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FINITE ELEMENT APPROXIMATION OF ELLIPTIC DIRICHLET OPTIMAL CONTROL PROBLEMS

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□ *We develop a priori error analysis for the finite element Galerkin discretization of elliptic Dirichlet optimal control problems. The state equation is given by an elliptic partial differential equation and the finite dimensional control variable enters the Dirichlet boundary conditions. We prove the optimal order of convergence and present a numerical example confirming our results.*

Keywords Dirichlet boundary control; Error estimates; Finite elements; Optimal control.

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1. INTRODUCTION

In this paper, we present *a priori* error analysis for the finite element discretization of elliptic optimal control problems, where a finite dimensional control variable enters the Dirichlet boundary conditions. The analysis of finite element approximations of optimization problems governed by partial differential equations is an area of active research, see, e.g., [1, 12, 17, 18]. The consideration of Dirichlet boundary control problems is more difficult than Neumann control or control by right-hand side, from both the theoretical and the numerical point of view, because the Dirichlet boundary conditions do not directly enter the variational setting. Moreover, the necessary optimality conditions described by an optimality system usually contain the normal derivative of the adjoint state, see, e.g., [16] or [8]. This fact complicates the achievement of the optimal order of convergence in the context of finite element discretization.

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The Dirichlet boundary optimal control problems play an important role in the context of the computational fluid dynamics, see, e.g., [3, 9, 13]. However, to our knowledge, there are only few published results discussing the question of the *a priori* error analysis for such problems, see [2, 11, 15]. Further, often the solution of the Dirichlet boundary optimal control problem is based on the direct discretization of the optimality system involving normal derivative of the adjoint state. This leads to *reduced order of convergence*, see the discussion in the sequel.

For clarity of presentation, we concentrate on the following optimal control problem: Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $L^2(\Omega)$ the corresponding Lebesgue space with inner product and norm denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, and $H^r(\Omega)$ the Sobolev space of order $r \in \mathbb{R}$. We denote $V = H^1(\Omega)$ and for a nonempty part of boundary $\Gamma_D \subset \partial\Omega$ consisting of some edges of $\partial\Omega$, where the Dirichlet boundary conditions will be posed, we set:

$$V^0 = \{v \in V \mid v = 0 \text{ on } \Gamma_D\}.$$

For the space $L^2(\Gamma_D)$ with the scalar product $(\cdot,\cdot)_{\Gamma_D}$ and the norm $\|\cdot\|_{\Gamma_D}$, the trace-operator $\gamma_D : V \to H^{1/2}(\Gamma_D)$ is a continuous linear operator.

Let $Q = \mathbb{R}^n$ be a finite dimensional control space with scalar product $(\cdot,\cdot)_Q$ and a norm $\|\cdot\|_Q$, where the *control variable* q will be sought. For covering box constraints on the control variable, we introduce an admissible set Q_{ad} given by:

$$Q_{ad} = \{q \in Q \mid q_- \le q \le q_+\},$$

where

$$q_-, q_+ \in Q, \quad q_- < q_+\,,$$

and the inequalities have to be understood component-wise. The state equation is formulated as follows: Find $u \in V$

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega,\\ u &= \gamma_D(Bq) && \text{on } \Gamma_D, \\ \partial_n u &= 0 && \text{on } \partial\Omega\backslash\Gamma_D, \end{aligned} \tag{1.1}$$

where $B : Q \to V$ is a continuous linear operator. The usual weak formulation of the state equation (1.1) is

$$u \in Bq + V^0 : (\nabla u, \nabla\phi) = (f, \phi) \quad \forall \phi \in V^0. \tag{1.2}$$

The optimal control problem is formulated as follows:

$$\text{Minimize } J(q,u) = \frac{1}{2}\|u-\bar{u}\|^2 + \frac{\alpha}{2}\|q\|_Q^2, \quad q \in Q_{ad}, \text{ subject to (1.2).} \tag{1.3}$$

Here, $\bar{u} \in L^2(\Omega)$ denotes a desired state and a cost term with $\alpha > 0$ added. An example of a typical choice of boundary operator B is

$$Bq = \sum_{i=1}^{n} q_i g_i, \quad g_i \in V.$$

The optimization problem (1.3) admits a unique solution. This can be shown with standard arguments and will be done in the next section for completeness.

