots, M$. We also set $k = \max k_m$ the maximal time step size. The temporal mesh is assumed to be regular in the sense of [26, Section 3.1].

Concerning the spatial discretization, let $\mathcal{T}_h = \{\,K\,\}$ be a mesh consisting of triangular or tetrahedral cells K that form a non-overlapping cover of the domain Ω. The corresponding spatial discretization parameter h is the cellwise constant function $h|_K = h_K$, where h_K is the diameter of the cell K. In addition, we set $h = \max h_K$. Let $V_h \subset H^1_0$ denote the subspace of continuous and cellwise linear functions. We assume that the L^2-projection onto V_h, denoted by $\Pi_h \colon L^2 \to V_h$, is stable in H^1. This is satisfied if the mesh is globally quasi-uniform, but weaker conditions are known; see [7]. We construct the space-time finite element space in a standard way by setting

$$X_{k,h} = \left\{ v_{kh} \in L^2(I; V_h) \colon v_{kh}|_{I_m} \in \mathcal{P}_0(I_m; V_h),\ m = 1, 2, \ldots, M \right\},$$

where $\mathcal{P}_0(I_m; V_h)$ is the space of constant functions on the time interval I_m with values in V_h. Moreover, for $\varphi_k \in X_{k,h}$ we set $\varphi_{k,m} \coloneqq \varphi_k(t_m)$ with $m = 1, 2, \ldots, M$, as well as $[\varphi_k]_m \coloneqq \varphi_{k,m+1} - \varphi_{k,m}$ for $m = 1, 2, \ldots, M-1$.

In order to introduce the discrete version to the state equation, consider the form $\mathrm{B} \colon \mathbb{R} \times X_{k,h} \times X_{k,h} \to \mathbb{R}$ defined as

$$\begin{aligned}\mathrm{B}(\nu, u_{kh}, \varphi_{kh}) \coloneqq \sum_{m=1}^{M} \langle \partial_t u_{kh}, \varphi_{kh} \rangle_{L^2(I_m; L^2)} \\ + \nu(\nabla u_{kh}, \nabla \varphi_{kh})_{L^2(I;L^2)} + \sum_{m=2}^{M} ([u_{kh}]_{m-1}, \varphi_{kh,m}) + (u_{kh,1}, \varphi_{kh,1}).\end{aligned}$$

Given $\nu \in \mathbb{R}_+$ and $q \in Q(0,1)$ the discrete state equation reads as follows: Find a state $u_{kh} \in X_{k,h}$ satisfying

$$\mathrm{B}(\nu, u_{kh}, \varphi_{kh}) = \nu(Bq, \varphi_{kh})_{L^2(I;L^2)} + (u_0, \varphi_{kh,1})_{L^2} \quad \text{for all } \varphi_{kh} \in X_{k,h}. \tag{4.1}$$

We also introduce the discrete Laplace operator $-\Delta_h \colon V_h \to V_h$ by

$$-(\Delta_h u_h, \varphi_h)_{L^2} = (\nabla u_h, \nabla \varphi_h)_{L^2}, \quad \varphi_h \in V_h.$$

Next, we introduce a discrete control variable. To consider different discretization schemes in one consistent notation, we introduce the operator I_σ onto the possibly discrete control space $Q_\sigma(0,1) \subset L^2(I \times \omega)$, where σ is abstract parameter for the control discretization. To simplify the discussion, we assume that in the case of a distributed control a subset denoted $\mathcal{T}_h^\omega$ of the mesh $\mathcal{T}_h$ is a non-overlapping cover of ω. Furthermore, we suppose that the optimal control $\bar{q}$ satisfies

$$\|B\left(\bar{q} - \mathrm{I}_\sigma \bar{q}\right)\|_{L^2(I;H^{-1})} \leq \sigma(k,h), \tag{4.2}$$

where $\sigma(k,h) \to 0$ as $k,h \to 0$ and $\mathrm{I}_\sigma Q_{ad}(0,1) \subset Q_{ad}(0,1)$. We also simply write $\mathrm{I}_\sigma(\nu,q) = (\nu, \mathrm{I}_\sigma q)$ using the same symbol and define $Q_{ad,\sigma}(0,1) = Q_\sigma(0,1) \cap Q_{ad}(0,1)$. Concrete discretization schemes for the control will be discussed at the end of this section.

We define the discretized optimal control problem corresponding to ($\hat{P}$) by

$$(\hat{P}_{kh}) \qquad \text{Minimize } \nu_{kh} \quad \text{subject to} \quad \nu_{kh} \in \mathbb{R}_+,\ q_{kh} \in Q_{ad,\sigma}(0,1), \quad g_{kh}(\nu_{kh}, q_{kh}) \leq 0,$$

where $g_{kh}(\nu_{kh}, q_{kh}) = G(i_1 S_{kh}(\nu_{kh}, q_{kh}))$ and S_{kh} denotes the control-to-state mapping for the discrete state equation (4.1). In the following, $\{(k,h)\}$ is always a sequence of positive mesh sizes converging to zero.

4.1. Error estimates for the terminal times. Similar as in [5] we construct two auxiliary sequences: First, we construct $(\nu_{kh}^\gamma, q_{kh}^\gamma)$ converging to $(\bar{\nu}, \bar{q})$ as $k,h \to 0$ that is feasible for ($\hat{P}_{kh}$). In particular, this ensures existence of a solution $(\bar{\nu}_{kh}, \bar{q}_{kh})$ to the discrete problem. Moreover, we obtain a first convergence result without rates. Thereafter, we construct another sequence $(\nu_{kh}^\tau, q_{kh}^\tau)$ approximating the discrete optimal solutions that is feasible for ($\hat{P}$). Since the solution operator to the state equation is continuous for right-hand sides from $L^2(I;H^{-1})$ into $W(0,1) \hookrightarrow C([0,1];L^2)$, we may use (4.2) for all estimates concerning the state or the linearized state. Note that all sequences constructed in [5] are independent of the cost parameter α.

The error estimates are based on the following discretization error estimates for the state equation.

Lemma 4.1 ([5, Lemma 4.6]). *Let $\nu \in \mathbb{R}_+$ and $Bq \in L^\infty((0,1);L^2)$. For $u = S(\nu,q)$ and $u_{kh} = S_{kh}(\nu,q)$ it holds*

$$\|u - u_{kh}\|_{L^\infty(I;L^2)} \leq c|\log k| \left(k + h^2\right) \left((1+\nu)\|Bq\|_{L^\infty(I;L^2)} + \nu^{-1}\|u_0\|_{L^2}\right), \tag{4.3}$$

$$\|u - u_{kh}\|_{L^\infty(I;L^2)} \leq c|\log k| \left(k + h^2\right)(1+\nu)\left(\|Bq\|_{L^\infty(I;L^2)} + \|\Delta u_0\|_{L^2}\right), \tag{4.4}$$

where the constant c is independent of ν, k, h, q, u_0, and u.