The state equation is discretized by a conforming finite element Galerkin method defined on a family $\{\mathcal{T}_h\}_{h>0}$ of shape regular quasi-uniform meshes $\mathcal{T}_h = \{K\}$ consisting of closed *cells* K that are either triangles or quadrilaterals. The straight parts that make up the boundary ∂K of a cell K are called *faces*. The mesh parameter h is defined as a cell-wise constant function by setting $h_{|K} = h_K$ and h_K is the diameter of K. Usually, we use the symbol h also for the maximal cell size, i.e.,

$$h = \max_{K\in\mathcal{T}_h} h_K. \tag{1.4}$$

On the mesh $\mathcal{T}_h$ we define finite element spaces $V_h \subset V$ consisting of linear or bilinear shape functions, see, e.g., [5] or [14]. Moreover, we introduce the space V_h^0 by:

$$V_h^0 = \{v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_D\}.$$

With these ingredients, the discretized optimization problem is formulated as follows:

$$\text{Minimize } J(q_h, u_h), \quad q_h \in Q_{ad}, \tag{1.5}$$

under the constraint

$$u_h \in B_h q_h + V_h^0 : (\nabla u_h, \nabla\phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h^0. \tag{1.6}$$

Here, B_h denotes a discrete version of the operator B, which is given by:

$$B_h q = i_h B q,$$

where $i_h : V \to V_h$ is a (H^1-stable) interpolation operator, see, e.g., [7].

The main purpose of this paper is to analyze the asymptotic behavior of the error $\|q - q_h\|_Q$ for h tending to zero. We will show that

$$\|q - q_h\|_Q = \mathcal{O}(h^2). \tag{1.7}$$

To obtain this result, one should take care of the solution of discretized problem (1.5)–(1.6). The solution of this problem is determined using an optimality system, describing first-order necessary optimality conditions. We will show that there are two possibilities to pose the optimality system on the continuous level, which are equivalent. The first (standard) possibility reads:

$$u \in Bq + V^0 : (\nabla u, \nabla\phi) = (f, \phi) \quad \forall\phi \in V^0, \tag{1.8}$$

$$z \in V^0 : (\nabla\phi, \nabla z) = (u - \bar{u}, \phi) \quad \forall\phi \in V^0, \tag{1.9}$$

$$q \in Q : \alpha(q, \delta q - q)_Q - (B(\delta q - q), \partial_n z)_{\Gamma_D} \geq 0 \quad \forall\delta q \in Q_{ad}. \tag{1.10}$$

However, the discrete solution based on the discrete counterpart of this system leads in general only to *first-order convergence*, i.e., $\mathcal{O}(h)$, as discussed in the sequel. To overcome this drawback, we will propose another formulation of the optimality system, which can be obtained from (1.8)–(1.10) by virtue of the integration by parts in (1.9) and an insertion in (1.10). This system is then given by:

$$u \in Bq + V^0 : (\nabla u, \nabla\phi) = (f, \phi) \quad \forall\phi \in V^0, \tag{1.11}$$

$$z \in V^0 : (\nabla\phi, \nabla z) = (u - \bar{u}, \phi) \quad \forall\phi \in V^0, \tag{1.12}$$

$$q \in Q : \alpha(q, \delta q - q)_Q + (u - \bar{u}, B(\delta q - q)) - (\nabla B(\delta q - q), \nabla z) \geq 0 \quad \forall\delta q \in Q_{ad}. \tag{1.13}$$

Although the systems (1.8)–(1.10) and (1.11)–(1.13) are equivalent on the continuous level, the direct translation of these systems to the discrete level *does not lead* to equivalent formulations. We will show that the solution to the discrete version of (1.11)–(1.13) allows for the promised *second order convergence* result (1.7).

Remark 1.1. We emphasize that the consideration of a finite dimension control space Q is on the one hand motivated by practical applications and on the other hand essential for achieving the second-order convergence result (1.7). In a recent work [6], the authors provide an error estimate of order $\mathcal{O}(h^{1-1/p})$ with some $p > 2$ for the Dirichlet control problem with an infinite dimensional control space $Q = L^2(\partial\Omega)$.

The outline of the paper is as follows: In the next section, we briefly discuss the existence and uniqueness of the solution to the optimization problem (1.3), state the optimality systems, and describe a reduced formulation of the problem. Thereafter, in Section 3 we prove our main result. In Section 4 we discuss a numerical example illustrating first order of convergence for the discretization based on (1.8)–(1.10) and the established optimal convergence in the case of using (1.11)–(1.13). Possible extensions are addressed in the last section.

2. OPTIMIZATION PROBLEM

In this section, we discuss the optimization problem under consideration. Throughout we make the following regularity assumption on the domain Ω and the Dirichlet part of the boundary Γ_D:

Assumption 2.1. For any $g \in L^2(\Omega)$, the weak solution $v \in V_0$ of

$$-\Delta v = g \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \Gamma_D, \qquad \partial_n v = 0 \quad \text{on } \partial\Omega \backslash \Gamma_D$$

is in $H^2(\Omega)$ and satisfies the following estimate:

$$\|v\|_{H^2(\Omega)} \leq c\|g\|_{L^2(\Omega)},$$

where the constant c does not depend on $g \in L^2(\Omega)$.

Remark 2.2. Assumption 2.1 is satisfied if the following conditions are fulfilled:

- the maximal corner between two "Dirichlet edges" is not larger than π;
- the maximal corner between two "Neumann edges" is not larger than π;
- the maximal corner between Dirichlet and Neumann edges ("mixed corner") is not larger than $\pi/2$.

See [10] for details. The assumption is also obviously satisfied if Ω is convex and $\Gamma_D = \partial\Omega$.

In addition, we assume the following regularity of the boundary operator B:

Assumption 2.3.

$$Bq \in H^{\frac{5}{2}}(\Omega) \quad \forall q \in Q_{ad}.$$

This assumption allows for H^2-regularity of the state variable and is required in the sequel for the error analysis. Note that due to the finite dimension of Q, this assumption implies the boundedness of the operator $B : Q \to H^{\frac{5}{2}}(\Omega)$. The corresponding operator norm is denoted by:

$$\|B\| = \sup_{q \in Q, \|q\|=1} \|Bq\|_{H^{\frac{5}{2}}(\Omega)}. \tag{2.1}$$

This assumption is fulfilled if $g_i \in H^{\frac{5}{2}}(\Omega)$, $i = 1, \ldots n$.

Proposition 2.4. *The optimization problem* (1.3) *admits a unique solution*

$$(u, q) \in H^2(\Omega) \times Q_{ad}.$$

Proof. The proof of the existence uses a standard variational argument, see, e.g., [16] or [19]. For a minimizing sequence $\{q_n, u_n\}$, we choose a subsequence $q_k \to q \in Q_{ad}$. This is possible due to the fact that $\alpha > 0$ and the sequence $\{q_n\}$ is bounded. Due to the continuous dependence of the Dirichlet boundary data, the corresponding subsequence $\{u_k\}$ converges to $u \in V$, which solves the state equation (1.2) for the boundary data Bq. This implies the existence. By virtue of Assumptions 2.1 and 2.3 we obtain $u \in H^2(\Omega)$.

The proof of uniqueness relies on the convexity of Q_{ad} and on the strict convexity of the cost functional $J(q, u)$. □

Let the continuous affine-linear operator $S : Q \to V$ denote the solution operator for the state equation (1.2). There holds:

$$S(q) \in Bq + V^0 : (\nabla S(q), \nabla \phi) = (f, \phi) \quad \forall \phi \in V^0 \;\; \forall q \in Q.$$

This allows us to eliminate the constraint of the state equation by introduction of the *reduced cost functional* $j : Q \to \mathbb{R}$:

$$j(q) = J(q, S(q)). \tag{2.2}$$

The reduced optimization problem is then formulated as:

$$\text{Minimize } j(q), \quad q \in Q_{ad}. \tag{2.3}$$

The necessary optimality condition for this optimization problem relies on the convexity of Q_{ad} and reads:

$$j'(q)(\delta q - q) \geq 0 \quad \forall \delta q \in Q_{ad}. \tag{2.4}$$

The second derivative $j''(q)$ does not depend on q any more due to the linear-quadratic structure of the optimization problem (1.3). Moreover, there holds:

$$j''(q)(p,p) \geq \gamma \|p\|_Q^2 \quad \forall p \in Q,$$

with a positive constant

$$\gamma \geq \alpha > 0.$$

Therefore, the functional j is convex and the condition (2.4) is also sufficient for the optimality of q.

In the next proposition, we give two representations of the derivatives $j'(q)$ leading to the optimality systems (1.8)–(1.10) and (1.11)–(1.13), respectively.