Proposition 4.2. *Let $\nu_{\max} > 0$ and $q \in Q_{ad}(0,1)$. Then*

$$\lim_{k,h \to 0} \sup_{\nu \in (0,\nu_{\max})} \|i_1 S_{kh}(\nu,q) - i_1 S(\nu,q)\|_{L^2} = 0.$$

Proof. Consider first the case $q = 0$. Let $\varepsilon > 0$. Density of H^2 in L^2 yields $u_{0,\varepsilon} \in H^2$ such that $\|u_0 - u_{0,\varepsilon}\| \leq \varepsilon$. Let u_ε and $u_{kh,\varepsilon}$ denote the corresponding continuous and discrete solutions to the state equation with initial value $u_{0,\varepsilon}$. The stability estimates for the state equation [5, Proposition 3.3] and the discrete state equation [5, Proposition 4.1], as well as the discretization error estimate (4.4) imply

$$\begin{aligned}\|u_{kh}(1) - u(1)\|_{L^2} &\leq \|u_{kh}(1) - u_{kh,\varepsilon}(1)\|_{L^2} + \|u_{kh,\varepsilon}(1) - u_\varepsilon(1)\|_{L^2} + \|u_\varepsilon(1) - u(1)\|_{L^2} \\ &\leq c\|\Pi_h (u_0 - u_{0,\varepsilon})\|_{L^2} + c|\log k|(k+h^2)\|\Delta u_{0,\varepsilon}\|_{L^2} + c\|u_{0,\varepsilon} - u_0\|_{L^2},\end{aligned}$$

with a constant c independent of k, h, ν, and ε. Moreover, stability of the projection Π_h in L^2, for $k, h > 0$ sufficiently small such that $|\log k|(k + h^2)\|\Delta u_{0,\varepsilon}\|_{L^2} \leq \varepsilon$ implies the estimate $\|u_{kh}(1) - u(1)\|_{L^2} \leq c\varepsilon$. In the case $u_0 = 0$, we can directly apply the discretization error estimate (4.3). The assertion follows by superposition. ∎

Moreover, we require the following stability and discretization error estimates for the reduced constraint g that are essentially based on [5, Propositions 4.4 and 4.8].

Proposition 4.3. *Let $0 < \nu_{\min} < \nu_{\max}$ be fixed. Then for all $\nu_{\min} \leq \nu \leq \nu_{\max}$ and $q \in Q_{ad}(0,1)$ we have*

$$|\partial_{\nu\nu} g_{kh}(\nu, q)| \leq c, \tag{4.5}$$

where $c > 0$ is a constant independent of ν, q, k, and h. Moreover,

$$|g(\nu, q) - g_{kh}(\nu, q)| \leq c|\log k|(k+h^2)\left(\|Bq\|_{L^\infty(I;L^2)} + \|u_0\|_{L^2}\right), \tag{4.6}$$

$$|\partial_\nu g(\nu, q) - \partial_\nu g_{kh}(\nu, q)| \leq c|\log k|(k+h^2)\left(\|Bq\|_{L^\infty(I;L^2)} + \|u_0\|_{H^1}\right), \tag{4.7}$$

where $c > 0$ is a constant independent of ν, q, k, and h.

Proposition 4.4. *Let $(\bar\nu, \bar q)$ be an optimal solution of problem ($\hat P$). There exists a sequence $(\nu^\gamma_{kh}, q^\gamma_{kh})$ of controls that are feasible for ($\hat P_{kh}$) for k and h sufficiently small and fulfill the estimate*

$$|\nu^\gamma_{kh} - \bar\nu| \leq c\left(\sigma(k,h) + |\log k|(k+h^2)\right),$$

where $\sigma(k,h)$ satisfies (4.2).

Proof. The sequence is chosen as $\nu^\gamma_{kh} = \bar\nu + \gamma(k,h)$ and $q^\gamma_{kh} = \mathrm{I}_\sigma \bar q$, where the constant $0 < \gamma(k,h) \leq 1$ can be chosen as in [5, Proposition 4.9], using Assumption 3.1.

In particular, Proposition 4.4 implies existence of feasible points for the discrete problem ($\hat P_{kh}$), which in turn guarantees existence of an optimal solution to the discrete problem. Even better, we obtain a first convergence result.

Lemma 4.5. *Let Assumption 3.1 hold. For k and h sufficiently small, the discrete problem ($\hat P_{kh}$) has an optimal solution $(\bar\nu_{kh}, \bar q_{kh})$. Moreover, $\bar\nu_{kh} \to \bar\nu$ and every weak accumulation point of $(\bar q_{kh})_{k,h>0}$ in $L^s(I \times \omega)$ for any $s < \infty$ is optimal for ($\hat P$).*

For the proof, we employ a complete continuity result for the control-to-state mapping.

Proposition 4.6. *Let $s > 2$. If $\nu_n \to \nu \geq 0$ and $q_n \rightharpoonup q$ in $L^s(I \times \omega)$ for $n \to \infty$, then $u_n = S(\nu_n, q_n) \to u = S(\nu, q)$ in $C([0,1]; L^2)$.*

Proof. If $\nu > 0$, we directly apply [2, Proposition A.20]. Otherwise, if $\nu = 0$, then the assertion follows from the variation of constants formula for the solution u to the parabolic state equation (3.1). ∎

Proof of Lemma 4.5. Existence of solutions follows by standard arguments, since the set of admissible controls is nonempty according to Proposition 4.4. Moreover, using optimality of $(\bar\nu_{kh}, \bar q_{kh})$, feasibility of $(\nu^\gamma_{kh}, q^\gamma_{kh})$, and $0 \le \gamma(k,h) \le 1$, we observe

$$0 \le \bar\nu_{kh} \le \nu^\gamma_{kh} = \bar\nu + \gamma(k,h) \le \bar\nu + 1.$$

Hence, $(\bar\nu_{kh}, \bar q_{kh})$ is uniformly bounded. Thus, there exists a subsequence denoted in the same way such that $\bar\nu_{kh} \to \nu^*$ and $q_{kh} \rightharpoonup q^*$ in $L^s(I\times\omega)$ with $q^* \in Q_{ad}(0,1)$ and some $s > 2$. Feasibility of $(\bar\nu_{kh}, \bar q_{kh})$ for $(\hat P_{kh})$ further yields

$$\begin{aligned} g(\nu^*, q^*) &\le g_{kh}(\bar\nu_{kh}, \bar q_{kh}) + |g(\bar\nu_{kh}, \bar q_{kh}) - g_{kh}(\bar\nu_{kh}, \bar q_{kh})| + |g(\nu^*, q^*) - g(\bar\nu_{kh}, \bar q_{kh})| \\ &\le c\|i_1 S(\bar\nu_{kh}, \bar q_{kh}) - i_1 S_{kh}(\bar\nu_{kh}, \bar q_{kh})\|_{L^2} + c\|i_1 S(\nu^*, q^*) - i_1 S(\bar\nu_{kh}, \bar q_{kh})\|_{L^2}, \end{aligned}$$

where we have used Lipschitz continuity of G on bounded sets in L^2. Going to the limit $k,h\to 0$, employing the convergence result Proposition 4.2 as well as complete continuity Proposition 4.6, we deduce that $g(\nu^*, q^*) \le 0$. In particular, $\bar\nu \le \nu^*$.

Optimality of $(\bar\nu_{kh}, \bar q_{kh})$ and feasibility of $(\nu^\gamma_{kh}, q^\gamma_{kh})$ from Proposition 4.8 for $(\hat P_{kh})$, leads to

$$\nu^* = \lim_{k,h\to 0} \bar\nu_{kh} \le \lim_{k,h\to 0} \nu^\gamma_{kh} = \lim_{k,h\to 0} \left(\bar\nu + c\left(\sigma(k,h) + |\log k|(k+h^2)\right)\right) = \bar\nu.$$

Hence, $\bar\nu = \nu^*$ and $(\bar\nu, q^*)$ is also optimal. Moreover, as the limit $\bar\nu$ is independent of the concretely chosen subsequence, the whole sequence converges. ∎

In addition, Lemma 4.5 implies that the sequence $\bar\nu_{kh}$ is uniformly bounded away from zero. Hence, the constants in the following error estimates can be chosen to be independent of $\bar\nu_{kh}$; cf. Lemma 4.1 and Proposition 4.3.