Proposition 2.5. *Let for $q \in Q$, the state variable $u = S(q)$ be the solution of the state equation* (1.2) *and $z \in V^0$ be the adjoint state determined by:*

$$(\nabla\phi, \nabla z) = (u - \bar{u}, \phi) \quad \forall \phi \in V^0. \tag{2.5}$$

Then, the derivative $j'(q)(\delta q)$ for a given direction δq admits the two following representations:

(i) $j'(q)(\delta q) = \alpha(q, \delta q)_Q - (B\delta q, \partial_n z)_{\Gamma_D}$,
(ii) $j'(q)(\delta q) = \alpha(q, \delta q)_Q + (u - \bar{u}, B\delta q) - (\nabla B\delta q, \nabla z)$.

Remark 2.6. Due to Assumption 2.1, we obtain $z \in V^0 \cap H^2(\Omega)$. Therefore, the equivalence of the representations (i) and (ii) can be verified in virtue of integration by parts of the equation (2.5).

Proof. Due to Remark 2.6, it is sufficient to prove (ii). Let δu denote the derivative of $u = S(q)$ in the direction δq. Then δu is determined by the following equation:

$$\delta u \in B\delta q + V^0 : (\nabla \delta u, \nabla \phi) = 0 \quad \forall \phi \in V^0.$$

Therefore, we obtain using chain rule:

$$\begin{aligned} j'(q)(\delta q) &= \alpha(q, \delta q)_Q + (u - \bar{u}, \delta u) \\ &= \alpha(q, \delta q)_Q + (u - \bar{u}, \delta u - B\delta q) + (u - \bar{u}, B\delta q). \end{aligned}$$

Due to the adjoint equation (2.5) and the fact that $\delta u - B\delta q \in V^0$, there holds:

$$j'(q)(\delta q) = \alpha(q, \delta q)_Q + (\nabla(\delta u - B\delta q), \nabla z) + (u - \bar{u}, B\delta q).$$

To complete the proof, we use the fact that $(\nabla \delta u, \nabla z)$ vanishes. □

The necessary optimality condition (2.4) together with the representations from the above proposition implies that the solution to the optimization problem (2.4) can be characterized equivalently by each of the optimality systems (1.8)–(1.10) or (1.11)–(1.13). In literature, see, e.g., [16], one finds optimality systems similar to (1.8)–(1.10). Therefore, this system is often used as a basis for the numerical solution of the underlying optimization problem. However, in the sequel, we will show that the two representations from the above proposition are *not equivalent* on the discrete level, and only the second one leads to optimal order of convergence.

3. *A PRIORI* ERROR ANALYSIS

In this section, we prove the optimal order of convergence for the finite element discretization of the optimization problem under consideration.

The state equation is discretized by finite element method leading to the discrete equation (1.6). The corresponding optimization problem (1.5)–(1.6) admits the unique solution (q_h, u_h). Similar to the continuous case, we introduce a discrete solution operator $S_h : Q \to V_h$ defined by:

$$S_h(q) \in B_h q + V_h^0 : (\nabla S_h(q), \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h^0 \ \forall q \in Q.$$

Then, we define a discrete reduced cost functional $j_h : Q \to \mathbb{R}$ by:

$$j_h(q) = J(q, S_h(q)).$$

The discrete problem (1.5)–(1.6) can be reformulated as:

$$\text{Minimize } j_h(q_h), \quad q_h \in Q_{ad}. \tag{3.1}$$

The corresponding optimality condition reads:

$$j_h'(q_h)(\delta q - q_h) \geq 0 \quad \forall \delta q \in Q_{ad}. \tag{3.2}$$

As before, there holds:

$$j_h''(q_h)(p,p) \geq \hat{\gamma} \, \|p\|_Q^2 \quad \forall p \in Q,$$

with a constant

$$\hat{\gamma} \geq \alpha > 0.$$

Remark 3.1. Using similar techniques as described below, one can show that

$$\hat{\gamma} \geq \max(\gamma - ch^2, \alpha),$$

with a positive constant c.