As the next step towards error estimates, we verify that the linearized Slater condition holds at $(\bar\nu_{kh}, \bar q_{kh})$ for the discrete problem.

PROPOSITION 4.7. *Let Assumption 3.1 hold. Moreover, let $(\bar\nu_{kh}, \bar q_{kh})$ be a sequence of optimal solutions of $(\hat P_{kh})$. Then, for $k,h\to 0$ we have*

$$\partial_\nu g_{kh}(\bar\nu_{kh}, \bar q_{kh}) \to \partial_\nu g(\bar\nu, \bar q) = -\bar\eta < 0.$$

Proof. Define $\bar\eta_{kh} = -\partial_\nu g_{kh}(\bar\nu_{kh}, \bar q_{kh})$. According to Lemma 4.5, we may select a subsequence, denoted for convenience again by $(\bar\nu_{kh}, \bar q_{kh})$, such that $\bar\nu_{kh} \to \bar\nu$ and $\bar q_{kh} \rightharpoonup \bar q$ in $L^s(I\times\omega)$, $s > 2$, where $(\bar\nu, \bar q)$ is an optimal solution of $(\hat P)$. Now, we use the representation of g', i.e. $\mu\,\partial_\nu g(\nu, q) = \int_0^1 \langle Bq + \Delta u, z\rangle$ from (3.4), where we choose $\mu = \bar\mu > 0$ from Lemma 3.2. Together with the discretization error estimate (4.7) we obtain that

$$\bar\mu\,|\bar\eta_{kh} - \bar\eta| \le c|\log k|\left(k + h^2\right) + \left|\int_0^1 \langle B\bar q_{kh} + \Delta\hat u, \hat z\rangle\,\mathrm{d}t - \int_0^1 \langle B\bar q + \Delta\bar u, \bar z\rangle\,\mathrm{d}t\right|,$$

where $\hat u = S(\bar\nu_{kh}, \bar q_{kh})$ and $\hat z$ denotes the dual state solving $-\partial_t\hat z - \bar\nu_{kh}\Delta\hat z = 0$ and $\hat z(1) = \bar\mu(\hat u(1) - u_d)$. Moreover, $\bar z$ is the optimal adjoint state from Lemma 3.2. The convergence result Lemma 4.5 and complete continuity of the control-to-observation mapping from Proposition 4.6 imply that $\hat u \to \bar u$ in $C([0,1], L^2)$ and thus also $\hat z \to \bar z$ in $W(0,1)$ for $k,h\to 0$. Employing that $B\bar q_{kh} \rightharpoonup B\bar q$ in $L^2(I; H^{-1})$ which implies $\hat u \rightharpoonup \bar u$ in $W(0,1)$, we deduce that $\bar\eta_{kh} \to \bar\eta$. Since the limit is independent of $\bar q$ (see Lemma 3.2) the whole sequence $\bar\eta_{kh}$ converges. ∎

PROPOSITION 4.8. *Let k and h be sufficiently small. Moreover, let $(\bar\nu, \bar q)$ be the unique optimal solution of $(\hat P)$ and let $(\bar\nu_{kh}, \bar q_{kh})$ be an optimal control of $(\hat P_{kh})$. Then there exists a sequence ν_{kh}^τ such that $(\nu_{kh}^\tau, \bar q_{kh})$ is feasible for $(\hat P)$ and*

$$|\nu_{kh}^\tau - \bar\nu_{kh}| \le c|\log k|(k + h^2). \tag{4.8}$$

Proof. The sequence can be constructed by choosing $\nu_{kh}^\tau = \bar\nu_{kh} + \tau(k, h)$ with $0 < \tau(k, h) \le c|\log k|(k + h^2)$ as in [5, Proposition 4.12]. ∎

LEMMA 4.9. *Let Assumption 3.1 hold. Moreover, let $(\bar\nu_{kh}, \bar q_{kh})$ be a sequence of optimal solutions of $(\hat P_{kh})$. Then, for k and h sufficiently small, we have*

$$|\bar\nu - \bar\nu_{kh}| \le c\left(\sigma(k, h) + |\log k|(k + h^2)\right),$$

where $c > 0$ is independent of k and h and $\sigma(k, h)$ satisfies (4.2). Moreover, there exists a unique Lagrange multiplier $\bar\mu_{kh} = \bar\mu_{kh}(\bar q_{kh}) > 0$ such that the optimality system is satisfied

$$\int_0^1 1 + \langle B\bar q_{kh}(t) + \Delta_h \bar u_{kh}(t), \bar z_{kh}(t)\rangle \,\mathrm{d}t = 0, \tag{4.9}$$

$$\int_0^1 \langle B^* \bar z_{kh}(t), q(t) - \bar q_{kh}(t)\rangle \,\mathrm{d}t \ge 0 \quad \text{for all } q \in Q_{ad,\sigma}(0, 1), \tag{4.10}$$

$$G(\bar u_{kh}(1)) = 0, \tag{4.11}$$

where $\bar u_{kh} = S(\bar\nu_{kh}, \bar q_{kh})$ and $\bar z_{kh} \in X_{k,h}$ is the solution to the discrete adjoint equation

$$\mathrm{B}(\bar\nu_{kh}, \varphi_{kh}, \bar z_{kh}) = \bar\mu_{kh}(\bar u_{kh}(1) - u_d, \varphi_{kh}(1)), \quad \varphi_{kh} \in X_{k,h}.$$

Proof. Because the pair $(\nu_{kh}^\tau, \bar q_{kh})$ is feasible for $(\hat P)$, we have

$$0 \le \nu_{kh}^\tau - \bar\nu = \nu_{kh}^\tau - \bar\nu_{kh} + \bar\nu_{kh} - \nu_{kh}^\gamma + \nu_{kh}^\gamma - \bar\nu \le \nu_{kh}^\tau - \bar\nu_{kh} + \nu_{kh}^\gamma - \bar\nu,$$

where the last inequality follows from optimality of the pair $(\bar\nu_{kh}, \bar q_{kh})$ for $(\hat P_{kh})$ and feasibility of $(\nu_{kh}^\gamma, q_{kh}^\gamma)$ for $(\hat P_{kh})$. Hence,

$$\begin{aligned} |\bar\nu_{kh} - \bar\nu| &\le |\bar\nu_{kh} - \nu_{kh}^\tau| + \nu_{kh}^\tau - \bar\nu \le 2|\bar\nu_{kh} - \nu_{kh}^\tau| + |\nu_{kh}^\gamma - \bar\nu| \\ &\le c\left(\sigma(k, h) + |\log k|(k + h^2)\right). \end{aligned}$$

where we have used Propositions 4.4 and 4.8 in the last step. Finally, the linearized Slater condition due to Proposition 4.7 yields the optimality conditions in qualified form as stated above. ∎

REMARK 4.10. *For each discrete optimal solution $(\bar\nu_{kh}, \bar q_{kh})$, there exists a unique Lagrange multiplier $\bar\mu_{kh}$. However, as the discrete control is not guaranteed to be unique, there might be different multipliers. Note that a similar argument as in Lemma 3.2 is not valid on the discrete level, since the condition (4.9) holds only in the integrated form; cf. (3.8). (For the proof of (3.8), variable time transformations $\nu \in L^\infty((0, 1))$ are employed, which cannot be used on the discrete level, since this would deform the mesh geometry.) Nevertheless, we can prove the a priori bound $\bar\mu_{kh} \le 2/\bar\eta$ for k and h sufficiently small using the optimality conditions for $(\hat P_{kh})$ and Proposition 4.7.*

4.2. Convergence of controls. Next, we prove convergence of the control variable based on the growth condition (3.14). In the following, we generally assume that k and h are chosen small enough for the results of the previous section, in particular Lemma 4.9, to hold.