The derivative $j_h'(q_h)$ is expressed in the following proposition:

Proposition 3.2. *For $q_h \in Q$, let the state variable $u_h = S_h(q_h)$ be the solution of the discrete state equation* (1.6) *and $z_h \in V_h^0$ be the discrete adjoint state determined by:*

$$(\nabla\phi_h, \nabla z_h) = (u_h - \bar{u}, \phi_h) \quad \forall \phi_h \in V_h^0. \tag{3.3}$$

Then the derivative $j_h'(q_h)(\delta q)$ in a direction δq is given by:

$$j_h'(q_h)(\delta q) = \alpha(q_h, \delta q)_Q + (u_h - \bar{u}, B_h\delta q) - (\nabla B_h\delta q, \nabla z_h). \tag{3.4}$$

Proof. The proof is similar to the proof of Proposition 2.5. □

The representation (3.4) is a direct translation of the representation (ii) from Proposition 2.5 to the discrete level. Therefore, the optimality condition (3.2) is equivalent to the discrete version of the optimality system (1.11)–(1.13):

$$u_h \in B_h q_h + V_h^0 : (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h^0, \tag{3.5}$$

$$z_h \in V_h^0 : (\nabla\phi_h, \nabla z_h) = (u_h - \bar{u}, \phi_h) \quad \forall \phi_h \in V_h^0, \tag{3.6}$$

$$q_h \in Q : \alpha(q_h, \delta q - q_h)_Q + (u_h - \bar{u}, B_h(\delta q - q_h)) - (\nabla B_h(\delta q - q_h), \nabla z_h) \geq 0 \quad \forall \delta q \in Q_{ad}. \tag{3.7}$$

Remark 3.3. The direct translation of the representation (i) from Proposition 2.5 to the discrete level does not in general lead to a correct expression for $j_h'(q_h)$. It can be shown that

$$j_h'(q_h)(\delta q) = \alpha(q_h, \delta q)_Q - (B_h\delta q, \partial_n z_h)_{\Gamma_D} + \mathcal{O}(h).$$

Therefore, a discretization of the optimality system (1.8)–(1.10) does not lead to the exact discrete gradient. This fact may slow down the corresponding optimization algorithm and explains the reduced order of approximation, if this system is used; see the numerical example in Section 4.

The main result of this article is given in the following theorem:

Theorem 3.4. *Let q be the solution of the optimization problem* (2.3) *and q_h of the discrete optimization problem* (3.1)*, then there holds:*

$$\|q - q_h\|_Q \leq \frac{c}{\hat{\gamma}} h^2 \left(\|f\| + \|\bar{u}\| + 1\right),$$

where the constant c depends neither on the discretization parameter h nor on $\hat{\gamma}$.

The proof of this theorem is divided into several steps and is given below.

First, we give an error estimate of the error $q - q_h$ by means of the difference between the gradients of j and j_h.

Proposition 3.5. *Let q be the solution of the optimization problem* (2.3) *and q_h of its discrete counterpart* (3.1). *Then there holds:*

$$\|q - q_h\|_Q \leq \frac{1}{\hat{\gamma}} \|\nabla j(q) - \nabla j_h(q)\|_Q.$$

Proof. Due to the positive definiteness of $j_h''(q_h)$ and the fact that j_h is quadratic, we obtain:

$$\|q - q_h\|_Q^2 \leq \frac{1}{\hat{\gamma}} j_h''(q_h)(q_h - q, q_h - q) = \frac{1}{\hat{\gamma}} \left(j_h'(q_h)(q_h - q) - j_h'(q)(q_h - q) \right).$$

By virtue of the optimality conditions (2.4) and (3.2) we have:

$$j_h'(q_h)(q_h - q) \leq 0 \leq j'(q)(q_h - q).$$

This implies:

$$\begin{aligned} \|q - q_h\|_Q^2 &\leq \frac{1}{\hat{\gamma}} (j'(q)(q_h - q) - j_h'(q)(q_h - q)) \\ &\leq \frac{1}{\hat{\gamma}} \|\nabla j(q) - \nabla j_h(q)\|_Q \|q - q_h\|_Q. \end{aligned}$$

This completes the proof. □

It remains to show the promised estimate for the difference between the derivatives of j and j_h for a given direction $r \in Q$, i.e., for

$$j'(q)(r) - j_h'(q)(r). \tag{3.8}$$

To this end, we introduce $\tilde{u}_h = S_h(q)$, i.e., as the solution of

$$\tilde{u}_h \in B_h q + V_h^0 : (\nabla \tilde{u}_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h^0. \tag{3.9}$$

Further, we define $\tilde{z}_h \in V_h^0$ as the solution of

$$(\nabla \phi_h, \nabla \tilde{z}_h) = (\tilde{u}_h - \bar{u}, \phi_h) \quad \forall \phi_h \in V_h^0.$$

This allows us to rewrite the derivative $j_h'(q)(r)$ in virtue of Proposition 3.2 as:

$$j_h'(q)(r) = \alpha(q, r)_Q + (\tilde{u}_h - \bar{u}, B_h r) - (\nabla B_h r, \nabla \tilde{z}_h).$$

The next proposition provides necessary estimates of the errors $u - \tilde{u}_h$ and $z - \tilde{z}_h$.