THEOREM 4.11. *Let Assumptions 3.1 and 3.2 hold. Moreover, let $(\bar\nu_{kh}, \bar q_{kh})$ be a sequence of optimal solutions of $(\hat P_{kh})$. Then, we have $\bar q_{kh} \to \bar q$ in $L^s(I \times \Omega)$ for any $s < \infty$, where $(\bar\nu, \bar q)$ is the unique solution of $(\hat P)$.*

Proof. With Assumption 3.2, the optimal solution $(\bar\nu, \bar q)$ is uniquely determined; see Proposition 3.3. Let ν^τ_{kh} be from Proposition 4.8. Since

$$|\nu^\tau_{kh} - \bar\nu| \le |\nu^\tau_{kh} - \bar\nu_{kh}| + |\bar\nu_{kh} - \bar\nu| \to 0,$$

and because the pair $(\nu^\tau_{kh}, \bar q_{kh})$ is feasible for $(\hat P)$, we may use the growth condition from Theorem 3.8 to deduce

$$\frac{\bar\nu}{4} \Psi^{-1}\left(c_0 \|\bar q_{kh} - \bar q\|_{L^1(I\times\omega)}\right) \|\bar q_{kh} - \bar q\|_{L^1(I\times\omega)} \le \nu^\tau_{kh} - \bar\nu. \tag{4.12}$$

Strict monotonicity and continuity of Ψ^{-1} imply $\bar q_{kh} \to \bar q$ in $L^1(I \times \omega)$. Finally, using Hölder's inequality, we obtain

$$\|\bar q_{kh} - \bar q\|^s_{L^s(I\times\omega)} \le \left(\|\bar q_{kh}\|^{s-1}_{L^\infty(I\times\omega)} + \|\bar q\|^{s-1}_{L^\infty(I\times\omega)}\right) \|\bar q_{kh} - \bar q\|_{L^1(I\times\omega)}$$

and due to $\bar q_{kh}, \bar q \in Q_{ad}(0,1)$, we conclude $\bar q_{kh} \to \bar q$ in $L^s(I \times \omega)$. □

REMARK 4.12. *If $\Psi(\varepsilon) = C\varepsilon^\kappa$, then in view of Remark 3.11 we obtain from (4.12) with similar arguments as in the proof of Lemma 4.9 the sub-optimal estimate*

$$c\|\bar q_{kh} - \bar q\|^{1+1/\kappa}_{L^1(I\times\omega)} \le \nu^\tau_{kh} - \bar\nu \le c\left(\sigma(k,h) + |\log k|(k + h^2)\right).$$

An improved estimate will be derived in the next section.

4.3. Improved error estimates for controls. We provide an improved error estimate that is directly based on the structural condition (3.11). For the proof we require Lipschitz-type estimates of the solution to the equation

$$\partial_t u - \nu\Delta u = \nu f, \quad u(0) = u_0 \tag{4.13}$$

with respect to the time transformation ν. In the case of a purely time-dependent control we will use the following result.

PROPOSITION 4.13 ([2, Proposition 5.29]). *Let $\nu_{\max} > \nu_{\min} > 0$. There is $c > 0$ such that for any $u_0 \in L^2$, $f \in L^2(I; H^{-1})$, and $\nu_1, \nu_2 \in [\nu_{\min}, \nu_{\max}]$ the corresponding solutions u_1, u_2 to (4.13) satisfy the estimate*

$$\|u_1 - u_2\|_{C([0,1];L^2)} \le c|\nu_1 - \nu_2| \left(\|f\|_{L^2(I;H^{-1})} + \|u_0\|_{L^2}\right),$$

where $c > 0$ is independent of ν_1, ν_2, f, and u_0.

In the case of a distributed control, we require estimates for the adjoint state, which are uniform in time and space. The required improved regularity follows from Proposition 2.3, using the embedding

$$V_{p,s} := (L^p, \mathcal{D}_{L^p}(-\Delta))_{1-1/s,s} \hookrightarrow L^\infty \quad \text{for } \frac{d}{2p} + \frac{1}{s} < 1, \tag{4.14}$$

where we recall that d denotes the spatial dimension; cf., e.g., [14, Theorem 3.1]. The required regularity $u_0 \in V_{p,s}$ is certainly fulfilled, if, e.g., $u_0 \in \mathcal{D}_{L^p}(-\Delta)$ with $p > d/2$.

PROPOSITION 4.14 ([2, Proposition 5.30]). *Let* $\nu_{\max} > \nu_{\min} > 0$ *and* $s, p \in (1, \infty)$ *such that* (4.14) *holds. There is a constant* $c > 0$ *such that for any* $u_0 \in V_{p,s}$, $f \in L^s(I; L^p)$, *and* $\nu_1, \nu_2 \in [\nu_{\min}, \nu_{\max}]$ *the corresponding solutions* u_1, u_2 *to* (4.13) *satisfy the estimate*

$$\|u_1 - u_2\|_{L^\infty(I\times\Omega)} \le c\|u_1 - u_2\|_{C([0,1];V_{p,s})} \le c|\nu_1 - \nu_2| \left(\|f\|_{L^s(I;L^p)} + \|u_0\|_{V_{p,s}}\right),$$

where $c > 0$ *is independent of* ν_1, ν_2, f, *and* u_0.

PROPOSITION 4.15. *Adopt the assumptions of Theorem 4.11. Moreover, we assume that* I_σ *is the orthogonal projection onto* $Q_\sigma(0,1)$ *in* $L^2(I \times \omega)$. *In case of a distributed control, suppose in addition that* $u_0, u_d \in V_{p,s}$ *for* $s, p \in (1, \infty)$ *with* (4.14). *There is a constant* $c > 0$ *independent of* k, h, $\bar\nu_{kh}$, *and* $\bar q_{kh}$ *such that*

$$\Psi^{-1}\left(c_0\|\bar q - \bar q_{kh}\|_{L^1(I\times\omega)}\right) \le c\Big(|\bar\nu - \bar\nu_{kh}| + \|(\mathrm{Id} - \mathrm{I}_\sigma)B^*\hat z_{kh}\|_{L^\infty(I\times\omega)} + \|B^*\left(\hat z_{kh} - \hat z\right)\|_{L^\infty(I\times\omega)}\Big),$$

where $\hat z \in W(0,1)$ *solves*

$$-\partial_t \hat z - \bar\nu_{kh}\Delta \hat z = 0, \quad \hat z(1) = \bar\mu\left(\hat u(1) - u_d\right), \quad \hat u = S(\bar\nu_{kh}, \bar q_{kh}),$$

and $\hat z_{kh} = (\bar\mu/\bar\mu_{kh})\bar z_{kh} \in X_{k,h}$ *solves*

$$\mathrm{B}(\bar\nu_{kh}, \varphi_{kh}, \hat z_{kh}) = \bar\mu(\bar u_{kh}(1) - u_d, \varphi_{kh}(1)), \quad \varphi_{kh} \in X_{k,h}. \tag{4.15}$$

Proof. As in the proof of [34, Theorem 31], in (3.13) we set $q = \bar q_{kh}$ to obtain

$$\frac{\bar\nu}{2}\Psi^{-1}\left(c_0\|\bar q - \bar q_{kh}\|_{L^1(I\times\omega)}\right)\|\bar q - \bar q_{kh}\|_{L^1(I\times\omega)} \le -\int_0^1 (B^*\bar z, \bar q - \bar q_{kh})_{L^2(\omega)}\,\mathrm{d}t. \tag{4.16}$$