Proposition 3.6. *The following estimates hold:*

(i) $\|u - \tilde{u}_h\|_2 + h\|\nabla(u - \tilde{u}_h)\|_2 \le ch^2 \left(\|f\| + \|\bar{u}\| + 1\right),$
(ii) $\|z - \tilde{z}_h\|_2 + h\|\nabla(z - \tilde{z}_h)\|_2 \le ch^2 \left(\|f\| + \|\bar{u}\| + 1\right),$

where c is a generic constant, which does not depend on h.

Proof. The proof uses some *a priori* bounds for u and z. We obtain by the standard elliptic regularity theory and by virtue of Assumptions 2.1 and 2.3:

$$\|u\|_{H^2(\Omega)} \le \|f\| + \|Bq\|_{H^2(\Omega)},$$
$$\|z\|_{H^2(\Omega)} \le \|u\| + \|\bar{u}\|.$$

Using the fact that q lies in the bounded set Q_{ad}, we obtain:

$$\|Bq\|_{H^2(\Omega)} \le c\|B\|,$$

where the constant c depends on q_-, q_+. This implies:

$$\|u\|_{H^2(\Omega)} \le c(\|f\| + \|\bar{u}\| + 1),$$
$$\|z\|_{H^2(\Omega)} \le c(\|f\| + \|\bar{u}\| + 1).$$

(i) The state u is determined by the state equation (1.2) and $\tilde{u}_h$ is defined in (3.9). Therefore, there holds the Galerkin orthogonality for the error $e = u - \tilde{u}_h$:

$$(\nabla e, \nabla \phi_h) = 0 \quad \forall \phi_h \in V_h^0.$$

Hence,

$$\|\nabla e\|^2 = (\nabla e, \nabla(u - Bq)) + (\nabla e, \nabla(Bq - B_h q)) + (\nabla e, \nabla(B_h q - \tilde{u}_h)). \quad (3.10)$$

Due to $u - Bq \in V^0$, we have for the first term:

$$\begin{aligned}(\nabla e, \nabla(u - Bq)) &= (\nabla e, \nabla((u - Bq) - i_h(u - Bq))) \\ &\le \|\nabla e\| \|\nabla((u - Bq) - i_h(u - Bq))\|.\end{aligned}$$

The third term in (3.10) vanishes and we obtain using interpolation estimates and recalling $B_h q = i_h Bq$:

$$\|\nabla e\| \le ch \left(\|f\| + \|Bq\|_{H^2(\Omega)}\right).$$

The assertion for $\|e\|$ uses similar techniques as before and a standard Aubin–Nitsche trick.

(ii) For proving the assertion for z, we introduce an additional discrete variable $\bar{z}_h \in V_h^0$ determined by the equation:

$$(\nabla\phi_h, \nabla\bar{z}_h) = (u - \bar{u}, \phi_h) \quad \forall\phi_h \in V_h^0.$$

The error $e_z = z - \tilde{z}_h$ is split like $e_z = e_1 + e_2$, with $e_1 = z - \bar{z}_h$ and $e_2 = \bar{z}_h - \tilde{z}_h$. For the Ritz-projection error e_1, there holds obviously:

$$\|e_1\| + h\|\nabla e_1\| \leq ch^2\|\nabla^2 z\|.$$

For the error $e_2 \in V_h^0$, there holds a perturbed Galerkin orthogonality:

$$(\nabla\phi_h, \nabla e_2) = (u - \tilde{u}_h, \phi_h) \quad \forall\phi_h \in V_h^0.$$

Hence,

$$\|\nabla e_2\|^2 = (u - \tilde{u}_h, e_2) \leq \|u - \tilde{u}_h\|\,\|e_2\|.$$

This implies:

$$\|\nabla e_2\| \leq c_p\|u - \tilde{u}_h\|,$$

where c_p is a constant of the Poincaré inequality. The assertion for $\|e_2\|$ uses again the Aubin–Nitsche trick. □

Let us now come back to the proof of Theorem 3.4.

Proof. We have to estimate (3.8). There holds:

$$j'(q)(r) - j_h'(q)(r) = (u - \bar{u}, Br - B_h r) - (\nabla(Br - B_h r), \nabla z) + (u - \tilde{u}_h, B_h r) - (\nabla B_h r, \nabla(z - \tilde{z}_h)). \quad (3.11)$$

For the first and second terms we obtain integrating by parts and exploiting the adjoint equation (2.5)

$$(u - \bar{u}, Br - B_h r) - (\nabla(Br - B_h r), \nabla z) = -(Br - B_h r, \partial_n z)_{\Gamma_D}.$$

Using an interpolation estimate and the trace theorem, we obtain:

$$(Br - B_h r, \partial_n z)_{\Gamma_D} \leq ch^2\|Br\|_{H^{5/2}(\Omega)}\|z\|_{H^2(\Omega)}.$$

For the third term in (3.11), we obtain using Proposition 3.6

$$(u - \tilde{u}_h, B_h r) \leq \|u - \tilde{u}_h\| \, \|B_h r\| \leq ch^2 \|Br\|_{H^2(\Omega)} \left(\|f\| + \|\bar{u}\| + 1\right).$$

For the last term in (3.11), we have:

$$-(\nabla B_h r, \nabla(z - \tilde{z}_h)) = -(\nabla Br, \nabla(z - \tilde{z}_h)) + (\nabla(Br - B_h r), \nabla(z - \tilde{z}_h)). \quad (3.12)$$

For the second term here, we obtain using interpolation estimates and Proposition 3.6

$$(\nabla(Br - B_h r), \nabla(z - \tilde{z}_h)) \leq ch^2 \|Br\|_{H^2(\Omega)} \left(\|f\| + \|\bar{u}\| + 1\right).$$

In order to estimate the first term in (3.12), we introduce an additional variable $v \in V^0$ determined by:

$$(\nabla v, \nabla \phi) = -(\nabla Br, \nabla \phi) \quad \forall \phi \in V^0.$$

There holds: $v \in V^0 \cap H^2(\Omega)$ and

$$\|v\|_{H^2(\Omega)} \leq \|Br\|_{H^2(\Omega)}.$$

Then, we obtain for the first term in (3.12):

$$\begin{aligned}
-(\nabla Br, \nabla(z - \tilde{z}_h)) &= (\nabla v, \nabla(z - \tilde{z}_h)) \\
&= (\nabla(v - i_h v), \nabla(z - \tilde{z}_h)) + (\nabla i_h v, \nabla(z - \tilde{z}_h)) \\
&= (\nabla(v - i_h v), \nabla(z - \tilde{z}_h)) + (i_h v, u - \tilde{u}_h) \\
&\leq ch^2 \|v\|_{H^2(\Omega)} \left(\|f\| + \|\bar{u}\| + 1\right).
\end{aligned}$$

To complete the proof, we note that due to the finite dimension of Q, B is a continuous operator as $B : Q \to H^{5/2}(\Omega)$ and therefore:

$$\|Br\|_{H^{5/2}(\Omega)} \leq \|B\| \, \|r\|,$$

see (2.1). □

4. NUMERICAL EXAMPLE

In this section, we present a numerical example confirming the a *priori* error estimates from the previous section. The computations are done on the basis of the package *RoDoBo* for treating optimization problems governed by partial differential equations and the finite element toolkit *Gascoigne3D*.

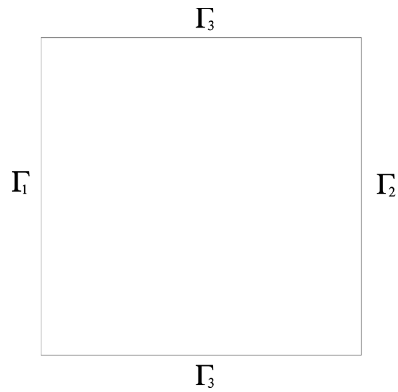

FIGURE 1 The computational domain.

The state equation (1.1) is considered on the computational domain $\Omega = (0,1)^2$, (see Fig. 1) and is given by:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= Bq && \text{on } \Gamma_1, \\ u &= 0 && \text{on } \Gamma_2, \\ \partial_n u &= 0 && \text{on } \Gamma_3. \end{aligned}$$

The Dirichlet control operator B is given by:

$$Bq = q_1 + q_2 \sin(2\pi y) + q_3 \cos(2\pi y)$$

and the control variable q is searched in $Q_{ad} = Q = \mathbb{R}^3$ by minimization of (1.3). The right-hand side f is given by:

$$f(x, y) = 10 \sin(2\pi x) \sin(2\pi y)$$

and the desired state is $\bar{u} = 1$.