The optimality condition (4.10) with $q = \mathrm{I}_\sigma \bar q$ multiplied by $\bar\mu/\bar\mu_{kh} > 0$ reads

$$0 \le \int_0^1 (B^*\hat z_{kh}, \mathrm{I}_\sigma\bar q - \bar q_{kh})_{L^2(\omega)}\,\mathrm{d}t, \tag{4.17}$$

where $\hat z_{kh} = (\bar\mu/\bar\mu_{kh})\bar z_{kh}$, i.e. $\hat z_{kh}$ fulfills the same discrete adjoint equation as $\bar z_{kh}$ but with multiplier $\bar\mu$ instead of $\bar\mu_{kh}$, as given in (4.15). Summation of (4.16) and (4.17) implies

$$\begin{aligned}
&\frac{\bar\nu}{2}\Psi^{-1}\left(c_0\|\bar q - \bar q_{kh}\|_{L^1(I\times\omega)}\right)\|\bar q - \bar q_{kh}\|_{L^1(I\times\omega)} \\
&\quad\le \int_0^1 \left(B^*\left(\hat z_{kh} - \bar z\right), \bar q - \bar q_{kh}\right)_{L^2(\omega)}\mathrm{d}t - \int_0^1 (B^*\hat z_{kh}, \bar q - \bar q_{kh})_{L^2(\omega)}\,\mathrm{d}t \\
&\qquad + \int_0^1 (B^*\hat z_{kh}, \mathrm{I}_\sigma\bar q - \bar q_{kh})_{L^2(\omega)}\,\mathrm{d}t \\
&\quad= \int_0^1 \left(B^*\left(\hat z_{kh} - \bar z\right), \bar q - \bar q_{kh}\right)_{L^2(\omega)}\mathrm{d}t + \int_0^1 (B^*\hat z_{kh}, \mathrm{I}_\sigma\bar q - \bar q)_{L^2(\omega)}\,\mathrm{d}t.
\end{aligned} \tag{4.18}$$

Concerning the first term of the right-hand side of (4.18), we have

$$\begin{aligned}
\int_0^1 \left(B^*\left(\hat z_{kh} - \bar z\right), \bar q - \bar q_{kh}\right)_{L^2(\omega)}\mathrm{d}t &= \int_0^1 \left(B^*\left(\hat z_{kh} - \hat z\right), \bar q - \bar q_{kh}\right)_{L^2(\omega)}\mathrm{d}t \\
&\quad + \int_0^1 \left(B^*\left(\hat z - \widetilde z\right), \bar q - \bar q_{kh}\right)_{L^2(\omega)}\mathrm{d}t + \int_0^1 \left(B^*\left(\widetilde z - \bar z\right), \bar q - \bar q_{kh}\right)_{L^2(\omega)}\mathrm{d}t,
\end{aligned} \tag{4.19}$$

where $\widetilde{z} \in W(0,1)$ is an additional adjoint state solving

$$-\partial_t \widetilde{z} - \bar{\nu}\Delta\widetilde{z} = 0, \quad \widetilde{z}(1) = \bar{\mu}\left(\widetilde{u}(1) - u_d\right), \quad \widetilde{u} = S(\bar{\nu}, \bar{q}_{kh}).$$

Note that all adjoint states appearing above correspond to the same multiplier $\bar{\mu}$. For the first term on the right-hand side of (4.19), Hölder's inequality yields

$$\int_0^1 \left(B^*\left(\hat{z}_{kh} - \hat{z}\right), \bar{q} - \bar{q}_{kh}\right)_{L^2(\omega)} \le \|B^*\left(\hat{z}_{kh} - \hat{z}\right)\|_{L^\infty(I\times\omega)} \|\bar{q} - \bar{q}_{kh}\|_{L^1(I\times\omega)}.$$

To estimate the second term on the right-hand side of (4.19), we introduce another auxiliary adjoint state $\breve{z}$ solving

$$-\partial_t \breve{z} - \bar{\nu}\Delta\breve{z} = 0, \quad \breve{z}(1) = \bar{\mu}\left(\hat{u} - u_d\right),$$

where $\hat{u} = S(\bar{\nu}_{kh}, \bar{q}_{kh})$, as in the definition of $\hat{z}$. Then, for purely time-dependent controls we estimate

$$\begin{aligned} \|B^*\left(\hat{z} - \widetilde{z}\right)\|_{L^\infty(I\times\omega)} &\le c\|\hat{z} - \breve{z}\|_{L^\infty(I;L^2)} + c\|\breve{z} - \widetilde{z}\|_{L^\infty(I;L^2)}, \\ &\le c\|\hat{z} - \breve{z}\|_{L^\infty(I;L^2)} + c\|\hat{u}(1) - \widetilde{u}(1)\|_{L^2}. \end{aligned}$$

Then, we can apply Proposition 4.13 for both terms on the right. For distributed controls, we replace L^2 by $V_{p,s}$ in the above estimate and then apply Proposition 4.14 twice. In both cases, we conclude that

$$\int_0^1 \left(B^*\left(\hat{z} - \widetilde{z}\right), \bar{q} - \bar{q}_{kh}\right)_{L^2(\omega)} \mathrm{d}t \le c|\bar{\nu}_{kh} - \bar{\nu}| \|\bar{q} - \bar{q}_{kh}\|_{L^1(I\times\omega)}.$$

The third term on the right-hand side of (4.19) is less than or equal to zero, because of affine linearity of $i_1 S(\bar{\nu}, q)$ with respect to q which implies

$$\begin{aligned} &\int_0^1 \left(B^*\left(\widetilde{z} - \bar{z}\right), \bar{q} - \bar{q}_{kh}\right)_{L^2(\omega)} \mathrm{d}t \\ &\quad = \bar{\mu}\Big(\Big(i_1\left(\partial_t - \bar{\nu}\Delta\right)^{-1}\Big)^* \left(\widetilde{u}(1) - \bar{u}(1)\right), B(\bar{q} - \bar{q}_{kh})\Big)_{L^2} = -\bar{\mu}\|\bar{u}(1) - \widetilde{u}(1)\|_{L^2}^2. \end{aligned}$$

where $\left(\partial_t - \bar{\nu}\Delta\right)^{-1}$ denotes the solution operator to the linear heat-equation with homogeneous initial data. Since I_σ is the $L^2(I\times\omega)$-projection onto $Q_\sigma(0,1)$ for the last term of the right-hand side of (4.18) we obtain

$$\int_0^1 \left(B^*\hat{z}_{kh}, \mathrm{I}_\sigma\bar{q} - \bar{q}\right)_{L^2(\omega)} \mathrm{d}t = \int_0^1 \left((\mathrm{Id} - \mathrm{I}_\sigma)B^*\hat{z}_{kh}, \bar{q}_{kh} - \bar{q}\right)_{L^2(\omega)} \mathrm{d}t.$$

In summary, we arrive at

$$\begin{aligned} \frac{\bar{\nu}}{2}\Psi^{-1}\left(c_0\|\bar{q} - \bar{q}_{kh}\|_{L^1(I\times\omega)}\right)\|\bar{q} - \bar{q}_{kh}\|_{L^1(I\times\omega)} &\le c\Big(|\bar{\nu}_{kh} - \bar{\nu}| \\ + \|(\mathrm{Id} - \mathrm{I}_\sigma)B^*\hat{z}_{kh}\|_{L^\infty(I\times\omega)} &+ \|B^*\left(\hat{z}_{kh} - \hat{z}\right)\|_{L^\infty(I\times\omega)}\Big)\|\bar{q} - \bar{q}_{kh}\|_{L^1(I\times\omega)}. \end{aligned}$$