TABLE 1 Convergence of $\|q - q_h\|$ for h tending to zero

	Based on (1.8)–(1.10)		Based on (1.11)–(1.13)	
h	$\|q - q_h\|$	Reduction rate	$\|q - q_h\|$	Reduction rate
2^{-2}	2.38e−1	—	1.47e−2	—
2^{-3}	1.21e−1	1.97	2.87e−3	5.12
2^{-4}	6.06e−2	2.00	6.81e−4	4.21
2^{-5}	3.03e−2	2.00	1.68e−4	4.05
2^{-6}	1.51e−2	2.01	4.17e−05	4.03
2^{-7}	7.56e−3	2.00	1.03e−05	4.05
2^{-8}	3.78e−3	2.00	2.55e−06	4.04

TABLE 2 Computation of the discrete gradients

h	g_1	$\lvert j_h'(q)(r) - g_1\rvert / \lvert j_h'(q)(r)\rvert$	g_2	$\lvert j_h'(q)(r) - g_2\rvert / \lvert j_h'(q)(r)\rvert$
2^{-2}	−1.34e−1	3.14e−1	−9.50e−2	1.46e−16
2^{-3}	−1.19e−1	3.37e−1	−7.91e−2	5.26e−15
2^{-4}	−1.00e−1	2.53e−1	−7.48e−2	1.11e−15
2^{-5}	−8.78e−2	1.61e−1	−7.37e−2	1.32e−14
2^{-6}	−8.08e−2	9.18e−2	−7.34e−2	2.61e−14
2^{-7}	−7.71e−2	4.94e−2	−7.33e−2	3.23e−14
2^{-8}	−7.52e−2	2.56e−2	−7.33e−2	1.10e−14

In Table 1, we collect the results concerning the convergence of $\|q - q_h\|$ for discretizations based the optimality systems (1.8)–(1.10) and (1.11)–(1.13). As expected, we observe the first-order convergence for (1.8)–(1.10) and the second-order convergence for (1.11)–(1.13) as stated in Theorem 3.4.

In Proposition 2.5 we have provided two representations of the gradient of the reduced cost functional. However, as mentioned in Remark 3.3, only the representation (3.4) from Proposition 3.2 leads to exact discrete gradients. The results collected in Table 2 confirm this fact. For $q = r = (1, 1, 1)^t \in Q$, we compute

$$g_1 = \alpha(q, r)_Q - (B_h r, \partial_n z_h)_{\Gamma_D}$$

and

$$g_2 = \alpha(q, r)_Q + (u_h - \bar{u}, B_h r) - (\nabla B_h r, \nabla z_h)$$

and compare them with $j_h'(q)(r)$ for different h.

5. CONCLUSIONS

In this paper, we have derived an *a priori* error estimate for finite element discretization of Dirichlet optimal control problem governed by elliptic partial differential equation. In the following, we list some possible directions of generalization of our result.

1. *Error estimation for the state variable.* In Theorem 3.4, we have established an error estimate for the discretization error in the control variable q. With similar arguments, the following estimate holds for the state variable:

$$\|u - u_h\| + h\|\nabla(u - u_h)\| \leq \frac{c}{\hat{\gamma}} h^2 \left(\|f\| + \|\bar{u}\| + 1\right).$$

2. *Higher-order approximation.* The result of Theorem 3.4 can be directly extended to the case of higher-order finite element, provided the solution possesses the corresponding regularity.

3. *More general linear elliptic equations.* Theorem 3.4 can be directly extended to more general elliptic equations of the form:

$$-\nabla \cdot (a(x)\nabla u + b(x)u) + c(x)u = f.$$

4. *Nonlinear elliptic equations.* The application of our techniques does not require the linear-quadratic structure of optimization problem. Therefore, the main result can be extended also to the case of nonlinear elliptic equations, provided that the derivative of the corresponding nonlinear differential operator is coercive in a neighborhood of the solution of the optimal control problem.

5. *Parabolic equations.* The generalization to the case of a parabolic state equation is not straightforward and requires additional amount of work. This will be done in a forthcoming paper based on space time finite element discretization, see [4].

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