Last, dividing by $\|\bar{q} - \bar{q}_{kh}\|_{L^1(I\times\omega)}$ yields the desired estimate. ◻

4.4. Concrete control discretization schemes. Before we apply the general results of the preceding subsections, we will verify the equivalence of a semi-variational and an explicit discretization of the controls. To this end, let $Q_h \subseteq Q$ be a finite dimensional subspace. In the following we consider for given Q_h the two choices of the control space $Q_\sigma(0,1)$: the discrete control space $Q_\sigma(0,1) = Q_{kh}(0,1)$, where

$$Q_{kh}(0,1) = \{v \in Q(0,1) \colon v|_{I_m} \in \mathcal{P}_0(I_m; Q_h),\ m = 1,2,\ldots,M\}\,, \tag{4.20}$$

and the semi-variational control space $Q_\sigma(0,1) = L^2(I;Q_h)$. Additionally, let Π_k denote the L^2-projection onto the piecewise constant functions in time. The problem ($\hat{P}_{kh}$) posed with $Q_\sigma(0,1) = Q_{kh}(0,1)$ is equivalent to ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = L^2(I;Q_h)$ in the following sense.

Proposition 4.16. *If $(\bar\nu_{kh}, \bar q_{kh})$ is an optimal solution to ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = Q_{kh}(0,1)$ then $(\bar\nu_{kh}, \bar q_{kh})$ is also optimal for ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = L^2(I;Q_h)$. Conversely, if $(\bar\nu^{\mathrm{v}}_{kh}, \bar q^{\mathrm{v}}_{kh})$ is an optimal solution to ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = L^2(I;Q_h)$, then $(\bar\nu^{\mathrm{v}}_{kh}, \Pi_k \bar q^{\mathrm{v}}_{kh})$ is also optimal for ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = Q_{kh}(0,1)$.*

Proof. First, since the variational admissible set $L^2(I;Q_h) \cap Q_{ad}(0,1)$ is larger than the fully discrete one $Q_{kh}(0,1) \cap Q_{ad}(0,1)$, we immediately obtain $\bar\nu^{\mathrm{v}}_{kh} \le \bar\nu_{kh}$ for the optimal times. Clearly, $\Pi_k \bar q^{\mathrm{v}}_{kh} \in Q_{kh}(0,1) \cap Q_{ad}(0,1)$ by the fact that Π_k can be computed explicitly on every interval I_m as the interval mean. In addition, by the orthogonality-properties of the L^2-projection Π_k and the definition of the state equation (4.1), $(\bar\nu^{\mathrm{v}}_{kh}, \Pi_k \bar q^{\mathrm{v}}_{kh})$ has the same associated discrete state as $(\bar\nu^{\mathrm{v}}_{kh}, \bar q^{\mathrm{v}}_{kh})$, which directly implies that $g_{kh}(\bar\nu^{\mathrm{v}}_{kh}, \Pi_k \bar q^{\mathrm{v}}_{kh}) \le 0$. Thus, $(\bar\nu^{\mathrm{v}}_{kh}, \Pi_k \bar q^{\mathrm{v}}_{kh})$ is feasible for ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = Q_{kh}(0,1)$ and therefore $\bar\nu_{kh} \le \bar\nu^{\mathrm{v}}_{kh}$. Hence, both problems have the same optimal time $\bar\nu_{kh} = \bar\nu^{\mathrm{v}}_{kh}$. Consequently, the optimal controls of both problems are given by all controls $q \in Q_{ad,\sigma}(0,1)$ such that $g_{kh}(\bar\nu_{kh}, q) \le 0$, with $Q_\sigma(0,1) = Q_{kh}(0,1)$ or $Q_\sigma(0,1) = L^2(I;Q_h)$, respectively. A similar argument as before yields the relation between the optimal controls as claimed. ∎

As we are interested in explicit rates of convergence, for the following considerations we assume that $\Psi(\varepsilon) = C\varepsilon^\kappa$ in (3.11). The proceeding results hold for a general function Ψ satisfying (3.11) with obvious modifications.

4.4.1. Purely time-dependent controls. In case of purely time-dependent controls we immediately derive an error estimate (that is optimal if $\kappa = 1$) using the $L^\infty(I;L^2)$ discretization error estimate for the variational control discretization. Note that besides theoretical advantages purely time-dependent controls are also interesting in practice as distributed controls are typically difficult to implement.

Theorem 4.17 (Time-dependent parameter control). *Adopt the assumptions of Theorem 4.11 and let (3.11) hold with $\Psi(\varepsilon) = C\varepsilon^\kappa$. Additionally, suppose purely time-dependent controls and let $(\bar\nu_{kh}, \bar q_{kh})$ be a sequence of optimal solutions of ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = Q_{kh}(0,1)$ and $Q_h = \mathbb{R}^{N_c}$. Then there is a constant $c > 0$ such that*

$$|\bar\nu - \bar\nu_{kh}| + \|\bar q - \bar q_{kh}\|^{1/\kappa}_{L^1(I\times\omega)} \le c|\log k|(k + h^2).$$

Proof. First, using Proposition 4.16, $(\bar\nu_{kh}, \bar q_{kh})$ is also a solution of ($\hat{P}_{kh}$) with $Q_\sigma(0,1) = L^2(I;\mathbb{R}^{N_c})$, and we may apply Lemma 4.9 with $\sigma(k,h) = 0$, which already yields the estimate for the optimal times. The estimate for the controls follows from

Proposition 4.15, where $\mathrm{I}_\sigma = \mathrm{Id}$ and

$$\begin{aligned}\|B^*\left(\hat{z}_{kh}-\hat{z}\right)\|_{L^\infty(I\times\omega)} &= \operatorname*{ess\,sup}_{t\in I}\max_{i\in\{1,\ldots,N_c\}}|(e_i,\hat{z}_{kh}(t)-\hat{z}(t))|\\ &\le c\|\hat{z}_{kh}-\hat{z}\|_{L^\infty(I;L^2)}\le c|\log k|(k+h^2),\end{aligned}$$

where we use the $L^\infty(I;L^2)$ discretization error estimate from Lemma 4.1 for the state and adjoint state. ∎

4.4.2. Distributed control with variational control discretization. Next, we discuss the case of a distributed control, i.e. $\omega \subset \Omega$. In order to apply Proposition 4.15 we require pointwise error estimates for the adjoint state equation. For simplicity, we only consider the particular case that the control domain ω has a strictly positive distance to the boundary $\partial\Omega$ of the spatial domain and smooth initial and desired states. Moreover, we assume in the remaining part of this section that the spatial mesh is quasi-uniform. Based on pointwise best approximations results from [23] we can obtain the following error estimate. For the proof of this estimate we refer to [2, Sections 5.5.3, 5.5.4].

PROPOSITION 4.18 ([2, Proposition 5.42]). *Let $\overline{\omega} \subset \Omega$. Suppose that $u_0, u_d \in \mathcal{D}_{L^\infty}(-\Delta)$. Then there exists a constant $c > 0$, independent of k, h, $\hat{z}_{kh}$, and $\hat{z}$, such that*

$$\|B^*\left(\hat{z}_{kh}-\hat{z}\right)\|_{L^\infty(I\times\omega)}\le c|\log k|^4|\log h|^7(k+h^2),$$

where $\hat{z}$ and $\hat{z}_{kh}$ are defined in Proposition 4.15.

We directly infer the following error estimate for the variational control discretization.

THEOREM 4.19 (Variational discretization). *Adopt the assumptions of Theorem 4.11 and let (3.11) hold with $\Psi(\varepsilon) = C\varepsilon^\kappa$. Moreover, suppose the variational control discretization, i.e. $Q_\sigma(0,1) = Q(0,1)$. In addition, assume $\overline{\omega} \subset \Omega$ as well as $u_0, u_d \in \mathcal{D}_{L^\infty}(-\Delta)$. Then there is a constant $c > 0$, independent of k, h, $\bar{\nu}_{kh}$, and $\bar{q}_{kh}$, such that*

$$|\bar{\nu}-\bar{\nu}_{kh}| + \|\bar{q}-\bar{q}_{kh}\|^{1/\kappa}_{L^1(I\times\omega)}\le c|\log k|^4|\log h|^7(k+h^2).$$

Proof. This result follows from Lemma 4.9 and Propositions 4.15 and 4.18, since for the variational control discretization we have $\mathrm{I}_\sigma = \mathrm{Id}$ and $\sigma(k,h) = 0$. ∎

4.4.3. Distributed control with cellwise constant control discretization. Last, we consider the discretization of the control by cellwise constant functions in space. Recall that $\sigma(k,h)$ denotes the projection error onto $Q_\sigma(0,1)$ measured $L^2(I;H^{-1})$; see (4.2). Since the control variable has a bang-bang structure, we cannot expect order k of convergence in L^2 in time. We therefore first consider a semi-variational control discretization and obtain the fully discrete result using Proposition 4.16. Let the discrete space of controls be defined as follows

$$Q_h = \left\{v \in L^2(\omega)\colon v|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathcal{T}_h^\omega\right\}, \quad Q_\sigma(0,1) = L^2(I;Q_h).$$

Hence, the controls are explicitly discretized in space but not explicitly discretized in time, which is equivalent to the discretization by piecewise and cellwise constant functions. Let $\Pi_{h,0}$ denote the $L^2(\omega)$-projection onto the cellwise constant functions. Moreover, for almost every $t \in [0,1]$ we set

$$\mathcal{S}_{h,t} \coloneqq \mathcal{T}_h^\omega \setminus \{K \in \mathcal{T}_h^\omega\colon \bar{q}(t)|_K \equiv q_a \text{ or } \bar{q}(t)|_K \equiv q_b\}.$$

We first establish error estimates for $\sigma(k,h)$ with $\mathrm{I}_\sigma = \Pi_{h,0}$.

PROPOSITION 4.20. *Suppose there are functions $\delta_h \in L^1(I)$, $h>0$, and a constant $c>0$ such that*

$$\sum_{K\in\mathcal{S}_{h,t}} |K| \le \delta_h(t), \quad a.e.\ t\in[0,1], \quad h>0, \tag{4.21}$$

and $\|\delta_h\|_{L^1(I)} \le ch$ for all $h>0$. Then the estimate

$$\|B\left(\Pi_{h,0}\bar q - \bar q\right)\|_{L^2(I;H^{-1})} \le ch^{3/2}, \tag{4.22}$$

holds with a constant $c>0$ not depending on h.

Proof. Since $\Pi_{h,0}$ is a projection, for any $v\in H^1$ and $K\in\mathcal{T}_h^\omega$ we have

$$(\Pi_{h,0}\bar q(t) - \bar q(t), v)_{L^2(K)} \le ch\|\Pi_{h,0}\bar q(t) - \bar q(t)\|_{L^2(K)}\|\nabla v\|_{L^2(K)}.$$

Using Hölder's inequality and the supposition (4.21) yields (4.22). ∎

We have the following sufficient condition for (4.21), which is proved along the lines of the proof of [9, Theorem 4.4].

PROPOSITION 4.21. *If $B^*\bar z \in L^1(I;C^1(\overline{\omega}))$ and (3.11) holds with $\Psi(\varepsilon) = C\varepsilon$, then (4.21) is valid.*

Finally, we provide error estimates for cellwise constant control discretization.

THEOREM 4.22 (Cellwise constant controls). *Adopt the assumptions of Theorem 4.11 and let (3.11) hold with $\Psi(\varepsilon) = C\varepsilon^\kappa$. Moreover, suppose the variational in time and cellwise constant control discretization in space, i.e. $Q_\sigma(0,1) = L^2(I;Q_h)$. In addition, assume $\overline{\omega}\subset\Omega$, $u_0, u_d \in \mathcal{D}_{L^\infty}(-\Delta)$, and that (4.21) is satisfied. There is a $c>0$ not depending on k, h, $\bar\nu_{kh}$, and $\bar q_{kh}$ such that*

$$|\bar\nu - \bar\nu_{kh}| \le c|\log k|(k + h^{3/2}),$$
$$\|\bar q - \bar q_{kh}\|_{L^1(I\times\omega)}^{1/\kappa} \le c|\log k|^4|\log h|^7(k+h).$$

Proof. The error estimate Proposition 4.18 and stability of $\mathrm{Id} - \Pi_{h,0}$ in L^∞ yield

$$\|(\mathrm{Id} - \Pi_{h,0})B^*\hat z_{kh}\|_{L^\infty(I\times\omega)} \le c|\log k|^4|\log h|^7(k+h^2) + \|(\mathrm{Id} - \Pi_{h,0})B^*\hat z\|_{L^\infty(I\times\omega)}$$

Moreover, employing elliptic regularity with some $p>d$, we have the estimate

$$\|(\mathrm{Id} - \Pi_{h,0})B^*\hat z\|_{L^\infty(I\times\omega)} \le ch\|\hat z\|_{L^\infty(I;W^{2,p}(\omega))} \le ch\|\hat z\|_{L^\infty(I;\mathcal{D}_{L^p}(-\Delta))} \le ch.$$

Hence, using Lemma 4.9, Propositions 4.15 and 4.18 as well as the estimates for σ from Proposition 4.20 we infer the desired estimate. ∎

Using Proposition 4.16, we immediately obtain the following result.

COROLLARY 4.23 (Piecewise and cellwise constant controls). *The result of Theorem 4.22 remains valid for $Q_\sigma(0,1) = Q_{kh}(0,1)$ under the same assumptions.*

5. Numerical examples. We verify the theoretical results by numerical examples. In order to solve the optimization problem ($\hat P$), we employ the equivalence of time- and distance optimal control problems (see [3]), and solve a sequence of optimization problems with a fixed time. The resulting convex sub-problems for a fixed time are solved by an accelerated conditional gradient method. In an outer loop the optimal time is determined by a Newton method. For further details we refer to [3]. The computations are performed in MATLAB.

5.1. Example with purely time-dependent control. We take the example from [5, Section 5.2] with purely time-dependent controls for fixed spatially dependent functions but without control costs in the objective functional. Let

$$\begin{aligned} \Omega &= (0,1)^2, \quad \omega_1 = (0,0.5)\times(0,1), \quad \omega_2 = (0.5,1)\times(0,0.5), \\ B&\colon \mathbb{R}^2 \to L^2(\Omega), \quad Bq = q_1\chi_{\omega_1} + q_2\chi_{\omega_2}, \\ Q_{ad}(0,1) &= \{q \in L^2(I;\mathbb{R}^2)\colon -1.5 \le q \le 0\}, \\ u_0(x) &= 4\sin(\pi x_1^2)\sin(\pi x_2^3), \quad u_d(x) = 0, \quad \delta_0 = 1/10, \end{aligned}$$

where χ_{ω_1} and χ_{ω_2} denote the characteristic functions on ω_1 and ω_2. The spatial mesh is chosen such that the boundaries of ω_1 and ω_2 coincide with edges of the mesh. We discretize the control by piecewise constant functions in time.

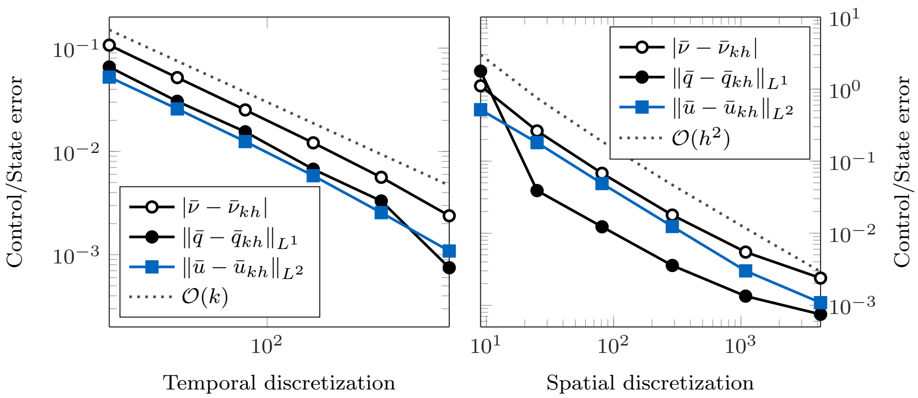

FIG. 5.1. *Discretization error for Example 5.1 with piecewise constant control discretization and refinement of the time interval for $N = 4225$ nodes (left) and refinement of the spatial discretization for $M = 640$ time steps (right). The reference solution is calculated for $N = 16641$ and $M = 1280$.*

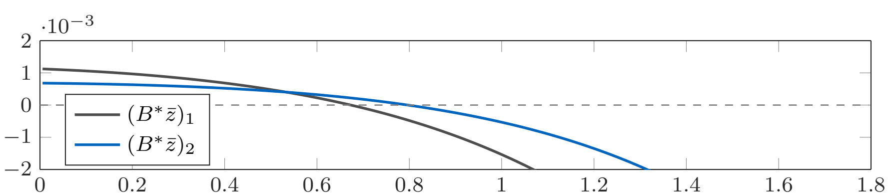

FIG. 5.2. *The switching function $B^*\bar z$ from Example 5.1 near zero.*

Since the exact solution is unknown, we calculate a numerical reference solution on an additionally refined grid. Comparing to the reference solution, we observe linear convergence with respect to k and quadratic order of convergence in h for all variables; see Figure 5.1. This is in accordance with Theorem 4.17, provided that (3.11) holds with $\Psi(\varepsilon) = C\varepsilon$. To evaluate this condition numerically, we plot the switching function in Figure 5.2, which appears to have only simple roots near the middle of the optimal time interval; see also the numerical test of (3.11) in Figure 5.5.

5.2. Example with distributed control on subdomain. Next, we consider the example from [5, Section 5.3] with distributed control on the subset $\omega = (0, 0.75)^2$

of the domain $\Omega = (0,1)^2$. As before we compare to a reference solution obtained numerically on a fine grid. The control bounds are $q_a = -5$, $q_b = 0$, and the data is

$$u_d(x) = -2\min\{x_1, 1-x_1, x_2, 1-x_2\}, \quad \delta_0 = 1/10,$$
$$u_0(x) = 4\sin(\pi x_1^2)\sin(\pi x_2)^3.$$

We consider the piecewise and cellwise constant discretization for the control variable. As in the first example we observe full order of convergence with respect to the terminal time. However, we do not have full order convergence for the control variable. From Figure 5.3 we approximately estimate the rate $k^{1/2}$ and h, respectively. In Figure 5.5, we evaluate the condition (3.11) numerically, and observe that

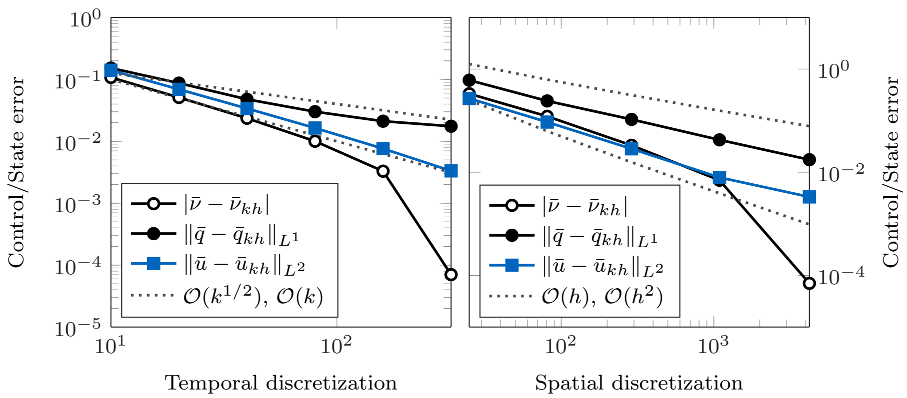

FIG. 5.3. *Discretization error for Example 5.2 with piecewise and cellwise constant control discretization and refinement of the time interval for $N = 4225$ nodes (left) and refinement of the spatial discretization for $M = 320$ time steps (right). The reference solution is calculated for $N = 16641$ and $M = 640$.*

the structural assumption does not appear to be satisfied with $\kappa = 1$ in this example; see also Figure 5.4 for a plot of the switching function. For this reason, we cannot expect the optimal rate $k + h$ for the control variable employing Theorem 4.22.

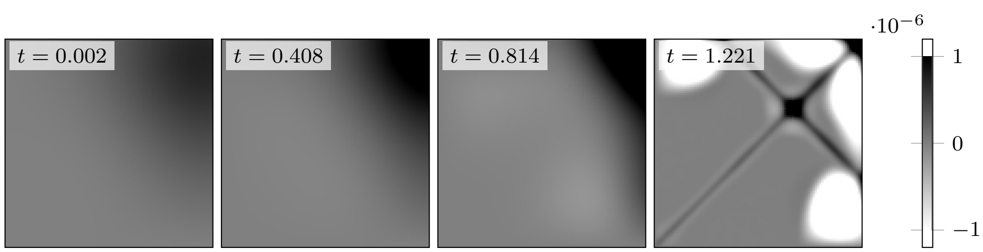

FIG. 5.4. *Snapshots of switching function $B^*\bar z$ from Example 5.2 (with color scale adapted to values below 10^{-6}).*

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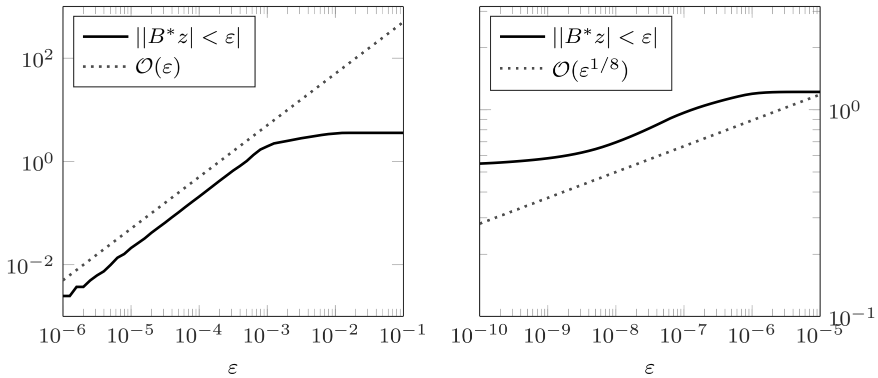

FIG. 5.5. *Numerical verification of structural assumption on adjoint state (3.11) for Example 5.1 (left) and Example 5.2 (right).*

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